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ON HYPONORMAL OPERATORS

I. H. SHETH

1. An operator T defined on a Hilbert space H is said to be hyponormal if $T^*T-TT^*\geqq 0$, or equivalently if $\|T^*x\|\leqq\|Tx\|$ for every $x\in H$. An operator T is said to be seminormal if either T or T^* is hyponormal. If T is hyponormal, then $T-zI$ is also hyponormal for all complex values of z.

The spectrum of an operator T, in symbols $\sigma(T)$, is the set of all those complex numbers z for which $T-zI$ is not invertible. A complex number z is said to be an approximate proper value for the operator T in case there exists a sequence x_n such that $\|x_n\|=1$ and $\|(T-zI)x_n\|\to 0$. The approximate point spectrum of an operator T, in symbols $\Pi(T)$, is the set of approximate proper values of T. The numerical range of an operator T, denoted by $W(T)$, is the set defined by the relation

$$W(T) = \{(Tx, x): \|x\| = 1\}.$$

Cl $(W(T))$ will, as usual, denote the closure of $W(T)$. An operator S is said to be similar to an operator T in case there exists an invertible operator A such that $S=A^{-1}TA$.

In this note, all the operators will relate to a Hilbert space H.

We shall prove the following theorem.

THEOREM. *Let N be a hyponormal operator. If for an arbitrary operator A, for which $0\notin$ Cl $(W(A))$, $AN=N^*A$, then N is self-adjoint.*

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For proving this theorem, we need certain results which we formulate in the form of lemmas.

2. LEMMA 1. *Let T be a hyponormal operator and let $z_1, z_2 \in \Pi(T)$, $z_1 \neq z_2$. If x_n and y_n are the sequences of unit vectors of H such that $\|(T-z_1I)x_n\| \to 0$ and $\|(T-z_2I)y_n\| \to 0$, then $(x_n, y_n) \to 0$.*

PROOF. See [1, p. 170].

LEMMA 2. *If T is hyponormal, $\sigma(T^*) = \Pi(T^*)$.*

PROOF. See [2].

LEMMA 3. *If T is a hyponormal operator such that $\sigma(T)$ is a set of real numbers, then T is self-adjoint.*

PROOF. See [3, Theorem 4, Corollary 1].

LEMMA 4. *If an operator A is similar to an operator B, then A is bounded below iff B is bounded below. In other words if A and B are similar, then $\Pi(A) = \Pi(B)$.*

PROOF. Let $A = T^{-1}BT$ for an invertible operator T. Now if B is bounded below, then $B^*B \geqq \alpha I$ for some constant $\alpha > 0$. Since T is invertible, there exist constants $\beta > 0$ and $\gamma > 0$ such that $T^*T \geqq \beta I$ and $(TT^*)^{-1} = T^{*-1}T^{-1} \geqq \gamma I$.

Now $A^*A = T^*B^*T^{*-1}T^{-1}BT = (BT)^*T^{*-1}T^{-1}BT \geqq (BT)^*\gamma IBT = \gamma T^*B^*BT \geqq \gamma T^*\alpha IT = \alpha\gamma T^*T \geqq \alpha\beta\gamma I$ i.e. A is bounded below. Since the above process is reversible, the stated result follows.

The relation $\Pi(A) = \Pi(B)$ follows from the following two observations.

(i) If A is similar to B, then $A - zI$ is similar to $B - zI$ for all complex numbers z.

(ii) $z \notin \Pi(A)$ iff $A - zI$ is bounded below.

3. PROOF OF THE THEOREM. Since $0 \notin \mathrm{Cl}\,(W(A))$, A is invertible. Hence $N = A^{-1}N^*A$ and it follows from Lemmas 2 and 4 that $\sigma(N) = \sigma(N^*) = \Pi(N^*) = \Pi(N)$.

In order to complete the proof of the theorem, it is sufficient, by virtue of Lemma 3, to prove that $\sigma(N)$ is real. Suppose on the contrary that there exists a $z \in \sigma(N)$ such that $z \neq \bar{z}$. Since $z \in \sigma(N) = \Pi(N)$, there exists a sequence x_n of unit vectors such that $\|(N^* - \bar{z}I)x_n\| \leqq \|(N - zI)x_n\| \to 0$.

Since $0 \notin \mathrm{Cl}\,(W(A))$, the relation $\|(N^* - \bar{z}I)x_n\| = \|(ANA^{-1} - \bar{z}I)x_n\| = \|A(N - \bar{z}I)A^{-1}x_n\| \to 0$ implies that $\|(N - \bar{z}I)A^{-1}x_n\| \to 0$. Hence $(x_n, A^{-1}x_n) = (AA^{-1}x_n, A^{-1}x_n) \to 0$ by Lemma 1. Put y_n

$=A^{-1}x_n/\|A^{-1}x_n\|$, then $\|y_n\|=1$ and $(Ay_n, y_n)\to 0$ i.e. $0\in \mathrm{Cl}\ (W(A))$, a contradiction. This completes the proof of the theorem.

We deduce, as a corollary, the following result.

COROLLARY. *Let N be a seminormal operator. If for an arbitrary operator A, for which $0\notin \mathrm{Cl}\ (W(A))$, $AN=N^*A$, then N is self-adjoint.*

PROOF. Suppose that N^* is hyponormal. The proof of the theorem shows that $0\notin \mathrm{Cl}\ (W(A))$ implies $0\notin \mathrm{Cl}\ (W(A^{-1}))$. Now $AN=N^*A$ implies $A^{-1}N^*=NA^{-1}$ i.e. $BM=M^*B$, where $M=N^*$ is hyponormal and $0\notin \mathrm{Cl}\ (W(B))=\mathrm{Cl}\ (W(A^{-1}))$. Hence $M=M^*$ by the theorem i.e. $N=N^*$.

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