CHAPTER IV

POSETS $DP(X)$, $IP(X)$ AND $K(X)$

In the third chapter, we obtained that if X and Y are countably compact T_3 spaces without isolated points, then $DP(X)$ and $DP(Y)$ are order isomorphic if and only if X and Y are homeomorphic. This result is similar to Theorem 1.3 and Theorem 1.17. This suggests that there might be a common generalization. We explore this in the present chapter. We obtain the relation of the poset $DP(X)$ with the poset $IP(X)$ of covering maps on a Hausdorff space X **[22]** and the relation of the poset $DP(X)$ with the poset $K(X)$ of compactifications of a locally compact space X **[17]**. We show that for a dense subset U of a compact space X, $DP(X,U)=IP(X,U)$ where $IP(X,U)$ (respectively $DP(X,U)$) is the poset of all covering (respectively density preserving) maps f on X satisfying $|f^{-1}(f(x))|=1$, for each x in U. In particular, for a locally compact space X we have $DP(\beta X,X)=IP(\beta X,X)$. Using this and a result due to Porter and Woods we obtain well-known Magill's result which states that for locally compact spaces X and Y, $K(X)$ and $K(Y)$ are order isomorphic if and only if $\beta X-X$ and $\beta Y-Y$ are homeomorphic.

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1. Poset $DP(X,A)$.

In this section for a topological space X we define the poset $DP(X,A)$ where $A \subseteq X$. We show that for a subset A of a compact Hausdorff space X containing all the isolated points of X, $DP(X,A)$ is a complete upper semilattice with the order relation defined in $DP(X)$. We also prove that if A_1, A_2 are subset of Hausdorff spaces X_1 and X_2 such that A_1 contains all the isolated points of X_1 and A_2 contains all the isolated points of X_2 then an order isomorphism $\psi : DP(X_1, A_1) \to DP(X_2, A_2)$ induces a cln-bijection $F : X_1 - A_1 \to X_2 - A_2$.

Definition 4.1.1. For a subset A of a topological space X define $DP(X,A) = \{f \in DP(X) \mid |f^{-1}(f(x))| = 1, \text{ for each } x \in A\}$.

Note. For a topological space X, $DP(X, \varphi) = DP(X)$.

Lemma 4.1.2. *For a subset A of a topological space X, $DP(X,A)$ is a poset with respect to the order defined in $DP(X)$.*

Proof. We show that $\leq$ is reflexive, antisymmetric and transitive.

'$\leq$' <u>is reflexive</u>: Take f in $DP(X,A)$ and consider the identity map $I_{Rf} : Rf \to Rf$. Clearly $I_{Rf} \circ f = f$. This proves $f \approx f$ for all f in $DP(X,A)$.

'$\leq$' <u>is antisymmetric</u>: Let f, g be in $DP(X,A)$ such that $f \leq g$ and $g \leq f$. Then we need to show $f \approx g$. By Definition 2.2.3 there exist continuous maps $h: Rg \to Rf$ and $k: Rf \to Rg$ such that $h \circ g = f$ and $k \circ f = g$. The composite maps $h \circ k$ and $k \circ h$ are the identity maps on Rf and Rg respectively. This implies that h and k are inverses of each other. Since h and k are bijective continuous maps, both h and k are homeomorphisms. This proves $f \approx g$.

'$\leq$' is transitive: Suppose $f \leq g$ and $g \leq h$. Then there exist continuous maps $k_1 : Rg \to Rf$ and $k_2 : Rh \to Rg$ such that $k_1 \circ g = f$ and $k_2 \circ h = g$. The composite map $k_1 \circ k_2 : Rh \to Rf$ is continuous and

$$(k_1 \circ k_2) \circ h = k_1 \circ (k_2 \circ h) = k_1 \circ g = f .$$

This proves $f \leq h$. Therefore $(DP(X,A),\leq)$ is a partially ordered set. We shall denote this poset by $DP(X,A)$.

Lemma 4.1.3. *For a subset A of a topological space X, if $g \in DP(X,A)$, $f \in DP(X)$ and $g \leq f$, then $f \in DP(X,A)$.*

Proof. Suppose $f \in DP(X)$ is such that $g \leq f$ for some $g \in DP(X,A)$. Then by Lemma 2.3.3 we have $\wp(f) \subseteq \wp(g)$, i.e., for each y in Rf there exists z in Rg such that $f^{-1}(y) \subseteq g^{-1}(z)$. Let $x \in f^{-1}(y) \subseteq g^{-1}(z)$. Then $f^{-1}(f(x)) \subseteq g^{-1}(g(x))$ for some $z = g(x) \in Rg$. In particular, if $x \in A$ then $g \in DP(X,A)$ and $g \leq f$ implies $|g^{-1}(g(x))| = 1$ and $\varphi \neq f^{-1}(f(x)) \subseteq g^{-1}(g(x))$. This proves $|f^{-1}(f(x))| = 1$ for each $x \in A$. Hence $f \in DP(X,A)$.

Theorem 4.1.4. *Let A be a subset of a compact space X containing all isolated points of X. Then $DP(X,A)$ is a complete upper semilattice.*

Proof. Let S be a non-empty subset of $DP(X,A)$ and let Z = $\prod_{f\in S} Rf$. Consider the natural evaluation map $g: X \to Z$ such that $\pi_f(g(p)) = f(p)$, where π_f :Z$\to Rf$ is the f^{th} projection map. Set $T = g(X)$, $\pi_f^{'} = \pi_f |_T$ and define $g^{'}: X \to T$ by $g^{'}(p) = g(p)$, $p \in X$. Continuity of $g^{'}$ follows from continuity of g. We complete the proof by showing $g^{'}$ is least upper bound of S.

Now $\pi^{'}_f \circ g^{'} = f$ for each f in S implies that $f \le g^{'}$ for each f in S. Therefore $g^{'}$ is an upper bound for S. Finally by Lemma 4.1.3 we get $g^{'} \in DP(X,A)$.

Next, we prove that $g^{'}$ is the least upper bound of S. Let k be another upper bound of S. Define $h: Rk \to Rg^{'}$ by $h(x) = g^{'}(y)$, where $y \in k^{-1}(x)$. We first observe that the map h is well defined. Let $h(x) = g^{'}(y_1)$ and $h(x) = g^{'}(y_2)$ for some $x \in Rk$. Since $f \le k$ for each f in S, we have $\wp(k) \subseteq \wp(f)$ for each f in S and hence $y_1, y_2 \in k^{-1}(x) \subseteq f^{-1}(z)$ for some z in Rf which implies $f(y_1) = f(y_2)$ for each f in S. This proves that $g^{'}(y_1) = g^{'}(y_2)$. Clearly $h \circ k = g^{'}$. Also h is continuous because k is a quotient map and $g^{'}$ is continuous. This also proves that $g^{'}$ is the least upper bound of S. Since $S \subseteq DP(X,A)$ is arbitrary, it follows that $DP(X,A)$ is a complete upper semilattice.

Remarks. (a) If X is a compact space without isolated point and $A \subseteq X$ then the proof that $DP(X,A)$ is a complete upper semilattice can be obtained by Theorem 2.3.5 as follows: Let S be a non-empty subset of $DP(X,A)$. Then S is a non-empty subset of $DP(X)$ and $DP(X)$ is a complete upper semilattice implies that S has a least upper bound say g in $DP(X)$. Since $f \leq g$ for all $f \in S$, by Lemma 4.1.3 we have $g \in DP(X,A)$.

(b) Recall that if X is a compact Hausdorff space without isolated points then $DP(X)$ is a complete lattice [Theorem 2.3.6]. On the other hand, if A is a dense set inside such a space X, then $DP(X,A)$ need not be a complete lattice. For example, consider the Stone-Čech compactification βQ of the usual space Q of rational numbers. Then the complete upper semilattice $DP(\beta Q,Q)$ is not a complete lattice as the greatest lower bound of the set $\{f \in DP(\beta Q,Q) \mid f$ is a dual obtained by identifying two distinct points of $\beta Q - Q\}$ does not exist in $DP(\beta Q,Q)$. In fact, if g is the greatest lower bound of S then we claim that it is primary. Assuming the claim in hand, we can write g as (g,K) where K is the only non-singleton member in $\wp(g)$. Clearly $\wp(f) \subseteq \wp(g)$, for all $f \in S$ which implies $\beta Q - Q \subseteq K$ and hence $K = \beta Q - Q$. But this is not possible as K is closed and $\beta Q - Q$ is not closed. For a proof of the claim suppose $\wp(g)$ contains non-singleton members H and K. Choose $a \in H$ and $b \in K$. The dual $(f;\{a,b\}) \in DP(\beta Q,Q)$, $g \leq f$ but $\wp(f) \not\subseteq \wp(g)$ gives a contradiction.

Theorem 4.1.6. *Let A_i be a subset of a Hausdorff topological space X_i containing all isolated points of X_i, $i = 1, 2$ and let $\psi : DP(X_1, A_1) \to DP(X_2, A_2)$ be an order isomorphism. Then there is a cln-bijection $F : X_1 - A_1 \to X_2 - A_2$ such that*

$$\wp(\psi(f)) = \{\{x\} \mid x \in A_2\} \cup \{F(A) \mid A \in \wp(f), A \subseteq X_1 - A_1\}.$$

Proof. We consider the following cases:

Case (i) Suppose $|X_1 - A_1| = 2$. Let $x, y \in X_1 - A_1$. Then $DP(X_1, A_1) = \{I_{X_1}, f\}$, where $f \approx (f; \{x, y\})$. Since ψ is an order isomorphism, $DP(X_2, A_2)$ contains exactly two members which implies that $|X_2 - A_2| = 2$. Let $a, b \in X_2 - A_2$. Then the map $F : X_1 - A_1 \to X_2 - A_2$ defined by $F(x) = a$ and $F(y) = b$ is the required map.

Case (ii) Suppose $|X_1 - A_1| \geq 3$. Let $p \in X_1 - A_1$. Choose distinct points $q, r \in X_1 - (A_1 \cup \{p\})$. By Theorem 3.1.6, $\psi(f; \{p, q\})$, $\psi(g; \{p, r\})$ are dual points of $DP(X_2, A_2)$, say $(\overline{f}; \{a, b\})$ and $(\overline{g}; \{c, d\})$ respectively. Note that $a, b, c, d \in X_2 - A_2$ because $\overline{f}$ and $\overline{g}$ are duals and both $\overline{f}|_{A_2}$ and $\overline{g}|_{A_2}$ are one-one. Clearly

$$(\overline{f}; \{a, b\}) \wedge (\overline{g}; \{c, d\}) = \psi(f \wedge g; \{p, q, r\}).$$

If $\{a, b\} \cap \{c, d\} = \varphi$ then

$$(\overline{f}; \{a, b\}) \wedge (\overline{g}; \{c, d\}) = (\overline{f} \wedge \overline{g}; \{a, b\}, \{c, d\}),$$

which is not possible since $(f; \{p, q\})$, $(g; \{p, r\})$, $(h; \{q, r\})$ are three dual points in $DP(X_1, A_1)$ greater than $(f \wedge g; \{p, q, r\})$ where as $(\overline{f}; \{a, b\})$,

$(\overline{g}; \{c, d\})$ are the only two dual points in $DP(X_2, A_2)$ greater than $(\overline{f} \wedge \overline{g}; \{a, b\}, \{c, d\})$. Therefore $\{a, b\} \cap \{c, d\} \neq \varphi$. Since ψ is an order isomorphism, it is a singleton, say $\{a\}$. Define $F: X_1 - A_1 \to X_2 - A_2$ by $F(p) = a$. Note that $a \in X_2 - A_2$. We now show that the choice of a does not depend upon the choices of r and q. Let $s \in X_1 - (A_1 \cup \{p, q, r\})$. Then there exist points y and z in $X_2 - A_2$ such that $\psi(k; \{p, s\}) = (\overline{k}; \{y, z\})$. We have $\psi(f; \{p, q\}) = (\overline{f}; \{a, b\})$. Assume $\psi(g; \{p, r\}) = (\overline{g}; \{a, c\})$. Using similar arguments, we conclude that $\{y, z\}$ intersects both $\{a, b\}$ and $\{a, c\}$ in exactly one point. As discussed in the proof of Lemma 3.2.3, this one point is precisely a. Thus for any $s \in X_1 - (A_1 \cup \{p\})$, if $\psi(k; \{p, s\}) = (\overline{k}; \{y, z\})$ then $a \in \{y, z\}$ and if s' is any other point in $X_1 - (A_1 \cup \{p, q\})$ and if $\psi(\sigma; \{p, s'\}) = (\overline{\sigma}; \{y', z'\})$ then $\{y', z'\} \cap \{y, z\} = \{a\}$. Thus F is well defined.

We now show ψ maps closed nowhere dense sets in $X_1 - A_1$ to closed nowhere dense sets in $X_2 - A_2$. Let H be a closed nowhere dense set in $X_1 - A_1$. Consider $f \in DP(X_1, A_1)$ of the form (f, H) and if $\psi(f; H) = \overline{f}$ then $\overline{f} = (\overline{f}; K)$ for some closed nowhere dense subset K of $X_2 - A_2$. Further, if $p, q \in H$, $p \neq q$. Then by Lemma 2.3.3 $(g; \{p, q\}) \geq (f; H)$ which implies $(\overline{g}; \{a, b\}) \geq (\overline{f}; K)$. This proves $F(\{p, q\}) = \{a, b\} \subseteq K$. Hence $F(H) \subseteq K$.

Similarly, using ψ^{-1} we can define $\overline{F}: X_2 - A_2 \to X_1 - A_1$. Giving similar arguments as above one can show that $\overline{F}(K) \subseteq H$.

We now prove that $\overline{F} \circ F$ is identity on $X_1 - A_1$. Let $p \in X_1 - A_1$ and $q \in X_1 - (A_1 \cup \{p\})$. Since $(f; \{p,q\})$ is a dual in $DP(X_1, A_1)$ therefore $\psi(\overline{f}; \{a, b\})$ is a dual in $DP(X_2, A_2)$ say $(\overline{f}; \{a, b\})$. We know $F(p) \in \{a, b\}$. Assume $F(p) = a$. Suppose $\overline{F}(a) \neq q \neq p$. Choose $r \in X_1 - (A_1 \cup \{p,q\})$. Then there exists $c \in X_2 - A_2$ such that $\psi(g; \{p, r\})$ is a dual point say $(\overline{g}; \{a, c\})$. Since $\overline{F}(a) \in \{p, r\}$ and $\overline{F}(a) \neq p$ therefore $\overline{F}(a) = r$, a contradiction since $\overline{F}(a) = q \neq r$. This proves $\overline{F} \circ F$ is identity on $X_1 - A_1$. Similarly, one can prove that $F \circ \overline{F}$ is identity on $X_2 - A_2$. Hence $F : X_1 - A_1 \to X_2 - A_2$ is a bijective map which preserves closed nowhere dense sets. Also, by the definition of the map F, it follows that $\wp(\psi(f)) = \{\{x\} \mid x \in A_2\} \cup \{F(B) \mid B \in \wp(f), B \subseteq X_1 - A_1\}$.

2. Density Preserving Maps and Irreducible Maps.

In this section we determine conditions under which a density preserving map is an irreducible map. We further discuss the natural question: When $DP(X) = IP(X)$? We first recall definition of an irreducible map.

Definition 4.2.1. Let $f : X \to Y$ be a surjective map. Then f is said to be *irreducible* if $f(F) \neq Y$ for every proper closed subset F of X **[32]**.

Theorem 4.2.2. *Let A be a dense subspace of a topological space X. Then every f in $DP(X,A)$ is irreducible.*

Proof. Let $f \in DP(X,A)$ and let F be a proper closed subset of X such that $f(F) = Rf$. Then A being dense in X, the open set $X - F$ has non-empty intersection with A. Let $y \in (X-F) \cap A$. Then $|f^{-1}(f(y))| > 1$ as $f(F) = Rf$ but this contradicts that $f|_A$ is one-one. This proves that f is irreducible.

Corollary 4.2.3. *If X is a compact Hausdorff space and A is a dense subset of X then $DP(X,A) = IP(X,A)$. In particular, if X is a locally compact Hausdorff space then $DP(\alpha X, X) = IP(\alpha X, X)$ where αX is a compactification of X.*

Proof. Clearly $IP(X,A) \subseteq DP(X,A)$. For the reverse containment, set $D_C(X,A) = \{f \in DP(X,A) \mid f$ is closed$\}$ and observe that $IP(X) \supseteq D_C(X,A) = DP(X,A)$. The particular case follows because if X is locally compact, then X is dense in αX.

Note. In general, if A is subset of a topological space X which is not dense then $D_C(X,A) \subseteq IP(X)$ need not be true. For example take $X =$ [0, 1], $A = [0, \frac{1}{2})$ and define $f : X \to X$ by

$$f(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ \frac{3}{2} - x, & \frac{1}{2} \le x \le 1 \end{cases}.$$

Then $f \in D_C(X,A)$ but f is not irreducible since $f([0, \frac{1}{2}]) = X$.

We recall the following result due to Porter and Woods **[22]** which relates the poset $IP(X)$ of all covering maps on a fixed domain X which is a locally compact Hausdorff space with the poset $K(X)$ of all Hausdorff compactifications of X.

Lemma 4.2.4. *Let X be a locally compact Hausdorff space. The function $\psi : IP(\beta X, X) \rightarrow K(X)$ defined by $\psi(f) = \beta X \mid \wp(f)$ is an order isomorphism, where $\beta X \mid \wp(f)$ is the natural compactification of X obtained by collapsing each fiber in $\wp(f)$ to a point.*

We now deduce the following result due to Magill [Theorem 1.3].

Theorem 4.2.5. *Let X and Y be locally compact spaces. Then $K(X)$ and $K(Y)$ are order isomorphic if and only if $\beta X - X$ and $\beta Y - Y$ are homeomorphic.*

Proof. Clearly for locally compact Hausdorff spaces X and Y, if $\beta X - X$ is homeomorphic to $\beta Y - Y$ then $K(X)$ is order isomorphic to $K(Y)$. Therefore it is sufficient to deduce a homeomorphism between $\beta X - X$ and $\beta Y - Y$ if $K(X)$ and $K(Y)$ are given to be order isomorphic. If $K(X)$ and $K(Y)$ are order isomorphic then by Corollary 4.2.3 and Lemma 4.2.4, $DP(\beta X, X)$ and $DP(\beta Y, Y)$ are order isomorphic and hence application of Theorem 4.1.6 gives us a cln-bijection $F : \beta X - X \rightarrow \beta Y - Y$. The bijection F is a closed map because all closed subsets in $\beta X - X$ are nowhere dense. Similarly F^{-1} is also a closed map. Hence F is a homeomorphism.