CHAPTER VI

NEARLY HAUSDORFF COMPACTIFICATIONS

Mathematicians including Alexandroff, Urysohn, Čech, Cartan, Wallman, Tychonoff and Lubben laid the foundation of the modern theory of Hausdorff extensions. Once the term "COMPACTNESS" was defined, it was a natural problem to try and extend a non-compact space to a compact space. The first general method in this direction was the one-point compactification in 1924 due to Alexandroff. In 1937 Čech developed a compactification having the maximal extension property by extending Tychonoff's idea of embedding a completely regular Hausdorff space X in a cube. Stone developed a similar compactification independently. This compactification is termed as Stone-Čech compactification and is denoted by βX for a space X. In fact, βX is maximal compactification of a Tychonoff space X. In 1938, Wallman gave a general method for constructing a T_1 compactification coinciding with βX.

A family B of subset of a space X is called a ring of sets if it is closed under finite intersections and finite unions. A subfamily α of non-empty members of a ring B is called a B-filter if α is closed under finite intersections and super sets. A B-filter α is a B-ultrafilter if it is not properly contained in any other B-filter. The filter concept was introduced in order to study convergence. Besides for describing convergence, collections of B-

ultrafilters have been used to construct topological spaces. Let $\varpi(B)$ denote the collection of all B-ultrafilters on X. For $z \in B$, let z^{ϖ} denote the members of $\varpi(B)$ which contain z. Taking $\{z^{\varpi} \mid z \in B\}$ as a base for closed sets we get topology on $\varpi(B)$ which is useful in the formation of compactifications. In 1938, Wallman considered the case in which B is the family of all closed sets of a T_1 space X. Wallman showed that under these conditions $\varpi(B)$ is not necessarily Hausdorff. We recall the definition of a Wallman base. A *Wallman base* L on a space X is a ring of subsets of X satisfying:

(i) $\varphi, X \in L$,

(ii) L is a closed base for X,

(iii) if $A \in L$ and $x \in X - A$ then there is a $B \in L$ such that $x \in B$, $A \cap B = \varphi$ and

(iv) if $A, B \in L$ such that $A \subseteq X - B$ then there are $C, D \in L$ such that $A \subseteq X - C \subseteq D \subseteq X - B$.

We have observed that for a T_1 topological space X having more than one point, the family $R(X)$ of all regular closed subsets of X, is not a ring in general but if we consider the family $Rf(X)$ of all finite intersections of members of $R(X)$ then the family $Rf(X)$ forms a ring. In this chapter, we attempt to construct a compactification rX for a non-Tychonoff space X by using the family $Rf(X)$. We observe that the resulting compactification rX is a non-Hausdorff T_1 space. A separation axiom stronger than T_1 but weaker than T_2 naturally exists on rX which we term as nearly Hausdorffness. In the section 1, of this chapter we define and study this separation axiom. In the section 2, we discuss the construction of the space rX and in the last section we discuss the natural question under what conditions $rX = \beta X$?

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1. Nearly Hausdorff Spaces.

In this section, we define and study "Nearly Hausdorffness" a separation axiom stronger than T_1 but weaker than T_2. We introduce a topological property π and note that a space with property π is a nearly Hausdorff space if and only if it is Urysohn. A flow diagram showing various implications about separation axioms supported by necessary counter examples is included in this section.

Definition 6.1.1. Distinct points x and y in a topological space X are said to be *separated* by subsets A and B of X if $x \in A - B$ and $y \in B - A$.

Definition 6.1.2. A topological space X is called a *nearly Hausdorff space* if for every pair of distinct points of X there exists a pair of regular closed sets in X separating them.

We observe that the notion of nearly Hausdorff spaces coincides with the notion of weakly Hausdorff spaces defined by Soundararajan in **[26]**. A topological space X is called *weakly Hausdorff* if each of its points is an intersection of regular closed sets. That a nearly Hausdorff space is weakly Hausdorff follows because for each pair of distinct points, there exist regular closed sets separating them and hence each point is an intersection of regular closed sets. Conversely, suppose X is a weakly Hausdorff space then each point of X is an intersection of regular closed sets. Hence for every pair of distinct points, there exists a pair of regular closed sets separating them. Thus a weakly Hausdorff space is a nearly Hausdorff space.

We introduce here a topological property π for a topological space X.

Definition 6.1.3. A topological space X is said to have *property* π if for every $F \in R(X)$ and $x \notin F$ there exists an $H \in R(X)$ such that $x \in IntH$ and $H \cap F = \varphi$.

We denote a topological space X with property π by $X(\pi)$.

We recall that a space X is called a *Urysohn space* if for every pair of distinct points x, y in X there exist open sets G and H containing x and y respectively such $ClG \cap ClH = \varphi$ **[33]**. Following flow diagram expresses the relationship of nearly Hausdorffness with other separation axioms.

$$Regular \Rightarrow Urysohn(\pi) \Leftrightarrow NearlyHausdorff(\pi)$$

$$\Downarrow$$

$$Urysohn \Rightarrow Hausdorff \Rightarrow NearlyHausdorff \Rightarrow T_1$$

Examples given below **[27, 33]** justify that unidirectional implications in the above flow diagram need not be revertible. In addition, example 6.1.4.(b) shows that nearly Hausdorffness is not a closed hereditary property.

Examples 6.1.4.(a) A T_1 space need not be nearly Hausdorff for example an infinite cofinite space is a T_1 space but not a nearly Hausdorff space.

6.1.4.(b) The following example justifies that a nearly Hausdorff space need not be a Hausdorff space: Consider **N**, the set of natural numbers with cofinite topology and **I** = [0, 1] with the usual topology. Let X = **N**×**I** and define a topology on X as follows:

(i) neighborhoods of the points of the form $(n, y), y \neq 0$ are usual neighborhoods $\{(n,z) \in X \mid y-\varepsilon < z < y+\varepsilon\}$ in $\mathbf{I}_n = \{n\} \times \mathbf{I}$ for small positive ε;

(ii) neighborhoods of the points of the form $(n,0)$ are of the form $\{(m,z) \in X \mid m \in U,\ 0 \leq z < \varepsilon_m\}$, where U is a neighborhood of n in **N** and ε_m is a small positive number for each $m \in U$.

The resulting space X is a non-Hausdorff space as the distinct points in X of the form $(n,0)$ and $(m,0)$ cannot be separated by disjoint open sets. We now observe that it is a nearly Hausdorff space. Let (n,x) and (m,y) be two distinct points in X. We consider the following cases:

Case (i) Let $m=n$. Then choose $\varepsilon<\frac{1}{2}|x-y|$. The regular closed sets $\{(n,z)\in X\,|\,x-\varepsilon\le z\le x+\varepsilon\}$ and $\{(m,z)\in X\,|\,y-\varepsilon\le z\le y+\varepsilon\}$ separates (n,x) and (m,y).

Case (ii) Let $m\ne n$. Then the regular closed sets $\{(n,z)\in X\,|\,0\le z<x\le\varepsilon\}$ and $\{(m,z)\in X\,|\,0\le z<y\le\delta\}$, where ε and δ are small positive numbers, separate (n,x) and (m,y).

Note. (1) The set $\{(n,0)\in X\,|\,n\in \mathbf{N}\}$ in the previous example is a closed subspace of X but not a nearly Hausdorff space. Thus a closed subspace of an nh-space need not be an nh-space.

(2) Also the space X does not possess property π because the set $F=\{(1,z)\in X\,|\,0\le z\le\frac{1}{2}\}$ is a regular closed set and $(2,0)\notin F$ but there does not exist a regular closed set H in X such that $(2,0)\in IntH$ and $H\cap F=\varphi$. Therefore a nearly Hausdorff space need not always have the property π.

6.1.4.(c) In the previous example we saw that a nearly Hausdorff space need not have property π. Now, we give an example showing that even a Hausdorff space need not have the property π.

Let A be the linearly ordered set $\{1,2,3,...,\omega,....,-3,-2,-1\}$ with the interval topology and let **N** be the set of natural numbers with the discrete topology. Define X to be $A\times\mathbf{N}$ together with two distinct points say a and $-a$, which are not in $A\times\mathbf{N}$. The topology $\Im$ on X is determined by the product topology on $A\times\mathbf{N}$ together with basic neighborhoods $M_n^+(a)=\{a\}\cup\{(i,j)|i<\omega,j>n\}$ and $M_n^-(-a)=\{-a\}\cup\{(i,j)|i>\omega,j>n\}$ about a and

$-a$. Resulting space X is a non Urysohn Hausdorff space without property π. The space X is not Urysohn because there do not exist disjoint open sets U and V containing a and $-a$ respectively such that $\overline{U} \cap \overline{V} = \varphi$. That X does not have property π follows from the fact that $a \notin \overline{M_n^-(-a)}$ and there does not exist a regular closed set F containing a such that $a \in IntF$ and $F \cap \overline{M_n^-(-a)} = \varphi$. Thus a Hausdorff space need not possess the property π.

6.1.4.(d) The following example justifies that a Urysohn space need not possess property π. Let S be the set of rational lattice points in the interior of the unit square except those whose $x-$coordinate is $\frac{1}{2}$. Define X to be $S \cup \{(0,\ 0)\} \cup \{(1,\ 0)\} \cup \{(\frac{1}{2},\ r\sqrt{2}) | r \in \mathbf{Q},\ 0 < r\sqrt{2} < 1\}$. Topologize X as follows: Local base for points in $S \subseteq X$ are same as those inherited from the Euclidean topology and for other points following local bases are taken:

$$U_n(0,0) = \{(x,y) \in X \,|\, 0 < x < \tfrac{1}{4},\ 0 < y < \tfrac{1}{n}\} \cup \{(0,\ 0)\},$$

$$U_n(1,0) = \{(x,y) \in X \,|\, \tfrac{3}{4} < x < 1,\ 0 < y < \tfrac{1}{n}\} \cup \{(1,\ 0)\},$$

$$U_n(\tfrac{1}{2}, r\sqrt{2}) = \{(x,y) \in X \,|\, \tfrac{1}{4} < x < \tfrac{3}{4} \text{ and } \left|y - r\sqrt{2}\right| < \tfrac{1}{n}\} \cup \{(0,\ 0)\}.$$

The resulting space X is a Urysohn space without property π. That space X does not have property π follows since $(0,0) \notin H = \{(x,y) \in S \,|\, \frac{1}{4} \le x \le \frac{3}{4}\}$ and there does not exist regular closed set F such that $(0,0) \in IntF$ and $IntF \cap H = \varphi$.

6.1.4.(e) The following example justifies that a Urysohn space with property π need not be a regular space: Let X be the set of real numbers with neighborhoods of any non-zero point as in the usual topology while neighborhoods of 0 will have the form $U-A$, where U is a neighborhood of 0 in the usual topology and $A=\{\frac{1}{n}\,|\,n\in\mathbf{N}\}$.

The resulting space X is a non-regular Urysohn space with property π. That the space X is not regular follows because $0\notin A$, A is closed in X but there do not exist disjoint open sets U and V such that $0\in U$ and $A\subseteq V$.

The space X has property π follows form the fact that the topology on X is finer than the usual topology on the set of real numbers.

The space X is a Urysohn space since for every pair of distinct points x and y in X, there exist disjoint open sets $(x-\eta, x+\eta)$ and $(y-\eta,\ y+\eta)$ having disjoint closures, where $\eta<\frac{1}{2}|x-y|$.

In an approach to unify the separation axioms between T_0 and completely Hausdorff, F. G. Arenas, J. Dontchev and M. L. Puertas in **[1]**, have studied the relation of the separation axiom weakly Hausdorffness with *kd*-space, *kc*-space, *us*-space, hT_1^R space. In **[1]**, authors have observed that a space X is weakly Hausdorff if its semiregularization is T_1, i.e., if each singleton is $\delta-$closed. We recall the following terms. A point x in a topological space X is called a $\delta-$*cluster point* of a subset A of X if $A\cap U\neq\varphi$ for every regular open set U containing x. The set of all $\delta-$cluster points of A is called the $\delta-$*closure* of A and is denoted by $Cl_\delta(A)$. If $A=Cl_\delta(A)$ then A is called $\delta-$closed. A topological space X is called *semiregular* if regular open sets form a base for the topology of X.

Further, for a topological space (X,τ), the family of all regular open sets forms a base for a new topology τ_S, coarser than τ, which is called the *semiregularization* of τ. Thus a space (X,τ) is semiregular if and only if $\tau_S = \tau$. We observe the following result:

Lemma 6.1.5. *A semiregular T_1 space is a nearly Hausdorff space.*

Proof. Let X be a semiregular T_1 space and let $x, y \in X$, $x \neq y$. Since X is a T_1 space, there exist open sets U and V separating x and y respectively. Further X is a semiregular space which implies there exist regular open sets G_x and G_y containing x and y respectively such that $G_x \subset U$ and $G_y \subset V$. The result now follows by observing that $X - G_y$ and $X - G_x$ are regular closed sets separating x and y.

Theorem 6.1.6. *A non-empty product of a nearly Hausdorff space is a nearly Hausdorff space if and only if each factor is a nearly Hausdorff space.*

Proof. Let $X = \prod_{\gamma \in \lambda} X_\gamma$, where $\{X_\gamma\}_{\gamma \in \lambda}$ is a family of nearly Hausdorff spaces, $\lambda \neq \varphi$. Consider two distinct points x, y in X. Then $x \neq y$

$\Rightarrow \quad x_\alpha \neq y_\alpha$ for some $\alpha \in \lambda$.

Since each X_α is a nearly Hausdorff space, for $x_\alpha \neq y_\alpha$ in X_α there exist regular closed sets F and H in X_α separating x_α and y_α. Define $U = \prod_{\gamma \in \lambda} U_\gamma$ and $V = \prod_{\gamma \in \lambda} V_\gamma$, where $U_\gamma = V_\gamma = X_\gamma$ for $\gamma \neq \alpha$ and $U_\alpha = IntF$,

$V_\alpha = IntH$. Then the regular closed sets ClU and ClV separate x and y respectively.

Conversely, suppose $X = \prod_{\gamma \in \lambda} X_\gamma$ is a nearly Hausdorff space. Let x_α, y_α be two distinct points in X_α. Choose points x, y in X such that they differ only in α^{th} co-ordinate and their α^{th} co-ordinates are x_α and y_α respectively. Since X is a nearly Hausdorff space, there exist regular closed sets F and H in X separating x and y respectively. Since $IntF$ and $IntH$ are open sets in X therefore $IntF = \prod_{\gamma \in \lambda} U_\gamma$ and $IntH = \prod_{\gamma \in \lambda} V_\gamma$, where U_γ and V_γ are open sets in X_γ for each γ and $U_\gamma = X_\gamma$, $V_\gamma = X_\gamma$ except for finitely many values of γ. The regular closed sets ClU_α and ClV_α separate x_α and y_α respectively. This proves that for each $\alpha \in \lambda$, X_α is a nearly Hausdorff space.

The following result was proved in **[8]** for a regular Hausdorff space. We now observe that it is true for nearly Hausdorff space also.

Theorem 6.1.7. *Let X be a nearly Hausdorff space and let $f : X \to Y$ be a density preserving epimorphism. Then*

(A) *for a regular closed set H of Y, we have $Clf(Clf^{-1}(IntH)) = H$ and hence $R(Y) = \{Clf(F) \mid F \in R(X)\}$.*

(B) *$Clf(F) \in R(Y)$ whenever $F \in R(X)$.*

Proof. **(A)** Clearly, $Clf(Clf^{-1}(IntH)) \subseteq H$. For the reverse containment, let $x \in H$. Then we consider the following cases:

Case (i) Let $x \in IntH$. Then we have $x \in Clf(Clf^{-1}(IntH))$ which implies $IntH \subseteq Clf(Clf^{-1}(IntH))$ and therefore $H \subseteq Clf(Clf^{-1}(IntH))$.

Case (ii) Let x be a limit point of H. Then every open set U_x containing x has a non-empty intersection with $IntH$. But this implies

$$f^{-1}(U_x) \cap Clf^{-1}(IntH) \neq \varphi$$

$$\Rightarrow \quad f\left(f^{-1}(U_x)\right) \cap f\left(Clf^{-1}(IntH)\right) \neq \varphi$$

$$\Rightarrow \quad U_x \cap f\left(Clf^{-1}(IntH)\right) \neq \varphi.$$

Therefore $x \in Clf(Clf^{-1}(IntH))$ and hence $H \subseteq Clf(Clf^{-1}(IntH))$.

(B) If $F = \varphi$ then the result follows trivially. Let $F \in R(X) - \{\varphi\}$. Then

$$ClInt(Clf(F)) \subseteq Clf(F). \qquad (1)$$

For the reverse containment, we shall show that $Clf(F) \cap (Y - ClIntClf(F)) = \varphi$. Let $G = Y - ClIntClf(F)$. Suppose $G \cap Clf(F) \neq \varphi$. Then G being open and f being a density preserving epimorphism, we have

$$G \cap f(F) \neq \varphi$$

$$\Rightarrow \quad f^{-1}(G) \cap F \neq \varphi$$

$$\Rightarrow \quad f^{-1}(G) \cap IntF \neq \varphi$$

Let $H = Cl(f^{-1}(G) \cap IntF)$. Then

$$\varphi \neq IntClf(H) = IntClf(Clf^{-1}(G) \cap IntF)$$

$$\subseteq G \cap IntClf(F) = \varphi,$$

which is a contradiction. Therefore our assumption that $G \cap f(F) \neq \varphi$ is wrong. Hence

$$Clf(F) \subseteq ClInt(Clf(F)). \qquad (2)$$

From (1) and (2), we have $ClInt(Clf(F)) = Clf(F)$, whenever $F \in R(X)$.

Note. (1) Observe that the first projection of the space **N**×**I** in Example 6.1.4 (b) shows that a continuous image of a nearly Hausdorff need not be a nearly Hausdorff space.

(2) In the same example if we consider the second projection of **N**×**I** onto [0, 1] with the cofinite topology then we get that even a continuous density preserving image of a nearly Hausdorff space need not be a nearly Hausdorff space.

2. The space rX.

In this section we obtain "βX like" compactification for a nearly Hausdorff space X with property π. Consider the family $Rf(X)$ of finite intersections of members of $R(X)$, where $R(X)$ is the family of all regular closed subsets of X. For a topological space X, an $\alpha \subseteq Rf(X) - \{\varphi\}$ is called an $r-$filter if it is closed under finite intersections and supersets. A maximal $r-$filter is called an $r-$ultrafilter. A filter α is said to be fixed (free) depending upon whether $\cap\alpha$ is non-empty (empty).

Lemma 6.2.1. *Let X be a nearly Hausdorff space. Then,*

(i) *for each* $x \in X$, *there exists a unique* $r-$*ultrafilter* α_x *such that* $\cap \alpha_x = \{x\}$, where $\alpha_x = \{F \in Rf(X) \mid x \in F\}$.

(ii) X *is a compact space if and only if each* $r-$*ultrafilter in* X *is fixed.*

Proof. (i) Follows from the fact that for each pair of distinct points of X there exist regular closed sets separating them.

(ii) If X is a compact space, then each $r-$ultrafilter being a family of closed sets with finite intersection property, has arbitrary intersection non-empty. Converse follows from the fact that for each $r-$ultrafilter α, the family of open sets $C_\alpha = \{X - F \mid F \in \alpha\}$ is such that if no finite sub collection of C_α covers X then C_α does not cover X.

For a nearly Hausdorff space X with property π, denote by rX, the set of all $r-$ultrafilters in X. Further for $F \in R(X)$ define $\overline{F} = \{\alpha \in rX \mid F \in \alpha\}$. Topologize the set rX by taking $B = \{\overline{F} \mid F \in R(X)\}$ as a base for closed sets in rX. We use the following result from **[3]** to show that B is a base for closed sets in rX.

A collection B of subsets of a set X is a closed base for a topological space X if and only if the following conditions are satisfied:

(i) The intersection of members of B is empty.

(ii) For each F_1 and F_2 in B and $x \notin F_1 \cup F_2$, there exists an F in B such that $x \notin F \supseteq F_1 \cup F_2$.

Lemma 6.2.2. *Let X be a nearly Hausdorff space with property π. Then the set $B = \{\overline{F} \mid F \in R(X)\}$ forms a base for closed sets in rX.*

Proof. Observe that for $F_1, F_2 \in B$, $\overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2}$. Let $\alpha \in \overline{F_1 \cup F_2}$. Then $F_1 \cup F_2 \in \alpha$ and α is an $r-$ultrafilter implies that either $F_1 \in \alpha$ or $F_2 \in \alpha$, i.e. either $\alpha \in \overline{F_1}$ or $\alpha \in \overline{F_2}$. Hence $\alpha \in \overline{F_1} \cup \overline{F_2}$. This proves that $\overline{F_1 \cup F_2} \subseteq \overline{F_1} \cup \overline{F_2}$. The reverse containment can be proved similarly. Since X is a nearly Hausdorff space it follows that intersection of members of B is empty. Hence B is a base for closed sets in rX.

Note. Let X be a nearly Hausdorff space with property π. Then for each $F \in R(X)$, $Cl_{rX} Int_{rX} \overline{F} = \overline{F}$. Clearly $Cl_{rX} Int_{rX} \overline{F} \subseteq \overline{F}$. To observe the reverse containment, let $\alpha \in \overline{F}$. If possible suppose $\alpha \notin Cl_{rX} Int_{rX} \overline{F}$. Then there exists an open set U in rX containing α such that

$$U \cap (Int_{rX} \overline{F}) = \varphi$$

$$\Rightarrow \quad U \cap Int_X F = \varphi$$

$$\Rightarrow \quad U \cap F = \varphi,$$

a contradiction since U is an open set containing α and $F \in \alpha$.

Theorem 6.2.3. *Let X be a nearly Hausdorff space with property π. Then the space rX of all $r-$ultrafilters in X is a compact nh-space which contains X as a dense subspace.*

Proof. Define $r : X \to rX$ by $r(x) = \alpha_x$ where $\alpha_x = \{F \in Rf(X) \mid x \in F\}$. We first prove that α_x is an $r-$ultrafilter. Clearly, $\varphi \notin \alpha_x$ and $X \in \alpha_x$. Also α_x is closed under finite intersections and supersets. We now prove that α_x is a maximal subfamily of $Rf(X)$ with finite intersection property.

Let $A \in Rf(X)$ be such that $A \cap F \neq \varphi$, for all $F \in \alpha_x$. Let $A = \bigcap_{i=1}^{n} A_i$ where $A_i \in R(X)$ for each $i \in \{1, 2, \dots, n\}$. Now,

$$A \cap F \neq \varphi \text{ for all } F \in \alpha_x$$

$$\Rightarrow \quad A_i \cap F \neq \varphi \text{ for all } F \in \alpha_x.$$

It is sufficient to prove that $A_i \in \alpha_x$ for each $i \in \{1, 2, \dots, n\}$. If possible, suppose $A_i \notin \alpha_x$ for some i. This implies $x \notin A_i$. Since X has property π, there exists H in $R(X)$ such that $x \in IntH$ and $H \cap A_i = \varphi$. Since $x \in H$ we have $H \in \alpha_x$. But this contradicts $A_i \cap F \neq \varphi$ for each $F \in \alpha_x$. Therefore our assumption that $A_i \notin \alpha_x$ for some i, is wrong. This prove α_x is an $r-$ultrafilter.

That the map r is well defined and is one-one follows by Lemma 6.2.1.(i). We now prove that $r(F) = \overline{F} \cap r(X)$, where $F \in R(X)$. Note that

$$\alpha_x \in \overline{F} \cap r(X) \Leftrightarrow F \in \alpha_x, \alpha_x \in r(X) \Leftrightarrow x \in F, \alpha_x \in r(X) \Leftrightarrow \alpha_x \in r(F).$$

The identity $r(F) = \overline{F} \cap r(X)$ implies $r^{-1}(\overline{F}) = F$, i.e. inverse image of every basic closed set in rX is closed in X. This proves r is continuous. That the map r is a closed map onto its image follows from the fact that

$r(F) = \overline{F} \cap r(X)$ and the fact that the family $Rf(X)$ form a base for closed sets in X.

We now prove that $Cl_{rX} r(F) = \overline{F}$, where $F \in R(X)$. The identity $r(F) = \overline{F} \cap r(X)$, $F \in R(X)$, implies $Cl_{rX} r(F) \subseteq \overline{F}$. For the reverse containment let $\overline{K}$ be a basic closed set in rX containing $r(F)$. Then $\overline{K} \supseteq \overline{F}$ since

$$r(F) \subseteq \overline{K}$$

$$\Rightarrow \quad \{\alpha_x \in rX \mid x \in F\} \subseteq \overline{K}$$

$$\Rightarrow \quad K \in \alpha_x \text{ for each } x \in F$$

$$\Rightarrow \quad x \in K \text{ for each } x \in F$$

$$\Rightarrow \quad F \subseteq K$$

$$\Rightarrow \quad \overline{F} \subseteq \overline{K}.$$

Therefore every basic closed set $\overline{K}$ containing $r(F)$ contains $\overline{F}$. Since $Cl_{rX} r(F)$ is the intersection of all closed sets in rX containing $r(F)$, it follows that $Cl_{rX} r(F) = \overline{F}$.

We now establish that the space rX is compact. Let $\{\overline{F}\}_{F \in \lambda}$ be a family of basic closed sets in rX with finite intersection property, where λ is a subfamily of $Rf(X)$. Observe that the family λ also has the finite intersection property. For if $\bigcap_{i=1}^{n} F_i = \varphi$, $F_i \in \lambda$ for each $i \in \{1, 2, \ldots, n\}$ then

$$\bigcap_{i=1}^{n} \overline{F_i} = \{\alpha \in rX \mid \bigcap_{i=1}^{n} F_i \in \alpha\}$$

$\Rightarrow \qquad \varphi \in \alpha,$

which is a contradiction. An r-ultrafilter is a maximal subfamily of $Rf(X)$ with finite intersection property. Hence λ is contained in some r-ultrafilter say α. Now

$$\alpha \in \bigcap_{F \in \alpha} \overline{F} \subseteq \bigcap_{K \in \lambda} \overline{K}$$

proves that $\bigcap_{k \in \lambda} \overline{K} \neq \varphi$. Hence rX is compact.

We note that the space rX is an nh-space. Let α and ζ in rX be two distinct r-ultrafilters. Then $\alpha \neq \zeta$ implies there exists $F \in \alpha$ such that $F \notin \zeta$. Now $F \notin \zeta$ implies that there exists $H \in \zeta$ such that $F \cap H = \varphi$. The regular closed sets $\overline{F}$ and $\overline{H}$ separate α and ζ respectively.

Note. If a space X with property π is a non-Hausdorff, nearly Hausdorff space then rX cannot be Hausdorff. For, if rX is Hausdorff then X being subspace of a compact Hausdorff space must be a completely regular Hausdorff space, which is a contradiction.

Example 6.2.4. Following example justifies that a one point compactification of a non-Urysohn Hausdorff space without property π can be a nearly Hausdorff. Consider the subspace $Y = \{(\frac{1}{n}, \frac{1}{m}) \mid n \in \mathbf{N}, |m| \in \mathbf{N}\} \cup \{(\frac{1}{n}, 0) \mid n \in \mathbf{N}\}$ of the usual Euclidean space $\mathbf{R}^2$. Set $X = Y \cup \{p, q\}$, where $p, q \notin Y$ and topologize X by taking sets open in Y as open in X and a set U containing p (respectively q) is open in X if for some $r \in \mathbf{N}$,

$\{(\frac{1}{n}, \frac{1}{m}) \mid n \geq r,\, m \in \mathbf{N}\} \subseteq U$ (respectively $\{(\frac{1}{n}, \frac{1}{m}) \mid n \geq r,\, -m \in \mathbf{N}\} \subseteq U$). The resulting space X is a non-Urysohn Hausdorff space without property π and its one point compactification is a nearly Hausdorff space.

The space X is not a Urysohn space follows from the fact that distinct points p and q cannot be separated by open sets such that their closures are disjoint.

Let $Z = X \cup \{(0,0)\}$. Topologize Z by declaring sets open in X as open in Z and the open sets about $(0,0)$ are those inherited from the subspace of Euclidean space $\mathbf{R}^2$. Resulting space Z is a compact nh-space.

Let μ be an open cover of Z. Then choose open sets U, V, W containing p, q and $(0,0)$ respectively. Let n be the largest natural number such that $(\frac{1}{n}, 0) \notin U \cup V \cup W$. Choose basic open sets U_r about each $(\frac{1}{r}, 0)$, $1 \leq r \leq n$. Then the open set $U \cup V \cup W \cup \left(\bigcup_{r=1}^{n} U_r \right)$ covers all but finitely many points of Z. The remaining finitely many points of Z are isolated points. This proves that Z is compact.

We now show that Z is a nearly Hausdorff space. Let $x, y \in Z$, $x \neq y$. Then we consider the following cases:

Case (i) Let $x, y \in Y \cup \{(0,0)\}$. Then x and y be separated by regular closed sets as $Y \cup \{(0,0)\}$ inherits the usual Euclidean space $\mathbf{R}^2$.

Case (ii) Let $x = p$ and $y = (0,0)$. Then the regular closed sets

$$F = \{(\tfrac{1}{n}, \tfrac{1}{m}) \mid 1 \leq m \leq r,\, n \in \mathbf{N}\} \cup \{p\}$$

and

$$H = \{(\tfrac{1}{n}, 0) \mid n \in \mathbf{N}\} \cup \{(\tfrac{1}{n}, \tfrac{1}{m}) \mid n \in \mathbf{N},\ -m \in \mathbf{N}\} \cup \{q\}$$

separate x and y.

Case (iii) Let $x = q$ and $y = (0,0)$. Then the regular closed sets

$$F = \{(\tfrac{1}{n}, \tfrac{1}{m}) \mid -r \leq -m \leq -1,\ n \in \mathbf{N}\} \cup \{q\}$$

and

$$H = \{(\tfrac{1}{n}, 0) \mid n \in \mathbf{N}\} \cup \{(\tfrac{1}{n}, \tfrac{1}{m}) \mid n \in \mathbf{N},\ m \in \mathbf{N}\} \cup \{p\}$$

separate x and y.

Case (iv) Let $x = p$ and $y = q$. Then the regular closed sets

$$F = \{(\tfrac{1}{n}, \tfrac{1}{m}) \mid 1 \leq m \leq r,\ n \in \mathbf{N}\} \cup \{p\}$$

and

$$H = \{(\tfrac{1}{n}, \tfrac{1}{m}) \mid -r \leq -m \leq -1,\ n \in \mathbf{N}\} \cup \{q\}$$

separate x and y.

Theorem 6.2.5. *Let the spaces X and rX be as in Theorem 6.2.3. Then X is C*- embedded in rX.*

Proof. Let $f \in C^*(X)$. Suppose image of $f \subseteq [0, 1]$ = **I**. For α in rX, define $f^{\#}(\alpha) = \{ H_1 \cup H_2 \in R \mid Cl_X f^{-1}(Int_I H_1 \cup Int_I H_2) \in \alpha \}$. Observe that $f^{\#}(\alpha)$ satisfies finite intersection property. In fact for F, H in $f^{\#}(\alpha)$, $Cl_X f^{-1}(IntH)$, $Cl_X f^{-1}(IntF) \in \alpha$ and therefore

$$Cl_X f^{-1}(IntH) \cap Cl_X f^{-1}(IntF) \neq \varphi$$

$$\Rightarrow \quad \varphi \neq f\big((Cl_X f^{-1}(IntH)) \cap (Cl_X f^{-1}(IntF))\big)$$

$$\subseteq f(Cl_X f^{-1}(IntH)) \cap f((Cl_X f^{-1}(IntF))$$

$$\subseteq Cl_I f(f^{-1}(IntH)) \cap Cl_I f(f^{-1}(IntF))$$

$$\subseteq H \cap F.$$

Hence $H \cap F \neq \varphi$. We now prove that if $H \cup F \in f^{\#}(\alpha)$, then either $H \in f^{\#}(\alpha)$ or $F \in f^{\#}(\alpha)$. Suppose $H \notin f^{\#}(\alpha)$. If possible, suppose $F \notin f^{\#}(\alpha)$. Then

$$H \notin f^{\#}(\alpha) \Rightarrow Cl_X f^{-1}(IntH) \notin \alpha$$

$\Rightarrow$ there exists $K_1 \in \alpha$ such that $Cl_X f^{-1}(IntH) \cap K_1 = \varphi$.

Also,

$$F \notin f^{\#}(\alpha) \Rightarrow Cl_X f^{-1}(IntF) \notin \alpha$$

$\Rightarrow$ there exists $K_2 \in \alpha$ such that $Cl_X f^{-1}(IntF) \cap K_2 = \varphi$.

Now $K_1, K_2 \in \alpha$ and α an $r-$ultrafilter imply that $K_1 \cap K_2 \in \alpha$. Moreover,

$$H \cup F \in f^{\#}(\alpha)$$

$$\Rightarrow \quad Cl_X f^{-1}(IntH \cup IntF) = Cl_X f^{-1}(IntH) \cup Cl_X f^{-1}(IntF) \in \alpha,$$

but $Cl_X f^{-1}(IntH \cup IntK) \cap (K_1 \cap K_2) = \varphi$ - a contradiction. Therefore if $H \cup F \in f^{\#}(\alpha)$ then either $H \in f^{\#}(\alpha)$ or $F \in f^{\#}(\alpha)$. Since $f^{\#}(\alpha)$ is a family of closed sets in I with finite intersection property therefore $\cap f^{\#}(\alpha) \neq \phi$. We assert that $\cap f^{\#}(\alpha)=\{t\}$, for some $t \in$ I. Define $rf : rX \to$ I by $rf(\alpha) = \cap f^{\#}(\alpha)$. Clearly, rf restricted to X is f. We show rf is continuous.

Let $\alpha \in rX$. Then choose an open set G of I such that $rf(\alpha) \in G$. If $rf(\alpha) = t$ then using regularity of I successively we obtain open sets G_1, G_2 satisfying

$$t \in G_1 \subseteq \overline{G_1} \subseteq G_2 \subseteq \overline{G_2} \subseteq G.$$

Set $F_t = Cl_I\, G_2$ and $H_t = Cl_I\, (I - Cl_I\, G_1)$. Since $Int_I\, F_t \cup Int_I\, H_t = I$, we have $F_t \cup H_t \in f^{\#}(\alpha)$ and as $t \notin H_t$, $F_t \in f^{\#}(\alpha)$ and $H_t \notin f^{\#}(\alpha)$. If $K_t = Cl_X\, f^{-1}(Int_I\, F_t)$ and $L_t = Cl_X\, f^{-1}(Int_I\, H_t)$ then $\alpha \notin \overline{L_t}$ and the open set $rX - \overline{L_t}$ contains $\{\alpha\}$. Finally the containment $rf(rX - \overline{L_t}) \subseteq G$ establishes the continuity of rf. For the assertion, one may use the above technique to note that $\{F \in R(I) \mid t \in Int_I F\} \subseteq f^{\#}(\alpha)$.

Theorem 6.2.6. *Let X be a nearly Hausdorff space with property π. Then there exists a compact nearly Hausdorff space rX in which X is densely C*-embedded.*

Proof. Follows from Theorem 6.2.3 and Theorem 6.2.5.

Corollary 6.2.7. *If X is a regular Hausdorff space, then it is densely C*-embedded in rX.*

3. When $rX = \beta X$?

In this section we answer the natural question when $rX = \beta X$? We observe that if $Rf(X)$ forms a Wallman base for a nearly Hausdorff space X then $rX = \beta X$. As a consequence we have that if X is normal or zero-dimensional then $rX = \beta X$.

Lemma 6.3.1. *Let X be a normal space and let $Rf(X)$ be the collection of all finite intersections of members of $R(X)$. Then $Rf(X)$ is a Wallman base.*

Proof. Clearly $Rf(X)$ is closed under finite intersections and finite unions. We observe the following:

(i) $\varphi, X \in Rf(X)$.

(ii) Note that $Rf(X)$ forms a closed base for X: Since X is a normal space, $\cap Rf(X) = \varphi$. Further for each $F, H \in Rf(X)$ such that $x \notin F \cup H$ implies that there exist disjoint open sets U and V such that $x \in U$, $F \cup H \subseteq V$ and $\overline{U} \cap \overline{V} = \varphi$. Clearly, $\overline{V} \in Rf(X)$ and $x \notin \overline{V} \supseteq F \cup H$.

(iii) Let $A \in Rf(X)$ and $x \in X - A$. Then X being a normal space, there exists an open set V such that $x \in \overline{V} \subset X - A$.

(iv) Let $A, B \in Rf(X)$ be such that $A \subseteq X - B$. Since X is a normal space, the closed set $A \subseteq X - B$ implies that there exists an open set U such that

$$A \subseteq U \subseteq \overline{U} \subseteq X - B \qquad (1)$$

Further,

$$A \subseteq U \Rightarrow X - U \subseteq X - A.$$

Since X is a normal space there exists open set W such that $X - U \subseteq W \subseteq \overline{W} \subseteq X - A$ which implies $X - U \subseteq \overline{W} \subseteq X - A$ which in turn gives

$$A \subseteq X - \overline{W} \subseteq U \subseteq \overline{U} \qquad (2)$$

From (1) and (2), it follows that $A \subseteq X - \overline{W} \subseteq \overline{U} \subseteq X - B$. Therefore for $A \subseteq X - B$, there exist $\overline{U}, \overline{W} \in Rf(X)$ such that $A \subseteq X - \overline{W} \subseteq U \subseteq \overline{U}$.

Lemma 6.3.2. *Let X be a nearly Hausdorff space such that $Rf(X)$ is a Wallman base. Then X is a regular space.*

Proof. Let $x \in X$ and let F be a closed subset of X such that $x \notin F$. Since $Rf(X)$ forms a base for closed sets in X, $F = \bigcap_{H \in \beta} H$, where β is some subfamily of $Rf(X)$. Now, $x \notin F$ implies that $x \notin H$ for some $H \in \beta$. Since $Rf(X)$ forms a Wallman base,

$$x \notin H,\ H \in Rf(X)$$

$$\Rightarrow \quad \text{there exists } K \in Rf(X) \text{ such that } x \in K \text{ and } K \cap H = \phi$$

Further, $K \cap H = \phi$ implies that $H \subset X - K$. Again using the fact that $Rf(X)$ is a Wallman base, there exist $C, D \in Rf(X)$ such that

$$H \subset X - C \subset D \subset X - K.$$

The open sets $X - C$ and $X - D$ are disjoint and contain F and x, respectively.

Theorem 6.3.3. *Let X be a nearly Hausdorff space such that $Rf(X)$ is a Wallman base. Then $rX = \beta X$.*

Proof. It is sufficient to show that rX is Hausdorff. Let $x, y \in rX, x \neq y$. Then there exist $r-$ultrafilters α_x and α_y in rX with limit points x and y, respectively. Since α_x and α_y are distinct r-ultrafilters, there exist $F \in \alpha_x$ such that $F \cap K = \phi$, for some $K \in \alpha_y$. Therefore $F \subset X - K$. Since $Rf(X)$ is a Wallman base therefore there exist E and H in $Rf(X)$ such that

$$F \subset X - H \subset E \subset X - K.$$

Then $rX - \overline{H}$ and $rX - \overline{E}$ are disjoint open sets containing x and y, respectively. Hence rX is a Hausdorff space.

Corollary 6.3.4. *A nearly Hausdorff space X for which $Rf(X)$ forms a Wallman base is a Tychonoff space.*

Proof. Follows from Theorem 6.3.3 since X is a subspace of βX.

Corollary 6.3.5. *If X is a normal space or zero-dimensional space then $rX = \beta X$.*

Proof. Follow from Lemma 6.3.1 and Theorem 6.3.3.