SUMMARY

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ON

LATTICES OF DENSITY PRESERVING MAPS, EXTENSIONS AND ABSOLUTES OF TOPOLOGICAL SPACES

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SUMMARY

In **[10]**, Porter and Woods have extensively studied the poset $IP(X)$ of covering maps on a fixed domain X. They are able to relate the sub-poset $IP(\beta X, X)$ of $IP(\beta X)$ with the well known poset $K(X)$ of all compactifications of a locally compact Hausdorff space X. The poset $IP(X)$ turns out to be a complete upper semilattice with the *partial order* defined by $g \leq f$, if there exists a continuous map $h: Rf \to Rg$ such that $h \circ f = g$. In **[10]**, Porter and Woods have partially answered the question: When $IP(X)$ is a complete lattice? In an attempt to obtain further results we study the poset $DP(X)$ of density preserving maps on a fixed domain X.

Mathematicians like Alexandroff, Uryshon, Čech, Cartan, Wallman, Tychonoff and Lubben laid the foundation of the modern theory of Hausdorff extensions. Once compactness was defined, it was a natural problem to try and extend a non-compact space to a compact space. The first general method in this direction was the one-point compactification in 1924 due to Alexandroff. In 1937 Čech developed a compactification having the maximal extension property by extending Tychonoff's idea of embedding a completely regular Hausdorff space X in a cube **[1]**. In **[14]**, Stone developed a similar compactification independently. This compactification is termed as Stone-Čech compactification and is denoted by βX for a Tychonoff space X. In fact, βX is a maximal compactification for a Tychonoff space X. In the

present thesis we obtain "βX like" compactification for a non-Tychonoff space X.

In **[5]**, S. Iliadis constructed for a Hausdorff space X an extremally disconnected regular Hausdorff space EX such that X is the image of EX under a perfect θ-continuous map. The space EX is unique upto homeomorphism. EX is called the *Iliadis absolute* of X. We construct here similar space for a non-Hausdorff space.

The material of the present thesis entitled 'Lattices of density preserving maps, extensions and absolutes of topological spaces' is the out come of researches carried out by the author mainly along these lines. There are seven chapters in the thesis.

Chapter 1 aims at providing the introduction to the subject matter of the thesis. In Chapter 2, we define and study the poset $DP(X)$ of density preserving continuous maps on a space X. A continuous map f from a topological space X into a topological space Y is called a *density preserving map* if $IntClf(A) \neq \varphi$, whenever $IntA \neq \varphi$, where A is a subset of X **[3]**. Two density preserving maps f and g defined over a topological space X with range Rf and Rg respectively are said to be *equivalent* if there exists a homeomorphism $h : Rf \to Rg$ satisfying $h \circ f = g$. We identify equivalent density preserving maps on a fixed domain X and denote by $DP(X)$, the set of all such equivalence classes of density preserving maps. Further, a *partial order* '$\leq$' is defined on $DP(X)$ such that $(DP(X), \leq)$ becomes a partially ordered set (poset). Having observed that the poset $IP(X)$ is naturally contained in the poset $DP(X)$, we study the poset $DP(X)$ to obtain further results in the direction of **[10]**. We prove the following theorem in this chapter.

Theorem 1. *Let X be a compact Hausdorff space without isolated points. Then $DP(X)$ is a complete lattice.*

Topology of a space X always determines order structure of the poset $DP(X)$, i.e. if spaces X and Y are homeomorphic then the posets $DP(X)$ and $DP(Y)$ are order isomorphic.

In Chapter 3, we study the converse problem: When order structure of the poset $DP(X)$ determines the topology of X? We introduce here the term cln-bijection for a bijective map. A bijection $f : X \to Y$ from a topological space X to a topological space Y is called a *cln-bijection* if the family $\{ f(A) \mid A$ is a closed nowhere dense subset of $X \}$ is precisely the family of all closed nowhere dense subsets of Y.

Theorem 2. *Let X and Y be Hausdorff spaces without isolated points and let φ: $DP(X) \to DP(Y)$ be an order isomorphism. Then there exists a cln-*

bijection $F: X \to Y$ *such that for each* $f \in$ $DP(X)$ *we have* $\wp(\varphi(f)) = \{F(A) \mid A \in \wp(f)\}$, where $\wp(f) = \{f^{-1}(y) \mid y \in Rf\}$.

We recall that topology of a countably compact T_3 space without isolated points is determined by closed nowhere dense sets **[10]**. Using these results we prove the following theorem:

Theorem 3. *Let* X *and* Y *be countably compact* T_3 *spaces without isolated points. Then* $DP(X)$ *and* $DP(Y)$ *are order isomorphic if and only if* X *and* Y *are homeomorphic.*

In Chapter 4, we determine some conditions under which a density preserving maps is an irreducible map. We further discuss the natural question: When $DP(X) = IP(X)$? We recall definition of an irreducible map. A surjection $f: X \to Y$ said to be *irreducible* if $f(F) \neq Y$ for every proper closed subset F of X **[18]**. Further for a subset A of a topological space X we define $DP(X, A) = \{f \in DP(X) \mid |f^{-1}(f(x))| = 1, \text{ for } x \in A\}$. We obtain the following result.

Theorem 5. *Let* A *be a dense subspace of a topological space* X. *Then every* f *in* $DP(X, A)$ *is irreducible.*

Corollary 6. *If X is a compact Hausdorff space and A is a dense subset of X, then $DP(X,A) = IP(X,A)$. In particular, if X is a locally compact Hausdorff space then $DP(\alpha X, X) = IP(\alpha X, X)$, where αX is a compactification of X.*

In particular, for a locally compact space X we have $DP(\beta X, X) = IP(\beta X, X)$. As a consequence of this we obtain Magill's result **[8]**. Using a result due to Porter and Woods **[10]** we obtain that $DP(\beta X, X)$ is order isomorphic to $K(X)$. As a consequence of this we are able to use lattice theoretic properties of the complete lattice $DP(\beta X, X)$ to obtain topological properties of $\beta X - X$ when X is a locally compact Hausdorff space.

An $f \in DP(X)$ is said to be a *dual* if the only non-singleton fiber of f is a doubleton. In Chapter 5, we introduce and study the notion of overlapping duals and also the notion of duals hinged with overlapping duals. We topologize the collection $\Im$ of all subset of the set of duals in $DP(X)$ which are hinged with overlapping duals in $DP(X)$ and study when the topological space $\Im$ is homeomorphic to X. We in fact prove here that $\Im$ is homeomorphic to X when X is a countably compact T_3 space without isolated points. To prove this result we introduce here the notion of $F-$closed sets and observe that for a locally compact Hausdorff space X, the notion of $F-$closed sets coincide with the notion of $F-$compact sets defined by Thrivikraman in **[15]**.

In Chapter 6, we construct a "βX like" compactification of a space X. We introduce here a topological property π for a topological space X. A space X is said to have *property* π if for every $F \in R(X)$ and $x \notin F$ there exists an $H \in R(X)$ such that $x \in IntH$ and $H \cap F = \varphi$. For a Urysohn space X with property π, a filter $\alpha \subseteq Rf(X) - \{\phi\}$ is called an $r-$*filter* if it is closed under finite intersection and supersets. A maximal $r-$filter is called an $r-$*ultrafilter.* The family of all $r-$ultrafilters in X is denoted by rX. Further for $F \in R(X)$ define $\overline{F} = \{\alpha \in rX \mid F \in \alpha\}$. Topologize the set rX by taking $B = \{\overline{F} \mid F \in R(X)\}$ as a base for closed sets in rX. The resulting space rX is a compact T_1 non-Hausdorff space. We observe that the space rX satisfies a separation axiom stronger than T_1 but weaker than T_2 which we term as nearly Hausdorff and have defined as follows: A space X is called a *nearly Hausdorff space* if for every pair of distinct points x and y in X, there exist regular closed sets F_x and F_y containing x and y respectively such that $x, y \notin F_x \cap F_y$. We observe that nearly Hausdorffness is productive but not closed hereditary. Also, we study the relation of nearly Hausdorffness with other separation axioms and we observe the following implications:

$$\begin{array}{l} Regular \Rightarrow Urysohn(\pi) \Leftrightarrow NearlyHausdorff(\pi) \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \Downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad Urysohn \Rightarrow Hausdorff \Rightarrow NearlyHausdorff \Rightarrow T_1 \end{array}$$

We include examples to justify that the unidirectional implications in the above flow diagram need not be reversible. We observe that a nearly Hausdorff space with property π is an Urysohn space. Hence the construction of rX described above can be done for a nearly Hausdorff space with property π. We observe the following results:

Theorem 7. *Let X be a nearly Hausdorff space with property π. Then the space rX of all r-ultrafilters in X is a compact nearly Hausdorff space which contains X as a dense C*-embedded subspace.*

Following results answers natural question: When rX = βX?

Theorem 8. *Let X be a nearly Hausdorff space such that $Rf(X)$ is a Wallman base. Then $rX = \beta X$.*

Corollary 9. *If X is a normal space or zero-dimensional space then $rX = \beta X$.*

In chapter 7, we describe the construction of projective cover (EX, h_X) for a compact nearly Hausdorff space X on the lines of Gleason's construction **[18]**. In this chapter we study projective lift and extension of density preserving epimorphism $f : X \to Y$. We prove the following theorem in this chapter.

Theorem 10. *Let X be a nearly Hausdorff space with property π. Then $r(EX) = E(rX)$.*

Some of the results of Chapters 2, 3 and 4, in its original form are being published in the Bulletin of the Australian Mathematical Society, Volume 72 (2005). Major results of Chapter 6 are accepted for its publication in the 'Applied General Topology' journal.

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