CHAPTER VII

ABSOLUTES FOR NEARLY HAUSDORFF SPACES

The major factor behind the study of absolutes was the problem of characterizing the projective objects in the category of compact T_2 spaces and continuous maps. In 1958, Gleason solved this problem by showing that the projective objects in the category of compact Hausdorff spaces and continuous maps are precisely the compact extremally disconnected spaces. Gleason constructed EX as a part of the solution to this problem. The space EX is a zero-dimensional extremally disconnected Hausdorff space, which can be mapped to X by a perfect irreducible surjection k_X. The space EX is unique in the sense that if Y is an extremally disconnected zero-dimensional space mapped to X by a perfect irreducible surjection then EX is homeomorphic to Y. The projective space EX together with the perfect irreducible mapping k_X is called the projective cover of X. Further, Gleason observed that every completely regular Hausdorff space has a projective cover, which is unique upto homeomorphism and as consequence of this it follows that $\beta(EX) = E(\beta X)$.

Along the lines of Gleason's construction we describe here the construction of projective cover (EX, h_X), for a given compact nearly Hausdorff space X. Further, we study projective lift $Ef : EX \to EY$ and extension $rf : rX \to rY$ of a dp-epimorphism $f : X \to Y$. We establish that E is a functor from the category of compact nearly Hausdorff spaces and dp-epimorphisms to the category of compact extremally disconnected Hausdorff spaces and continuous maps. We also establish that r is a functor form the category of nearly Hausdorff spaces with property π and dp-epimorphisms to the category of compact nearly Hausdorff spaces and continuous maps. We finally obtain commutativity of the functors E and r.

1. The functor r.

In this section we construct the extension $rf : rX \to rY$ of a density preserving epimorphism $f : X \to Y$ and observe that the map associating to each nearly Hausdorff space X with property π, the nearly Hausdorff compactification rX [Chapter 6, Section 2] and to each density preserving epimorphism f from a nearly Hausdorff space X with property π to a nearly Hausdorff space Y with property π, the continuous map $rf : rX \to rY$, is a covariant functor from the category of nearly Hausdorff spaces with property π and density preserving epimorphisms to the category of compact nearly Hausdorff spaces and continuous maps. We begin with the following result:

Theorem 7.1.1. *Let X and Y be nearly Hausdorff spaces with property π and let $f : X \to Y$ be a continuous dp-epimorphism. Then for $\alpha \in rX$, the set $f_{\#}(\alpha) = \{Clf(F) \mid F \in \alpha \cap R(X)\}$ generates a unique $r-$ultrafilter.*

Proof. Clearly $\varphi \notin f_{\#}(\alpha)$ and $X \in f_{\#}(\alpha)$. Next, we observe that $f_{\#}(\alpha)$ is closed under supersets. Let $H \in R(Y)$ be such that $Clf(F) \subseteq H$ for some $F \in \alpha \cap R(X)$. Then $IntClF \subseteq IntClf^{-1}(f(F)) \subseteq f^{-1}(IntH)$ which implies $F \subseteq Clf^{-1}(IntH)$. Since $F \in \alpha$ and $F \subseteq Clf^{-1}(IntH)$, $Clf^{-1}(IntH) \in \alpha$. Therefore $Clf(Clf^{-1}(IntH)) = H \in f_{\#}(\alpha)$. Suppose $f_{\#}(\alpha)$ generates r-ultrafilter η.

We now prove uniqueness of this $r-$ultrafilter. If possible, suppose $f_{\#}(\alpha)$ generates two distinct $r-$ultrafilters say η and ξ. Then $\eta \neq \xi$ implies there exists $F \in \eta$ such that $F \cap H = \varphi$ for some $H \in \xi$. Since η and ξ are $r-$ultrafilters generated by $f_{\#}(\alpha)$ we have

$$F = \bigcap_{i=1}^{n} Clf(F_i) \text{ and } H = \bigcap_{j=1}^{m} Clf(H_j),$$

where $F_i, H_j \in \alpha \cap R(X)$ for all i and j. Now, $F \cap H = \varphi$

$$\Rightarrow \quad \left(\bigcap_{i=1}^{n} f^{-1}(Clf(F_i))\right) \cap \left(\bigcap_{j=1}^{m} f^{-1}(Clf(H_j))\right) = \varphi$$

$$\Rightarrow \quad \left(\bigcap_{i=1}^{n} F_i\right) \cap \left(\bigcap_{j=1}^{m} H_j\right) = \varphi,$$

which is a contradiction as $\bigcap_{i=1}^{n} F_i, \bigcap_{j=1}^{m} H_j \in \alpha$. Hence $f_{\#}(\alpha)$ generates unique $r-$ultrafilter.

Theorem 7.1.2. *Let X and Y be nearly Hausdorff spaces with property π and let $f : X \to Y$ be a continuous dp-epimorphism. Then the map $rf : rX \to rY$*

defined by $rf(\alpha)=\alpha^{\#}$*, where* $\alpha^{\#}$ *is the unique* $r-$*ultrafilter generated by* $f_{\#}(\alpha)$*, is a continuous map.*

Proof. Let $\alpha \in rX$ and let $V = rY-\overline{F}$ be a basic open set containing $rf(\alpha)=\alpha^{\#}$, where $F \in R(Y)$. Now $\alpha^{\#} \in rY-\overline{F}$ implies $\alpha^{\#} \notin \overline{F}$. Thus there exists $H \in \alpha^{\#}$ such that $H \cap F=\varphi$. Suppose $H=\bigcap_{j=1}^{m} Clf(H_j)$, where $H_j \in \alpha \cap R(X)$ for all j.

Then $F \cap \left(\bigcap_{j=1}^{m} Clf(H_j)\right)=\varphi$ implies $Clf^{-1}(IntF) \cap \left(\bigcap_{j=1}^{m} H_j\right)=\varphi$.

This proves $Clf^{-1}(IntF) \notin \alpha$. Set $K=Clf^{-1}(IntF)$. Then $\alpha \in rX-\overline{K}$ and $rf(rX-\overline{K}) \subseteq rY-\overline{F}$. Clearly $U=rX-\overline{K}$ is an open set containing α such that $rf(U) \subseteq V$. This proves continuity of rf.

Note. The map $rf: rX \to rY$ defined in Theorem 7.1.2 is unique. If possible, suppose there are two maps $g': rX \to rY$ and $g'': rX \to rY$ such that the following diagram commutes:

$$\begin{array}{ccc} rX & \xrightarrow{rf} & rY \\ \uparrow & & \uparrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

Note that $g'|_X = f = g''|_X$. If possible suppose $g'(\alpha) \neq g''(\alpha)$ for some $\alpha \in rX$. Then there exists $H \in g'(\alpha)$ such that $H \cap F=\varphi$ for some $F \in g''(\alpha)$. Since g' and g'' are maps generated by $f_{\#}(\alpha)$, we have

$$F=\bigcap_{i=1}^{n} Clf(F_i) \text{ and } H=\bigcap_{j=1}^{m} Clf(H_j),$$

where F_i, $H_j \in \alpha \cap R(X)$ for all i and j. Now observe that $H \cap F = \varphi$ implies

$$\left(\bigcap_{i=1}^{n} F_i\right) \cap \left(\bigcap_{j=1}^{m} H_j\right) = \varphi,$$

which is a contradiction. This proves $g' = g''$.

Lemma 7.1.3. *Let X and Y be nearly Hausdorff spaces with property π and let $f: X \to Y$ and $g: Y \to Z$ be density preserving epimorphisms. Then $g \circ f$ is a density preserving map and $r(g \circ f) = rg \circ rf$.*

Proof. Let $\alpha \in rX$. Suppose $r(g \circ f)(\alpha) = \alpha^{\#}$ where $\alpha^{\#}$ is the r-ultrafilter generated by $(g \circ f)_{\#}(\alpha) = \{Cl(g \circ f)(F) \mid F \in \alpha \cap R(X)\}$. Observe that

$$Cl\, g(Clf(F)) \subseteq Cl\, g(f(F)),\ \forall\ F \in \alpha \cap R(X).$$

Since $(g \circ f)_{\#}$ generates a unique r-ultrafilter, it follows that $r(g \circ f)(\alpha) = (rg \circ rf)(\alpha)$ for each $\alpha \in rX$.

Theorem 7.1.4. *Let $\mathcal{N}(\pi)$ be the category of nearly Hausdorff spaces with property π and continuous density preserving epimorphisms and let $\mathcal{N}(\mathcal{K})$ be the category of compact nearly Hausdorff spaces and continuous maps. Then $r: \mathcal{N}(\pi) \to \mathcal{N}(\mathcal{K})$ assigning rX to X and $rf: rX \to rY$ to $f: X \to Y$ is a covariant functor.*

Proof. Follows from Theorem 7.1.2, Lemma 7.1.3 and the previous Note.

2. The functor E.

Let X be a nearly Hausdorff space with property π. Then the Iliadis absolute $E(rX)$ of the nearly Hausdorff compactification rX of the space X is defined to be the Stone space $S(R(rX))$ of the Boolean algebra $R(rX)$. The elements of $E(rX)$ are ultrafilters on $R(rX)$. The set $\{\lambda(A) \mid A \in R(rX)\}$, where $\lambda(A) = \{\alpha \in E(rX) \mid A \in \alpha\}$, forms a base for the topology on $E(rX)$. The set $E(rX)$ with this topology is a compact Hausdorff extremally disconnected space.

Lemma 7.2.1. *Let X be a nearly Hausdorff space and let $R(X)$ be the set of all regular closed subsets of X. Then $R(X)$ is a complete Boolean algebra with the following operations:*

(i) $A \leq B$ *if and only if* $A \subseteq B$.

(ii) $\vee_{\alpha} A_{\alpha} = Cl(\cup_{\alpha} Int A_{\alpha})$.

(iii) $\wedge_{\alpha} A_{\alpha} = Cl(Int(\cap_{\alpha} A_{\alpha}))$.

(iv) $A' = Cl(X - A)$.

Proof. Follows along the lines of proof of Proposition 2.3 of **[32]**.

We now include results about absolutes of nearly Hausdorff spaces. These can be proved along the lines of similar results proved for completely regular Hausdorff spaces.

Theorem 7.2.2. *Let X be a nearly Hausdorff space with property π. Then the Iliadis absolute $E(rX)$ is a compact Hausdorff extremally disconnected space. Further, the natural projection $h_X : E(rX) \to rX$ defined by $h_{rX}(\alpha) = \cap\alpha$ is a perfect irreducible θ-continuous map.*

Proof. Follows along the lines of Proof of Theorem 6.6(e) of **[21]**.

Theorem 7.2.3. *Let X be a compact nearly Hausdorff space and let (Y, f) be a pair consisting of a compact extremally disconnected zero-dimensional space Y and a perfect irreducible θ-continuous surjection f from Y to X. Then there exists a homeomorphism h from EX to Y such that $f \circ h = h_X$, where $h_X : EX \to X$ is the natural projection map.*

Proof. Follows along the lines of proof of Theorem 6.7(a) of **[21]**.

Following result has been proved in **[8]** for regular Hausdorff spaces. We include here details of the proof for nearly Hausdorff spaces.

Theorem 7.2.4. *Let X and Y be compact nearly Hausdorff spaces and let $f : X \to Y$ be a density preserving epimorphism. Then the map $Ef : EX \to EY$ sending $\alpha \in EX$ to the fixed ultrafilter $f_{\#}(\alpha) = \{Clf(F) \mid F \in \alpha\}$ is the projective lift of f.*

Proof. Clearly $\varphi \notin f_{\#}(\alpha)$ and $X \in f_{\#}(\alpha)$. We now observe that $f_{\#}(\alpha)$ is closed under supersets. Let $H \in R(Y)$ be such that $Clf(F) \subseteq H$ for some $F \in R(X)$. Then

$$Clf(F) \subseteq H$$

$$\Rightarrow \quad IntClf(F) \subseteq IntH$$

$$\Rightarrow \quad IntClf^{-1}(Clf(F)) \subseteq Intf^{-1}(Clf(F)) \subseteq f^{-1}(IntH)$$

$$\Rightarrow \quad IntF \subseteq f^{-1}(IntH)$$

$$\Rightarrow \quad F \subseteq Clf^{-1}(IntH).$$

Since $F \in \alpha$ and $F \subseteq Clf^{-1}(IntH)$, we obtain $Clf^{-1}(IntH) \in \alpha$. Therefore $Clf(Clf^{-1}(IntH)) = H \in f_{\#}(\alpha)$.

Further, if $Clf(F)$, $Clf(H) \in f_{\#}(\alpha)$, then

$$\varphi \neq Clf(F \wedge H) \subseteq Clf(F) \wedge Clf(H)$$

$$\Rightarrow \quad Clf(F) \wedge Clf(H) \in f_{\#}(\alpha).$$

This proves that $f_{\#}(\alpha)$ is closed under finite meet. We now establish maximality of $f_{\#}(\alpha)$. Suppose $H \wedge Clf(F) \neq \varphi$ for all $F \in \alpha$, where $H \in R(Y)$. Then

$$H \wedge Clf(F) \neq \varphi \text{ for all } F \in \alpha$$

$$\Rightarrow \quad Clf^{-1}(IntH) \wedge F \neq \varphi \text{ for all } F \in \alpha$$

$$\Rightarrow \quad Clf^{-1}(IntH) \in \alpha$$

$$\Rightarrow \quad H = Clf(Clf^{-1}(IntH)) \in f_{\#}(\alpha).$$

This proves $f_{\#}(\alpha)$ is an ultrafilter. We now prove continuity of the map Ef. Let $EY - \lambda(A)$ be an open set containing $Ef(\alpha) = f_{\#}(\alpha)$. Then $A \notin f_{\#}(\alpha)$ which implies that there exists B in $f_{\#}(\alpha)$ such that $A \wedge B = \varphi$ and hence $IntA \cap IntB = \varphi$.

Further, $B \in f_{\#}(\alpha)$ implies that $B = Clf(D)$ for some D in α. Thus $IntA \cap IntB = \varphi$ implies

$$IntA \cap Int(Clf(D)) = \varphi$$

$$\Rightarrow \quad Clf^{-1}(IntA) \wedge D = \varphi.$$

Set $K = Clf^{-1}(IntA)$. Then $K \in R(X)$ and $Ef(EX - \lambda(K)) \subseteq EY - \lambda(A)$. This proves continuity of the map Ef. The projective lift Ef is unique. Suppose $g : EX \to EY$ is a continuous map such that the following diagram commutes

$$\begin{array}{ccc} EX & \xrightarrow{Ef} & EY \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}.$$

If $Ef \neq g$ then there exists $\alpha \in EX$ such that $Ef(\alpha) \neq g(\alpha)$. This implies there exists $Clf(H) \in Ef(\alpha)$ such that $Clf(H) \wedge Clf(F) = \varphi$ for some $Clf(F) \in g(\alpha)$. Therefore $\varphi \neq IntClf(F \wedge H) \subseteq IntClf(F) \cap IntClf(H) = \varphi$ - a contradiction.

Theorem 7.2.5. *The map associating the compact extremally disconnected space EX to each compact nh-space X and a continuous map $Ef : EX \to EY$ to each density preserving epimorphism $f : X \to Y$, is a covariant functor from the category of compact nearly Hausdorff spaces and dp-epimorphisms to the category of compact extremally disconnected Hausdorff spaces and continuous maps.*

Proof. Follows from the Theorem 7.2.4 and the preceding Note.

3. Commutativity of E and r.

In this section, we establish that $E(rX) = r(EX)$, where X is a nearly Hausdorff space with property π. Let the space $E(rX)$ and the map h_{rX} be same as in Theorem 7.2.2. Define EX to be the subspace $h_{rX}^{-1}(X)$ of $E(rX)$.

Lemma 7.3.1. *The subspace EX defined above is dense in $E(rX)$.*

Proof. If U is an open subset of $E(rX)$ such that $U \cap EX = \varphi$ then $h_{rX}(E(rX) - U)$ is a closed subset of rX containing X and hence $h_{rX}(E(rX) - U) = rX$. Since h_{rX} is irreducible, $E(rX) - U = E(rX)$. This proves $U = \varphi$.

Lemma 7.3.2. $r(EX) = \beta(EX)$, *where X is a nearly Hausdorff space with property π.*

Proof. The space EX being a dense subspace of the compact Hausdorff extremally disconnected space $E(rX)$, is a regular extremally disconnected zero-dimensional space. Hence by Corollary 6.3.5 we have $r(EX) = \beta(EX)$.

Theorem 7.3.3. *Let the space $E(rX)$ and the map h_{rX} be as in Theorem 7.2.2. where X is a nearly Hausdorff space with property π and let $EX = h_{rX}^{-1}(X)$.*

Then the map $h: EX \to X$ obtained by restricting h_{rX} to EX, is a perfect irreducible $\theta-$continuous surjection.

Proof. Follows as h_{rX} is a perfect irreducible $\theta-$continuous surjection and EX is dense in $E(rX)$.

Note. (1) Recall that a nearly Hausdorff space with property π is a Hausdorff space. Thus by the uniqueness of the Iliadis absolute for a Hausdorff space it follows that EX is the Iliadis absolute of the nearly Hausdorff space X with property π.

(2) Since a nearly Hausdorff space with property π need not be regular, the map h in Theorem 7.3.3 need not be continuous.

Theorem 7.3.4. *Let X be a nearly Hausdorff space with property π. Then $r(EX) = E(rX)$.*

Proof. As observed in Theorem 7.2.2, the space $E(rX)$ is a compact Hausdorff space. Hence $E(rX)$ is a Tychonoff space. The space EX being a dense subspace of extremally disconnected space $E(rX)$, is densely C*-embedded in $E(rX)$. By the uniqueness of the Stone-Čech compactification of a Tychonoff space, it follows that $\beta(EX) = E(rX)$. Further by Theorem 7.3.2 we have $r(EX) = \beta(EX)$. This proves $r(EX) = E(rX)$.

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