CHAPTER II

LATTICE $DP(X)$

Attaching an algebraic object to a topological space in order to study the later is a well-known procedure. Two examples of such techniques are the use of the rings of continuous functions to study the space X and the use of the semilattice of compactifications of a Tychonoff space X to study the remainder $\beta X - X$ of its Stone-Čech compactification βX. Posets of compactifications of a completely regular Hausdorff space have been extensively studied by various authors including Magill **[17]**, Rayburn **[23]**, Visliseni and Flaksmaier **[31]**, Thrivikraman **[29, 30]**, Kannan **[12]**. In **[22]** Porter and Woods have extensively studied the poset $IP(X)$ of covering maps on a fixed domain X, where by a covering map we mean a perfect irreducible continuous surjection. The poset $IP(X)$ turns out to be a complete upper semilattice with the partial order '$\leq$' defined by $f \leq g$ if there exists a continuous map $h : Rg \to Rf$ such that $h \circ g = f$. Besides obtaining several interesting results, Porter and Woods have partially answered in **[22]** the question: When $IP(X)$ is a complete lattice?

In an attempt to obtain further results in this direction, we have observed that $IP(X)$ is naturally contained in the poset $DP(X)$ of the density preserving maps. In the first section of this chapter we recall the definition of density preserving maps **[8]** and give some examples. In the second section, we define partial order on the set $DP(X)$ of density preserving maps and prove that $DP(X)$ is a poset with this partial order. In the last section of this chapter we answer the question: When $DP(X)$ is a complete lattice?

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1. Density Preserving Maps: Definition and Examples.

In this section we recall the definition of a density preserving map and give several examples of such maps. Throughout, our maps are continuous.

Definition 2.1.1. A continuous map f from a topological space X into a topological space Y is called *density preserving* if $IntClf(A) \neq \varphi$, whenever $IntA \neq \varphi$, where A is a subset of X **[8]**.

Examples 2.1.2. (a) An open map is a density preserving map.

2.1.2. (b) An RC-preserving map is a density preserving map. (A continuous map f from a topological space X into a topological space Y is called an *RC-preserving map* if image of a regular closed set in X is a regular closed in Y) **[7]**.

2.1.2. (c) A closed irreducible surjection is a density preserving map. In fact, let f from a topological space X into a topological space Y be a closed irreducible surjection and let $A \subseteq X$ be such that $Int\,A \neq \varphi$. Then the proof for the case $A = X$ follows trivially. Suppose $[X - Int\,A]$ is a nonempty proper closed subset of X. Since f is closed and irreducible, $f(X - Int\, A)$ is a proper closed subset of Y. Observe that the nonempty open set $[Y - f(X - Int\, A)]$ is contained in $f(Int\, A)$ and hence $IntClf(A) \neq \varphi$. This proves that f is density preserving. In particular, an irreducible surjection from a compact Hausdorff space to a Hausdorff space is density preserving.

2.1.2. (d) The map $f:[0, 1]\to [0, 1]$ defined by

$$f(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ (2-2x), & \frac{1}{2} \le x \le 1 \end{cases}$$

is a continuous perfect RC-preserving surjection and hence a density preserving map. Since $f[0, \frac{1}{2}]=[0, 1]$, f is not irreducible and hence is not a covering map.

2.1.2. (e) The restriction of the continuous map f defined in the above example 2.1.2(d) on [0, 1) is a density preserving surjection but is none of the following: RC-preserving, closed, perfect and irreducible.

2.1.2. (f) The map $f:[0, 1)\to S^1$ defined by $f(x) = e^{2\pi ix}$ is a bijective continuous density preserving map. In fact f is irreducible. On the other hand f is not RC-preserving as f maps the regular closed set $[\frac{1}{2},1)$ onto a non-regular closed set in S^1. Also, f is not an open map as the image of the open set $[0,\frac{1}{2})$ under f is not an open subset of S^1. Since f is not a closed map, it is not a covering map.

2.1.2. (g) The natural inclusion of the usual space of rational numbers into the usual space of real numbers is a density preserving epimorphism.

2.1.2. (h) Let X be a topological space and let K be a closed nowhere dense subset of X. Then the natural quotient map $q:X \to X|_K$, where $X|_K$ is the space obtained by collapsing K to a point, is a density preserving map. Let us see a proof of this fact: Take $A \subset X$ with $Int\, A \neq \varphi$. Observe that either $Int\, A \cap K = \varphi$ or $Int A \cap K \neq \varphi$.

Case 1: Suppose $Int\, A \cap K = \varphi$. Then $q(Int\, A)$ is an open set in $X|_K$ satisfying $q(Int\, A) \subseteq Clq(A)$. This proves $Int\, Clq(A) \neq \varphi$.

Case 2: Suppose $Int A \cap K \neq \varphi$. Since K is a nowhere dense set, $Int A \not\subset K$. Choose x in $Int A - K$. Observe that we can find an open set $U \subset A$ such that $x \in U$ and $U \cap K = \varphi$; for otherwise x will become a limit point of K and hence x will be in K. Clearly $q(U)$ is an open set in $X|_K$ satisfying $q(U) \subset Clq(A)$ and hence $Int\, Clq(A) \neq \varphi$.

2. Poset $DP(X)$.

We define here a notion of equivalence on the set of density preserving maps defined on a fixed topological space X. We identify equivalent density preserving maps on a fixed domain X and denote by $DP(X)$, the set of all such equivalent classes of density preserving maps. An order relation $\leq$ is defined on $DP(X)$ such that $(DP(X), \leq)$ becomes a partially ordered set (poset). We also observe behavior of density preserving maps on isolated points.

Definition 2.2.1. Two density preserving maps f and g defined on a topological space X and range Rf and Rg respectively are said to be *equivalent* if there exists a homeomorphism $h: Rf \to Rg$ satisfying $h \circ f = g$.

Notation. Two such equivalent maps f and g are denoted by $f \approx g$.

Examples 2.2.2. (a) Let X be a topological space and let $f: X \to Rf$ be a homeomorphism. Then $f \approx I_X$ because the homeomorphism $h: X \to Rf$ defined by $h(x) = f(x)$ satisfies $h \circ I_X = f$. Thus all homeomorphisms on X represent one single member in $DP(X)$. Similarly the family of all constant maps defined on X represents a single member in $DP(X)$.

2.2.2. (b) Consider **R**, the usual space of all real numbers. Define f: **R** $\to [0,\infty)$ by $f(x) = x^2$ and g: **R** $\to [0,\infty)$ by $g(x) = |x|$. Clearly both f and g are continuous density preserving maps. Observe that the map $h:[0,\infty) \to [0,\infty)$ defined by $h(x) = x^2$ is continuous and $h \circ g = f$. Thus $f \approx g$.

Definition 2.2.3. Let X be a topological space and let f, g be in $DP(X)$, we define an order relation $\le$ on $DP(X)$ by $g \le f$ if there exists a continuous map $h: Rf \to Rg$ satisfying $h \circ f = g$.

Examples 2.2.4. (a) Consider I_X and c in $DP(X)$ where I_X represents the class of all homeomorphisms on X and c represents the class of all constant maps on X. Observe that $c \le I_X$ in $DP(X)$.

2.2.4. (b) Consider the usual space **R** of real numbers and take closed nowhere dense sets $K = \{1, 2, 3\}$ and $L = \{1, 2\}$ in **R**. Consider the natural quotient maps $q: \mathbf{R} \to \mathbf{R}|_K$ and $p: \mathbf{R} \to \mathbf{R}|_L$ as defined in Example 2.1.2 (h). Then $q \le p$ because the map $h: Rp \to Rq$ defined by

$$h(p(x)) = \begin{cases} x, & x \in R - \{1, 2, 3\} \\ q(x), & x \in \{1, 2, 3\} \end{cases}$$

is continuous and satisfies $h \circ p = q$.

2.2.4. (c) Along the lines of above example if we consider an increasing finite chain $\{K_i\}_{i=1}^n$ of closed nowhere dense sets in a topological space X and if $\{q_i\}_{i=1}^n$ is the corresponding family of quotient maps obtained by identifying K_i to a point then $q_n \le q_{n-1} \le \dots\dots\dots \le q_1$ in $DP(X)$.

Theorem 2.2.5. *Let X be a topological space.* Then $(DP(X), \le)$ *is a partially ordered set, where $\le$ is an order relation as defined in* 2.2.3.

Proof. We show that $\le$ is reflexive, antisymmetric and transitive.

'$\le$' <u>is reflexive</u>: Take f in $DP(X)$ and consider the identity map $I_{Rf}: Rf \to Rf$. Clearly $I_{Rf} \circ f = f$. This proves $f \approx f$, for all f in $DP(X)$.

'$\le$' <u>is antisymmetric</u>: Let f, g be in $DP(X)$ such that $f \le g$ and $g \le f$. Then we need to show $f \approx g$. By Definition 2.2.3 there exist continuous maps $h: Rg \to Rf$ and $k: Rf \to Rg$ such that $h \circ g = f$ and $k \circ f = g$. The composite maps $h \circ k$ and $k \circ h$ are identity maps on Rf and Rg respectively. This implies that h and k are inverses of each other. Since h and k are bijective continuous maps, both h and k are homeomorphism. This proves $f \approx g$.

'$\leq$' is transitive: Suppose $f \leq g$ and $g \leq h$. Then there exist continuous maps $k_1 : Rg \to Rf$ and $k_2 : Rh \to Rg$ such that $k_1 \circ g = f$ and $k_2 \circ h = g$. The composite map $k_1 \circ k_2 : Rh \to Rf$ is continuous and

$$(k_1 \circ k_2) \circ h = k_1 \circ (k_2 \circ h) = k_1 \circ g = f.$$

This proves $f \leq h$.

Lemma 2.2.6. *Let X be a topological space and let f, g be in $DP(X)$ with $g \leq f$. Then the map $h : Rf \to Rg$ satisfying $h \circ f = g$ is also density preserving.*

Proof. Let $A \subseteq Rf$ be such that $Int\,A \neq \varphi$. Then $f^{-1}(Int\,A)$ is a nonempty open subset of X. Also, $f^{-1}(Int\,A) \subseteq f^{-1}(A)$ implies $Int f^{-1}(A) \neq \varphi$. Set $f^{-1}(A) = A^*$. Since g is density preserving, $Int\,Cl\,g(A^*) \neq \varphi$. That h is a density preserving map follows from the following fact:

$$\begin{aligned} \varphi &\neq Int\,Cl\,g(A^*) \\ &= Int\,Cl\,(h \circ f)(A^*) = IntCl\,(h \circ f)(f^{-1}(A)) \\ &\subseteq IntClh(A). \end{aligned}$$

Lemma 2.2.7. *Let $f : X \to Y$ be a density preserving map from a topological space X into a T_1 space Y. Then f maps isolated points to isolated points.*

Proof. Let x be an isolated point in X. If $f(x)$ is not an isolated point of Y then $IntClf(\{x\}) = Intf(\{x\}) = \varphi$. But this is not possible because $f : X \to Y$ is density preserving and $Int\,\{x\} \neq \varphi$.

LEMMA 2.2.8. *Let $f : X \to Y$ be a density preserving map from a topological space X into a T_1 space Y. If Y is without isolated points then fibers $f^{-1}(y)$, $y \in Y$ are closed nowhere dense subsets of the space X.*

Proof. Continuity of the map f guarantees that each fiber $f^{-1}(y)$ is a closed subset in the space X. If possible, suppose for some y in Y, $f^{-1}(y)$ is not a nowhere dense set in the space X. Then

$$Int\,Clf^{-1}(y) = Intf^{-1}(y) \neq \varphi .$$

Set $A = f^{-1}(y)$. Note that $IntA \neq \varphi$ but

$$IntClf(A) = IntClf(f^{-1}(y)) = IntCl\{y\} = \varphi ,$$

which contradicts that f is density preserving.

3. Completion of $DP(X)$.

In this section we prove that if X is a compact Hausdorff space without isolated points, then $DP(X)$ is a complete lattice with respect to the partial order defined in Definition 2.2.3. For this we first define dp-partition for a density preserving map f and then use it to characterize equivalent members in $DP(X)$.

Definition 2.3.1. Let X be a topological space and let $f \in DP(X)$. Then the partition $\wp(f) = \{f^{-1}(y) \mid y \in Rf\}$ of X is called the *dp-partition* generated by f.

Example 2.3.2. Consider the density preserving map f: **R** $\to [0,\infty)$ defined by $f(x) = |x|$. Then $\wp(f) = \{\{-x, x\} | x \in [0,\infty)\}$ is the dp-partition of **R** generated by f.

We recall that a partition P of a set X is said to be refined by a partition P^* of X if for every A in P there exists B in P^* such that $B \subseteq A$. We denote this by $P^* \subseteq P$. On the same lines we define $\wp(g) \subseteq \wp(f)$, where f and g are density preserving maps on a space X and $\wp(f)$ and $\wp(g)$ are the corresponding dp-partitions of X generated by f and g respectively.

Now onwards we assume that members of $DP(X)$ are quotient maps. In case X is compact, this condition is automatically satisfied.

The following lemma relates order on $DP(X)$ with the dp-partition.

Lemma 2.3.3. *Let X be a topological space and let $f, g \in DP(X)$. Then $f \leq g$ if and only if $\wp(g) \subseteq \wp(f)$.*

Proof. Suppose $f \leq g$. Then by Definition 2.2.3 there exists a continuous map $h : Rg \to Rf$ satisfying $h \circ g = f$. Let $y \in Rg$. Then we need to find x in Rf such that $g^{-1}(y) \subseteq f^{-1}(x)$. Observe that we have an x in Rf such that $h(y) = x$. Set $A = g^{-1}(y) \in \wp(g)$. Clearly $A \subseteq (h \circ g)^{-1}(x) = f^{-1}(x)$. This proves $\wp(g) \subseteq \wp(f)$.

Conversely, suppose $\wp(g) \subseteq \wp(f)$, then for z in Rg we find a unique y in Rf for which $g^{-1}(z) \subseteq f^{-1}(y)$. Define $h : Rg \to Rf$ by $h(z) = y$. The map h is well defined and $h \circ g = f$. Continuity of h follows from the fact that g is a quotient map and f is continuous. Hence $f \leq g$.

Lemma 2.3.4. *Let X be a topological space and let $f, g \in DP(X)$. Then f and g are equivalent if and only if $\wp(f) = \wp(g)$.*

Proof. Suppose f is equivalent to g. Then $f \leq g$ and $g \leq f$. By Lemma 2.3.3, $\wp(g)$ is a refinement of $\wp(f)$ and is refined by $\wp(f)$. Hence $\wp(f) = \wp(g)$.

Conversely, suppose $\wp(f) = \wp(g)$. Then for each $z \in Rg$ take the unique $y \in Rf$ for which $g^{-1}(z) = f^{-1}(y)$ and define $h : Rg \to Rf$ by $h(z) = y$. Observe that h is bijective. Continuity of h as well as of h^{-1} is proved on the same lines as that of Lemma 2.3.3. Hence h is a homeomorphism and therefore f and g are equivalent.

Theorem 2.3.5. *Let X be a compact Hausdorff space without isolated points. Then the poset $DP(X)$ is a complete upper semi-lattice.*

Proof. Let S be a non-empty subset of $DP(X)$. Consider the product space $Z = \prod_{f \in S} Rf$ and the natural evaluation map $g : X \to Z$ satisfying $\pi_f(g(p)) = f(p)$, where π_f is the natural f^{th} projection of the product space Z onto the space Rf. Set $T = g(X)$, $\pi'_f = \pi_f|_T$ and define $g' : X \to T$ by $g'(p) = g(p)$, $p \in X$. We complete the proof by showing that g' is least upper bound of S.

Observe that $\pi'_f \circ g' = f$ for all f in S implies that $f \le g'$ for each f in S. This proves that g' is an upper bound for S.

We now prove that g' is the least upper bound of S. Let k be another upper bound for S. Define $h : Rk \to Rg'$ by $h(x) = g'(y)$ where $y \in k^{-1}(x)$. We first observe that the map h is well defined. For $x \in Rk$ if $h(x) = g'(y_1)$ and $h(x) = g'(y_2)$ then $g'(y_1) = g'(y_2)$ because $f \le k$ implies $\wp(k) \subseteq \wp(f)$ for each f in S and hence $y_1, y_2 \in k^{-1}(x) \subseteq f^{-1}(z)$ for some z in Rf implies $f(y_1) = f(y_2)$ for each f in S. By definition of map h it follows that $h \circ k = g'$. The continuity of map h follows as k is a quotient map and g' is continuous. This also proves that g' is the least upper bound for S. Since $S \subseteq DP(X)$ is arbitrary it follows that every nonempty subset of $DP(X)$ has least upper bound. Hence $DP(X)$ is a complete upper semilattice.

Theorem 2.3.6. *Let X be a compact Hausdorff space without isolated points. Then $DP(X)$ is a complete lattice.*

Proof. Since a constant map onto its image is a density preserving map and any two such maps are equivalent, $DP(X)$ has the minimal element. The required result now follows from Theorem 2.3.5 and the fact that a complete upper semilattice with minimal element is a complete lattice.