CHAPTER V

TOPOLOGY ON A FAMILY OF DUALS IN $DP(X)$

In **[29]**, Thrivikraman has constructed homeomorphism between 'the collection F of all subsets of the set of dual atoms in $K(X)$ which are hinged with overlapping dual atoms in $K(X)$' and the underlying locally compact Hausdorff space X by inducing a natural topology on the set F. In section 1 of this chapter, we introduce and study the notion of overlapping duals and also the notion of duals hinged with overlapping duals. In section 2, we topologize collection $\Im$ of all subsets of the set of duals in $DP(X)$ which are hinged with overlapping duals in $DP(X)$ and study when the topological space $\Im$ is homeomorphic to X. In fact, we prove that $\Im$ is homeomorphic to X when X is a countably compact T_3 space without isolated points. To prove this result we introduce the notion of $F-$closed sets and observe that for a locally compact Hausdorff space X, the notion of $F-$closed sets coincides with the notion of $F-$compact sets defined by Thrivikraman in **[29]**. Finally in section 3, we use lattice theoretic properties of the complete lattice $DP(\beta X, X)$ to obtain topological properties of $\beta X - X$ when X is a locally compact Hausdorff space.

1. Set D of duals and its subsets.

For a topological space X, we shall denote the collection of all dual members in $DP(X)$ by D.

Definition 5.1.1. Two distinct duals in D are said to *overlap* if there are precisely three dual members in D greater than their meet.

Example 5.1.2. Consider the usual space **R** of real numbers and let $(f;\{1,2\})$ and $(g;\{1,3\})$ be duals in $DP(\mathbf{R})$. Then by Theorem 3.1.5, $f \wedge g \approx (f \wedge g;\{1,2,3\})$ and the duals greater than $f \wedge g$ in $DP(\mathbf{R})$ are f, g and $(h;\{2,3\})$. Hence f and g are overlapping duals because f, g and h are the only duals in $DP(\mathbf{R})$ greater than $f \wedge g$. If $k \geq f \wedge g$ then by Lemma 2.3.3, $\wp(k) \subseteq \wp(f \wedge g)$.

Definition 5.1.3. An h in D is said to be *hinged* with two overlapping duals f and g in D if h overlaps both f and g and there are precisely six duals in D greater than $f \wedge g \wedge h$.

Example 5.1.4. Consider the usual space **R** of real numbers and let $(f;\{1,2\})$ and $(g;\{1,3\})$ be duals in $DP(\mathbf{R})$. Then the dual $(h;\{1,4\})$ in $DP(\mathbf{R})$ overlaps both f and g. By Theorem 3.1.5, $f \wedge g \wedge h \approx (f \wedge g \wedge h;\{1,2,3,4\})$ and the duals greater than $f \wedge g \wedge h$ in $DP(\mathbf{R})$ are f, g, h, $(\gamma;\{2,3\})$, $(\eta;\{2,4\})$ and $(\psi;\{3,4\})$. Another application of Lemma 2.3.3 justifies that f, g, h, γ, η and ψ are the only duals in $DP(\mathbf{R})$ greater than $f \wedge g \wedge h$. Thus h is hinged with f and g.

Definition 5.1.5. For overlapping duals f and g in D, define the point $|fg|$ to be the set containing f and g along with all those duals in D which are hinged with f and g.

Note. Let X be a topological space and let $(f;\{x,y\})$, $(g;\{w,z\})$ be two overlapping duals in $DP(X)$. Then f and g are overlapping duals implies $\{x,y\} \cap \{w,z\} \neq \varphi$, for otherwise there will be only two duals in $DP(X)$ greater than $f \wedge g$. Since f and g are also distinct duals, we have $\{x,y\} \cap \{w,z\} = \{x = w\}$. Let h be a dual hinged with f and g. Then h overlaps both f and g. By the above discussion and using the fact that there are precisely six duals in $DP(X)$ greater than $f \wedge g \wedge h$, it follows that the non-singleton member of $\wp(h)$ contains x as one of its member. Thus, for f and g as described above, the point $|fg| = \{(h;\{x,y\}) \mid y \in X - \{x\}\}$. Also, if $h,k \in |fg|$, $h \neq k$ then h and k are overlapping duals and $|hk| = |fg|$.

Lemma 5.1.6. *Let X be a topological space and let D be the set of all duals in $DP(X)$. Then for overlapping duals f and g in D, the point $|fg|$ uniquely determines a point of X.*

Proof. Since f and g are overlapping duals in D, we have $f \approx (f;\{a,b\})$ and $g \approx (g;\{a,c\})$ for $a,b,c \in X$, $b \neq c$. By the previous Note, the point $|fg|$ contains all the duals of the form $(h;\{a,d\})$, where $d \in X - \{a\}$. Thus

$$\{a\} = \bigcap_{\substack{h \in |fg|, \\ |h^{-1}(y)|=2}} h^{-1}(y)$$

is the point of X determined by the point $|fg|$.

For a topological space X, we denote by $\Im$, the set of all subsets of D of the form $|fg|$, where f and g are overlapping duals in D.

Definition 5.1.7. An f in D is said to be *determined* by a subset A of $\Im$ if there exist distinct points $|g_1 g_2|$ and $|h_1 h_2|$ in A satisfying $\{f\} = |g_1 g_2| \cap |h_1 h_2|$.

Example 5.1.8. Let X be a topological space. Suppose the points $|g_1 g_2|$ and $|h_1 h_2|$ determine points x and y of X respectively. Then the dual $(f;\{x,y\})$ belongs to $|g_1 g_2|$ and $|h_1 h_2|$ both: Since the dual f overlaps g_1, g_2, h_1, h_2, it follows that there are precisely six duals in $DP(X)$ greater than $f \wedge g_1 \wedge g_2$ and $f \wedge h_1 \wedge h_2$. That $\{f\} = |g_1 g_2| \cap |h_1 h_2|$ follows from the fact that if $g \in |g_1 g_2| \cap |h_1 h_2|$, $g \neq f$, then g has to overlap h_1, h_2, g_1 and g_2, which is not possible.

Definition 5.1.9. A subset A of $\Im$ is said to be $F-$*closed* if $\underset{f \in \lambda}{\wedge} f$ exists and $\lambda = \eta$, where η is the collection of all duals $\geq \underset{f \in \lambda}{\wedge} f$ in $DP(X)$.

Example 5.1.10. Consider the usual space **R** of real numbers and let F be the collection of all duals in D hinged with overlapping duals in DP(**R**). Let $A=\{|fg|, |hk|\}$, where $f\approx(f;\{1,2\})$, $g\approx(g;\{1,3\})$, $h\approx(h;\{4,5\})$, $k\approx(k;\{4,6\})$. Then $A\subseteq F$ is $F-$closed because the points $|fg|$ and $|hk|$ determine the dual $(\psi;\{1,4\})$ in DP(**R**) and hence in this case $\lambda=\{\psi\}$ and $\underset{f\in\lambda}{\wedge} f=\psi$. Note that the collection of all duals $\geq \underset{f\in\lambda}{\wedge} f$ in DP(**R**) is equal to λ.

2. Topology of $\Im$ and the order structure of $DP(X)$.

Let X be a Hausdorff space without isolated points and let $\Im$ and D be as defined in section 1. In this section, we show that there exists a bijection between $\Im$ and X which maps $F-$closed sets in $\Im$ to closed nowhere dense sets in X. We topologize the set $\Im$ and obtain conditions under which $\Im$ is homeomorphic to X. Consequently we derive Theorem 1.9 due to Thrivikraman **[29]** and Theorem 1.3 due to Magill **[17]**. We first prove the following result:

Theorem 5.2.1. *Let X be a Hausdorff space without isolated points and let $\Im$ be the collection of all duals hinged with overlapping duals in $DP(X)$. Then there exists a bijective map from $\Im$ onto X which maps $F-$closed sets in $\Im$ to closed nowhere dense sets in X.*

Proof. Define $\xi:\Im\rightarrow X$ by $\xi(|fg|)=a$, where a in X is the unique point determined by $|fg|$. By Lemma 5.1.6, map ξ is one-one. Further, to observe that the map ξ is onto, for $a \in X$ choose distinct points $b, c \in X - \{a\}$. The duals $(f;\{a, b\})$ and $(g;\{a, c\})$ are overlapping in $DP(X)$ and the point $|fg|$ in $\Im$ satisfies $\xi(|fg|) = a$.

We now prove that ξ maps $F-$closed sets in $\Im$ to closed nowhere dense subsets of X. Let A be a $F-$closed subset of $\Im$. Consider the following cases:

Case (i) If $A = \{\, |fg| \,\}$, where the point $|fg|$ determines point a in X, then A is $F-$closed and by definition of ξ, $\xi(A) = \{a\}$, which is a closed nowhere dense subset of X.

Case (ii) Let A be a non-singleton $F-$closed subset of $\Im$ and let λ be the set of all duals determined by A. Since A is $F-$closed, $\underset{f \in \lambda}{\wedge} f$ exists and the collection η of all duals $\geq \underset{f \in \lambda}{\wedge} f$ is equal to λ. We denote $\underset{f \in \lambda}{\wedge} f$ by h.

We first observe that h is a primary member of $DP(X)$. We assume the contrary. Let $\wp(h)$ contains more than one non-singleton members, say K and H. Choose points $a, b \in K$, $a \neq b$, $c, d \in H$, $c \neq d$ and consider the duals $(f;\{a,b\})$ and $(g;\{c,d\})$. By Lemma 2.3.3, it follows that $h \leq f$ and $h \leq g$. The duals f and g belong to λ as $\eta = \lambda$. Further, $f, g \in \lambda$ implies f and g are determined by the points of A. Hence there exist points $|k_1 k_2|$ and $|k_3 k_4|$ determining f; $|k_5 k_6|$ and $|k_7 k_8|$ determining g. Suppose $|k_1 k_2|$, $|k_3 k_4|$, $|k_5 k_6|$, $|k_7 k_8|$ determine points a, b, c, d in X. The set intersection of points $|k_1 k_2|$ and $|k_5 k_6|$ in A determines a dual equivalent to $(k;\{a,c\})$. Therefore

$k \in \lambda$. This further implies $k \geq h$ and hence we have $\{a,c\} \subseteq H$ or $\{a,c\} \subseteq K$ which contradicts the choice of points a and c. Therefore our assumption that $\wp(h)$ contains more than one non-singleton member is wrong. Hence h is a primary member say $(h;H)$, where H is a closed nowhere dense subset of X. We complete the proof by showing $\xi(A) = H$. Let $x \in \xi(A)$. Then there exists point $|h_1h_2| \in A$ such that $\xi(|h_1h_2|) = x$. Take another point $|h_3h_4| \in A$. Suppose it determines y in X. Since $\{(\sigma;\{x,y\})\} = |h_1h_2| \cap |h_3h_4|$, σ is determined by A. Clearly $\sigma \in \lambda$ and $\sigma \geq h$ which implies $\wp(\sigma) \subseteq \wp(h)$ and hence $x \in H$. This proves that $\xi(A) \subseteq H$. To establish the reverse containment let $z \in H$. Since A is a non-singleton member of $\Im$, $\xi(A) \subseteq H$ and ξ one-one implies that H is a non-singleton subset of X. Choose $w \in H - \{z\}$. Clearly the dual $(\psi;\{z, w\})$ belongs to λ since $\wp(\psi) \subseteq \wp(h)$. Thus there exist points $|l_1l_2|$ and $|l_3l_4|$ in A such that $|l_1l_2| \cap |l_3l_4| = \{\psi\}$, point $|l_1l_2|$ determines point z in X and the point $|l_3l_4|$ determines point w in X. But this gives $|l_1l_2|$ in A such that $\xi(|l_1l_2|) = z$. Hence $\xi(A) \supseteq H$. This proves that ξ is a bijective map, which maps $F-$closed sets in $\Im$ to closed nowhere dense sets in X.

Lemma 5.2.2. *Let the space X and the set $\Im$ be as in Theorem 5.2.1. Then the family $\{A \subseteq \Im \,|\, A$ is $F-$closed$\}$ contains φ and $\Im$ and is also closed under finite union and arbitrary intersections.*

Proof. That φ is $F-$closed follows vacuously and the set $\Im$ is $F-$closed follows because $\lambda = D$ and $\underset{f\in\lambda}{\wedge} f$ exists, which is the constant map.

Next, let A_1 and A_2 be $F-$closed subsets of $\Im$. Then $\underset{f\in\lambda_i}{\wedge} f$ exists, where λ_i is the collection of all duals determined by A_i, $i=1,\ 2$ and $\eta_i = \lambda_i$, $i=1,\ 2$, where η_i is the collection of all duals $\geq \underset{f\in\lambda_i}{\wedge} f$. As observed in Theorem 5.2.1 each of the meets $\underset{f\in\lambda_i}{\wedge} f$, $i=1,\ 2$, is a primary member in $DP(X)$ say $(g_i;H_i)$, $i=1,\ 2$, where H_1, H_2 are closed nowhere dense subsets of X. We denote by λ the collection of all duals determined by $A_1 \cup A_2$. Let $g \approx (g;H_1 \cup H_2)$. We shall show that $\underset{f\in\lambda}{\wedge} f \approx g$.

Since $\wp(f) \subseteq \wp(g)$, $g \leq f$ for each $f \in \lambda_1 \cup \lambda_2$. Further, if f is a dual determined by point $|k_1k_2|$ in A_1 and $|k_3k_4|$ in A_2 then also $g \leq f$ as $\wp(f) \subseteq \wp(g)$. Thus $f \in \lambda$ implies $g \leq f$. This proves that g is a lower bound for the set λ. We now establish that g is the greatest lower bound for the set λ. Let $k \in DP(X)$ be such that

$$g < k < f \text{, for each } f \in \lambda. \qquad (1)$$

Now

$$g < k$$

$$\Rightarrow \quad \wp(k) \subseteq \wp(g)$$

$$\Rightarrow \quad \text{if } K_1 \in \wp(k) \text{ and } |K_1| > 1 \text{, then } K_1 \subseteq H_1 \cup H_2.$$

We first observe that k is a primary member in $DP(X)$. If possible, let K_1, K_2 be non-singleton members of $\wp(k)$. Then $K_1 \cup K_2 \subseteq H_1 \cup H_2$.

Choose distinct points a, $b \in K_1$; c, $d \in K_2$. Then there exist $(f_1;\{a,b\})$, $(f_2;\{c,d\})$ in λ, which imply existence of $|g_1g_2|$, $|g_3g_4|$, $|g_5g_6|$, $|g_7g_8|$ in $A_1 \cup A_2$, determining the points a, b, c, d in X. Therefore $(f_3;\{a,c\}) \in \lambda$ as $\{f_3\} \in |g_1g_2| \cap |g_5g_6|$.

Since $\wp(f_3) \not\subseteq \wp(k)$, $f_3 \not\geq k$ which is a contradiction in view of (1). Therefore our assumption that k is not a primary member is wrong. Let H be the non-singleton member in $\wp(k)$. We now claim that $H = H_1 \cup H_2$. Clearly $H \subseteq H_1 \cup H_2$. If possible, suppose

$$H_1 \cup H_2 \not\subseteq H$$

$\Rightarrow$ there exists $a \in H_1 \cup H_2$ such that $a \notin H$.

Choose $b \in H$. Then the dual $(f_1;\{a,b\}) \in \lambda$, but $f_1 \not\geq k$ which is a contradiction in view of (1). This proves that $H = H_1 \cup H_2$ and hence $g \approx (g; H_1 \cup H_2)$ is the greatest lower bound for the set λ.

We now observe that the arbitrary intersection of $F-$closed sets is $F-$closed set. Let $\{A_\alpha\}_{\alpha\in\mu}$ be a family of $F-$closed sets. Then $(g_\alpha; H_\alpha) \approx \underset{f\in\lambda_\alpha}{\wedge} f$ exists for each $\alpha \in \mu$, where λ_α is the collection of all duals determined by A_α and $\eta_\alpha = \lambda_\alpha$, where η_α is the collection of all duals $\geq g_\alpha$. We now consider the following cases:

Case(i) $\bigcap_{\alpha\in\mu} A_\alpha$ is a singleton. In this case $\bigcap_{\alpha\in\mu} A_\alpha$ is $F-$closed.

Case(ii). $\bigcap_{\alpha \in \mu} A_\alpha$ is not a singleton. Let $A = \bigcap_{\alpha \in \mu} A_\alpha$. Then the set λ of all duals determined by A is contained in λ_α for each $\alpha \in \mu$. Let $g \approx (g;H)$, where $H = \bigcap_{\alpha \in \mu} H_\alpha$. We shall show that $g \approx \wedge_{f \in \lambda} f$. That g is a lower bound for λ follows because $(f;\{a,b\}) \in \lambda$ implies $f \in \lambda_\alpha$ for all $\alpha \in \mu$. But this gives $\{a,b\} \subseteq H_\alpha$ for each $\alpha \in \mu$ and hence $\{a,b\} \subseteq H$. This proves $g \leq f$.

We now show that g is the greatest lower bound of the set λ. Let $k \in DP(X)$ be such that

$$g < k < f \text{, for each } f \in \lambda. \qquad (2)$$

Now $g < k$ implies $\wp(k) \subseteq \wp(g)$. If $K_1 \in \wp(k)$ and $|K_1| > 1$ then we have $K_1 \subseteq H$. As discussed earlier k is a primary member in $DP(X)$. Let K be the non-singleton member in $\wp(k)$. We now claim that $K = H$. If possible, suppose $a \in H$ is such that $a \notin K$. Choose $b \in K$. Then the dual $(f_1;\{a,b\}) \in \lambda$ but $f_1 \ngeq k$, which is a contradiction in view of (2). This proves that $H \subseteq K$. The reverse containment $K \subseteq H$ can be easily proved. Hence $g \approx (g;H)$ is the greatest lower bound for the set λ.

Lemma 5.2.3. *Let X be a Hausdorff space without isolated points. Then the complements of F-closed sets in $\Im$ forms a base for the topology of $\Im$, where $\Im$ is the collection of all duals hinged with the overlapping duals in $DP(X)$.*

Proof. Follows from Lemma 5.2.2.

Recall that a subset A of countably compact T_3 space X without isolated points is closed if and only if whenever $B \subseteq A$ and $Cl_X B$ is nowhere dense in X then $Cl_X B \subseteq A$ **[22]**.

Theorem 5.2.4. *Let X be a countably compact T_3 space without isolated points and let F be the collection of subsets of D consisting of duals hinged with overlapping duals, with the topology described by declaring complements of F-closed sets in $\Im$ as open sets in $\Im$. Then $\Im$ is homeomorphic to X.*

Proof. By Theorem 5.2.1 there exists a bijection $\xi : F \to X$, which maps F-closed sets in $\Im$ to closed nowhere dense sets in X. Since F-closed sets are closed sets in $\Im$, ξ is a closed map. That ξ^{-1} is a closed map follows from the fact that closed nowhere dense subsets of X determine the topology of X. Therefore ξ is a bijective closed map whose inverse is also closed. Hence ξ is a homeomorphism.

Note. By Corollary 4.2.3 and Lemma 4.2.4 we have $DP(\beta X, X)$ is order isomorphic to $K(X)$. Let D and $\Im$ be subsets of $DP(\beta X, X)$ as defined earlier. Then in this case our notion of F-closed sets defined in 5.1.9 coincides with the notion of F-compact sets defined by Thrivikraman in **[29]**. Hence as a corollary to the above result we have the following result due to Thrivikraman **[29]**.

Corollary 5.2.5. *Let X be a locally compact space. Then there is a bijection from $\Im$ to $\beta X - X$ which carries F – compact sets in $\Im$ to compact sets of $\beta X - X$ and vice-versa. Further, the complements of F – compact sets of $\Im$ form a topology for $\Im$ if and only if X is locally compact. In this case $\Im$ is homeomorphic to $\beta X - X$.*

As a consequence of the above result we also obtain the well-known Magill's result [Theorem 1.3] concerning the Stone-Čech remainder of a locally compact Hausdorff space.

3. Lattice $DP(\beta X, X)$ and the remainder $\beta X - X$.

In this section we deduce topological properties of $\beta X - X$ from the lattice theoretic properties of $DP(\beta X, X)$. We recall that for $A \subseteq X$, $DP(X, A) = \{f \in DP(X) \mid |f^{-1}(f(x))| = 1$, for each $x \in A\}$.

Theorem 5.3.1. *Let X be a locally compact space. Then $DP(\beta X, X)$ is distributive if and only if $|\beta X - X| < 3$.*

Proof. Observe that $|\beta X - X| < 3$ if and only if $|DP(\beta X, X)| < 3$. Hence $DP(\beta X, X)$ is distributive if $|\beta X - X| < 3$. Suppose $|\beta X - X| \geq 3$. Then choose distinct points a, b, $c \in \beta X - X$ and consider members $I_{\beta X}$, $(f;\{a,b\})$, $(g;\{b,c\})$, $(h;\{a,c\})$ and $(k;\{a,b,c\})$ in $DP(\beta X, X)$. Clearly

$$(f \vee g) \wedge h \approx h$$

and

$$(f \wedge h) \vee (g \wedge h) \approx k\,.$$

Thus $(f \vee g) \wedge h \neq (f \wedge h) \vee (g \wedge h)$. This proves that $DP(\beta X, X)$ is not distributive.

Theorem 5.3.2. *Let X be a locally compact Hausdorff space. Then $DP(\beta X, X)$ has a minimal element but no atom if and only if $\beta X - X$ is connected.*

Proof. By Corollary 4.2.3 and Lemma 4.2.4, $DP(\beta X, X)$ is order isomorphic to $K(X)$. Therefore $DP(\beta X, X)$ has a minimal element if and only if X is locally compact. Further X is locally compact if and only if $\beta X - X$ is compact. Hence $DP(\beta X, X)$ has a minimal element if and only if $\beta X - X$ is compact.

We now complete the proof by establishing $DP(\beta X, X)$ has an atom if and only if $\beta X - X$ is disconnected. Let f be an atom in $DP(\beta X, X)$. Then observe that for such an f, $\wp(f)$ contains precisely two non-singleton members say H_1 and H_2 such that their union is $\beta X - X$. For, if $\wp(f)$ contains more than two non-singleton members, say H_1, H_2 and H_3 then the map g such that $\wp(g)$ contains $H_1 \cup H_2$ and H_3 belongs to $DP(\beta X, X)$ and $g < f$ as $\wp(f) \subseteq \wp(g)$. This contradicts that f is an atom. Therefore if $DP(\beta X, X)$ has an atom f then $\wp(f)$ contains exactly two non-singleton members say H_1 and H_2 such that $\beta X - X = H_1 \cup H_2$. Clearly H_1 and H_2 are clopen sets making $\beta X - X$ disconnected.

Conversely, suppose $\beta X - X$ is disconnected. Then there exist non-empty disjoint clopen sets H_1 and H_2 in $\beta X - X$ such that $\beta X - X = H_1 \cup H_2$. The natural quotient map q obtained by identifying H_1 and H_2 to distinct points, is an atom in $DP(\beta X, X)$.

Theorem 5.3.3. *Let X be a locally compact Hausdorff space. If $DP(\beta X, X)$ is complemented then $\beta X - X$ is totally disconnected.*

Proof. Let $x, y \in \beta X - X$, $x \neq y$. Then consider the dual member $(f;\{x,y\})$ in $DP(\beta X, X)$. Since $DP(\beta X, X)$ is complemented, there exists g in $DP(\beta X, X)$ such that $f \wedge g \approx \omega$ and $f \vee g \approx I_{\beta X}$, where ω is the minimal element in $DP(\beta X, X)$. Since $f \wedge g \approx \omega$, $\wp(g)$ can contain atmost two non-empty members. Further, $f \vee g \approx I_{\beta X}$ implies that $\wp(g)$ contains exactly two non-empty members say H and K such that $x \in H$ and $y \in K$. Since H and K are the only members of $\wp(g)$, we have $\beta X - X = H \cup K$. Therefore for each pair of distinct points x and y in $\beta X - X$, there exist disjoint closed sets H and K such that $x \in H$, $y \in K$ and $\beta X - X = H \cup K$. Hence $\beta X - X$ is totally disconnected.

We recall that a lattice L is *modular* if $a \leq c \Rightarrow (a \vee b) \wedge c = a \vee (b \wedge c)$, where $a, b, c \in L$.

Theorem 5.3.4. *Let X be a locally compact Hausdorff space. Then $DP(\beta X, X)$ is modular if and only if $|\beta X - X| < 4$.*

Proof. It is easy to observe that if $|\beta X - X| \leq 3$ then $DP(\beta X, X)$ is modular. Suppose $|\beta X - X| \geq 4$. Choose distinct points a, b, c and d in $\beta X - X$ and consider members $I_{\beta X}$, $(f;\{a,b\})$, $(g;\{a,b,c\})$, $(h;\{c,d\})$ and $(\omega;\{a,b,c,d\})$ in $DP(\beta X, X)$. Observe that

$$(g \vee h) \wedge f \approx f$$

and

$$g \vee (f \wedge h) \approx g$$

That $DP(\beta X, X)$ is not modular follows from the facts that $g \leq f$ and $(g \vee h) \wedge f \neq g \vee (h \wedge f)$.