CHAPTER III

NUMERICAL RANGE OF HYPONORMAL OPERATORS

The relation $\Sigma(T) \subseteq Cl(W(T))$ is true for every operator T defined on a Hilbert space H. It was M. H. Stone[1] who first proved that the reverse inclusion (and hence $\Sigma(T) = Cl(W(T))$) holds in the above relation in case T is a normal operator (bounded or unbounded). Because of this property, a normal operator enjoys many other interesting properties. While giving a new proof of the result that for a bounded normal operator T, $\Sigma(T) = Cl(W(T))$, S. K. Berberian[2] conjectured in the year 1964 that the equality $\Sigma(T) = Cl(W(T))$ holds good in case of hyponormal operators also. This conjecture of Berberian has been proved to be true by J. G. Stampfli[3], C. R. Putnam[4], T. Yoshino and T. Saito[5]. We shall give a new proof of this result. We may mention here that the result can also be derived as a corollary of a general theorem established in chapter IV of the present thesis.

Berberian's conjecture is formulated as:

THEOREM 3.1. If T is hyponormal, then $\Sigma(T) = Cl(W(T))$.

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1) M. H. Stone [29] 2) S. K. Berberian [4] 3) J.G.Stampfli [27]
4) C. R. Putnam [21] 5) T. Yoshino and T. Saito [23]

The proof of this theorem will essentially hinge on the following result due to G. H. Orland[1]:

THEOREM 3.A. *If* T *is an operator satisfying* $\|R_\alpha\| \leq d(\alpha, \Sigma(T))^{-1}$ *for all* $\alpha \notin \Sigma(T)$, *then* $\Sigma(T) = Cl(W(T))$.

For proving theorem 3.1, we shall need two more results which are being given below in the form of two lemmas.

If T is an arbitrary non-zero operator, then s(T) is a non-empty compact set in the complex plane. Hence the set S_T defined by the relation

$$S_T = \{ |\alpha| \ ; \ \alpha \in s(T) \}$$

is also a compact set on the real line and therefore it attains its bounds. Let $T_1 = \inf S_T$ and $T_L = \sup S_T$. Then both T_1 and T_L are in S_T. Now we can state our first lemma.

LEMMA 3.1. *If* T *is a positive semi-definite self-adjoint operator, then*

$$T_1 \|y\| \leq \|Ty\| \leq T_L \|y\| \quad \text{for all } y \in H.$$

PROOF: The first inequality follows from the facts that $T_1 = \min_{\|y\|=1} (Ty, y)$ and $(Ty, y) \leq \|Ty\| . \|y\|$ for all $y \in H$.

The second inequality follows from the definition of $\|T\|$ and the fact that s(T) contains a positive real number α such that $\|T\| = |\alpha|$.

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1) G. H. Orland [18]

This lemma yields the following corollaries:

COROLLARY 3.1. *If T is hyponormal, then*

$$T_1\|y\| \leq \|Ty\| \leq T_L\|y\| \text{ for all } y \in H.$$

PROOF: The first inequality is obvious for a non-invertible T. If T is invertible, let $T = UR$ be its polardecomposition. Then

$$R_1\|y\| \leq \|Ry\| \leq R_L\|y\|$$

for all $y \in H$ by lemma 3.1. Since $\|Ty\| = \|Ry\|$ for all $y \in H$, it will be sufficient to prove that $R_1 = T_1$ and $R_L = T_L$. But $T_L = \|T\| = \|R\| = R_L$ by lemma 2.3 of chapter II. Also, by corollary 2.1 of chapter II, T^{-1} is hyponormal. Hence

$$\frac{1}{R_1} = \|R^{-1}\| = \max_{\|y\|=1} \|R^{-1}y\| = \max_{\|y\|=1} \|T^{-1}Uy\|$$

$$= \max_{\|x\|=1} \|T^{-1}x\|$$

$$= \frac{1}{T_1},$$

again by lemma 2.3 of chapter II. Thus $T_1 = R_1$. This completes the proof of the corollary.

COROLLARY 3.2. *If T is hyponormal and s(T) lies on the unit circle, then T is unitary.*

PROOF: Since $0 \notin s(T)$, T is invertible. Moreover, as $T_L = T_1 = 1$, we have $\|Ty\| = \|y\|$ for all $y \in H$ by corollary 3.1 i.e. T is an isometry as well as an invertible operator. Hence T is unitary.

It may be remarked here that this result can also be derived as a corollary of the following theorem of J. G. Stampfli[1]:

THEOREM 2.B. *If T is hyponormal and s(T) lies on an arc, then T is normal.*

We also need another lemma:

LEMMA 3.2. *If T is hyponormal, then*
$\|R_\alpha\| \leq [d(\alpha, \Sigma(T))]^{-1}$ *for all* $\alpha \notin \Sigma(T)$.

PROOF: Since $\alpha \notin \Sigma(T)$, R_α exists. Also, by definition,

$$\|R_\alpha\| = \max_{\|y\|=1} \frac{\|R_\alpha y\|}{\|y\|}$$

$$= \max_{x \neq \theta} \frac{\|x\|}{\|(T - \alpha I)x\|}$$

$$= \max_{\|x\|=1} \frac{1}{\|(T - \alpha I)x\|}$$

$$= \frac{1}{\min_{\|x\|=1} \|(T - \alpha I)x\|}$$

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1) J. G. Stampfli [27]

But $T - \alpha I$ is an invertible hyponormal operator and $s(T - \alpha I) = \{\beta - \alpha \; ; \; \beta \in s(T)\}$. Hence $\|(T - \alpha I)x\| \geq \min \{|\beta - \alpha| \; ; \; \beta \in s(T)\}$ for all $x \in H$ with norm 1 by corollary 3.1 . Hence

$$\|R_\alpha\| = \frac{1}{\min\limits_{\|x\|=1} \|(T - \alpha I)x\|}$$

$$\leq \frac{1}{\min\limits_{\beta \in s(T)} |\beta - \alpha|}$$

$$= [d(\alpha \, , \, s(T))]^{-1} .$$

Moreover, the relation $s(T) \subseteq \Sigma(T)$ implies that $d(\alpha \, , \, s(T)) \geq d(\alpha \, , \, \Sigma(T))$ for every $\alpha \notin s(T)$. Hence

$$\|R_\alpha\| \leq [d(\alpha \, , \, s(T))]^{-1} \leq [d(\alpha \, , \, \Sigma(T))]^{-1} \quad \text{for}$$

$\alpha \notin \Sigma(T)$ as required.

The following corollary follows immediately from the proof of lemma 3.2 :

COROLLARY 3.3. *If T is hyponormal, then T satisfies the condition G_1.*

We can now easily prove theorem 3.1 .

PROOF OF THEOREM 3.1: It follows from lemma 3.2 that $\|R_{\alpha}\| \leq \left[d(\alpha , \Sigma(T))\right]^{-1}$ for all $\alpha \notin \Sigma(T)$. To complete the proof we have only to appeal to G. H. Orland's theorem referred to above which immediately gives $\Sigma(T) = Cl(W(T))$.