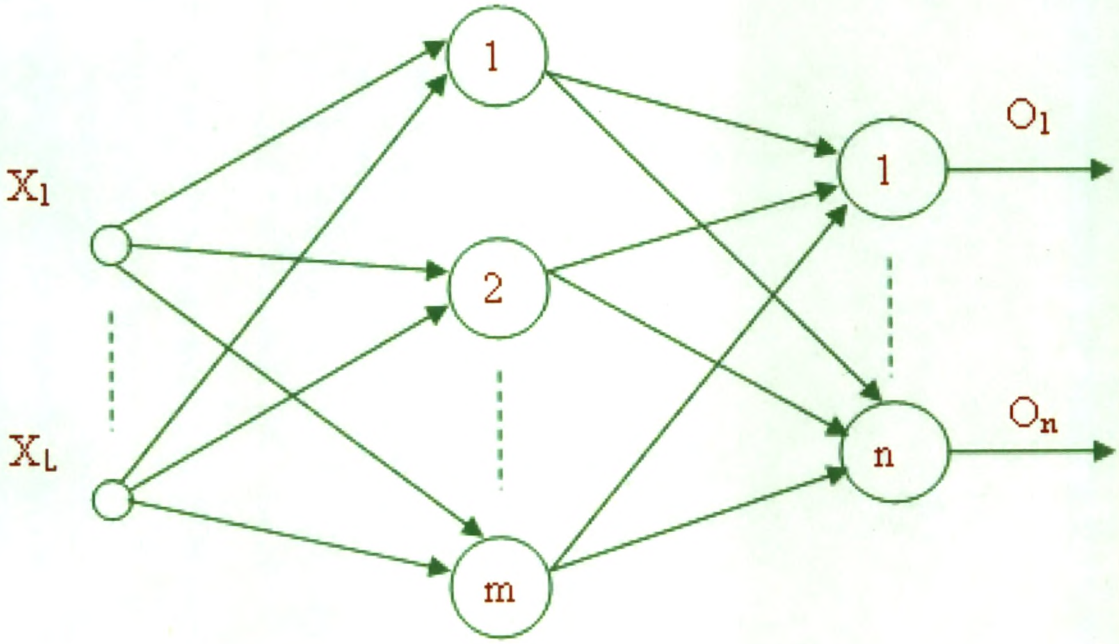
Chapter 4

ANN Based Steering Control of Semilinear Continuous Time System



Chapter 4

ANN BASED STEERING CONTROL OF SEMILINEAR CONTINUOUS TIME SYSTEM

4.1 Introduction

In this chapter, we develop the steering control for the continuous semilinear systems and implement it using Artificial Neural Network. To justify the need for such ANN steering control, we take up the first problem, form Chemical Industry. We start with the modeling of mixing tank process, which is a subprocess for many chemical processes.

The mixing tank subprocess is mathematically modeled into a continuous time-invariant semilinear dynamical system. For this model we first develop a local controller, as traditionally done in case of nonlinear systems and demonstrate its implementation using a multilayered feed-forward NN.

In the following sections we derive the steering control for a general semilinear dynamical system and apply it to the mixtank problem for the controllability. The simulation results for the ANN controller are provided in the last section.

4.2 Mathematical Modeling of Mixtank Process

Consider the process of synthesis of Ethyl Acetate. It has three major sub-processes:

- mixtank: In this subprocess the fresh and recycled Acetic Acid is mixed thoroughly and outputted.
- reactor: Here the ethanol is mixed with Acetic Acid in the ratio 2:3 which produces Ethyl Acetate.
- separator: It separates the produced Ethyl Acetate and the unused Acetic Acid to be recycled.

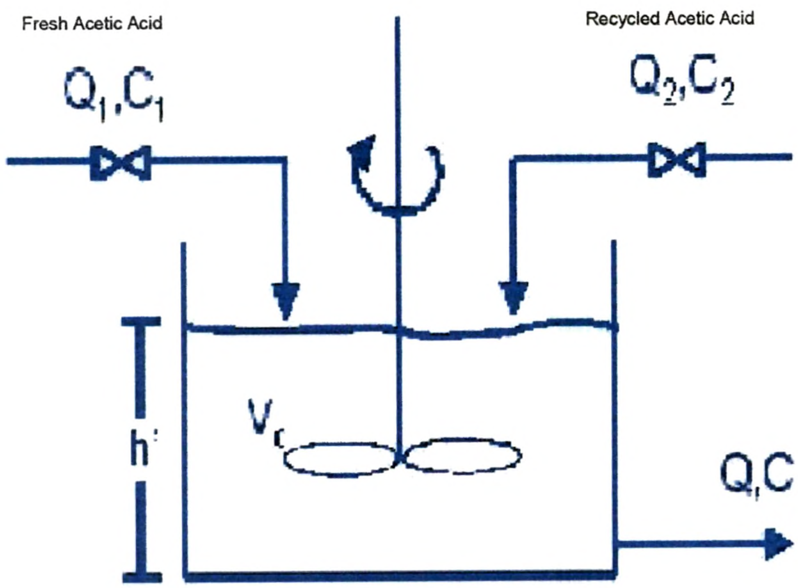


Figure 4.1: A mixtank sub-process

To produce Ethyl Acetate, Ethanol and Acetic Acid are required in 1:1 mole proportion but, practically it is mixed in the ratio 2:3 hence, some portion of Acetic Acid remains unused which is recycled into the process as shown in the Figure 4.1.

Assume that the fresh Acetic Acid and recycled Acetic Acid are fed in the mixing tank with the concentration C_1, C_2 and at the rate of $Q_1(t)$ and $Q_2(t)$, respectively, which are continuously mixed by the stirring rod. The outflow from the mixing tank is at the rate $Q(t)$ with the concentration $C(t)$. It is assumed, that stirring causes perfect mixing so that the concentration of the solution (Acetic Acid) in the tank is uniform throughout and is same as that of the flow coming out of the tank. Also, it is assumed that the density of the solution in the tank remains constant. Let $V(t)$ be the volume of Acetic Acid in the tank at time t, which is assumed to be nonzero for all t.

Now, considering the dynamics of the phenomenon, observed from time $t = t_0$, the mass balance and mole balance equations are:

$$\frac{dV(t)}{dt} = Q_1(t) + Q_2(t) - Q(t)$$

$$\frac{d(C(t)V(t))}{dt} = C_1Q_1(t) + C_2Q_2(t) - C(t)Q(t) \tag{4.2.1}$$

The outflow $Q(t)$, is characterized by the turbulent flow relation

$$Q(t) = k\sqrt{h(t)} = k\sqrt{\frac{V(t)}{A_c}} \tag{4.2.2}$$

where, $h(t)$ is the head of the liquid in the tank, A_c is the cross sectional area of the tank and k is a constant.

Putting (4.2.2) in (4.2.1), the system becomes

$$\frac{dV(t)}{dt} = Q_1(t) + Q_2(t) - k\sqrt{\frac{V(t)}{A_c}}$$

$$\frac{dC(t)}{dt} = [C_1 - C(t)]\frac{Q_1(t)}{V(t)} + [C_2 - C(t)]\frac{Q_2(t)}{V(t)} \tag{4.2.3}$$

i.e.

$$\begin{bmatrix} \dot{V}(t) \\ \dot{C}(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix} + \begin{bmatrix} -k\sqrt{\frac{V(t)}{A_c}} \\ \frac{(C_1-C(t))Q_1(t)}{V(t)} + \frac{(C_2-C(t))Q_2(t)}{V(t)} \end{bmatrix} \tag{4.2.4}$$

The above equation can be put in the standard form:

$$\dot{x}(t) = Bu(t) + f(x(t), u(t)) \tag{4.2.5}$$

$$x(t_0) = x_0$$

where,

$$x(t) = \begin{bmatrix} V(t) \\ C(t) \end{bmatrix}, \; u(t) = \begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix}, \; B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix};$$

and the nonlinear function $f(x(t), u(t))$ is given by

$$f(x(t), u(t)) = f\left(\begin{bmatrix} V(t) \\ C(t) \end{bmatrix}, \begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix}\right) = \begin{bmatrix} -k\sqrt{\frac{V(t)}{A_c}} \\ \frac{(C_1 - C(t))Q_1(t)}{V(t)} + \frac{(C_2 - C(t))Q_2(t)}{V(t)} \end{bmatrix} \quad (4.2.6)$$

The equation (4.2.5) represents the dynamics of state for the phenomenon. It is the continuous time invariant semilinear dynamical system. Here, we are concentrating only on the state equation, whereas, in general, the system may also have the associated output equation.

The nonlinear function given by (4.2.6) is Lipschitz in the neighborhood of equilibrium (x_0, u_0). This can be shown as below:

Expansion of the nonlinear function f using the Taylor's series expansion up to the linear term is given by

$$f(x + \delta x, u + \delta u) - f(x, u) = \delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial u}\delta u$$

That is,

$$\delta f = \begin{bmatrix} \frac{-k}{\sqrt{A_c}}\left(\frac{1}{2\sqrt{V(t)}}\right) & 0 \\ -\frac{(C_1 - C(t))Q_1(t)}{V^2(t)} - \frac{(C_2 - C(t))Q_2(t)}{V^2(t)} & -\frac{Q_1(t) + Q_2(t)}{V(t)} \end{bmatrix} \delta x + \begin{bmatrix} 0 & 0 \\ \frac{C_1 - C(t)}{V(t)} & \frac{C_2 - C(t)}{V(t)} \end{bmatrix} \delta u$$

The value of δf, at some $x(t) = x = (V, C)$ and $u(t) = u = (Q_1, Q_2)$, is given by

$$\delta f = \begin{bmatrix} \frac{-k}{\sqrt{A_c}}\left(\frac{1}{2\sqrt{V}}\right) & 0 \\ -\frac{(C_1 - C)Q_1}{V^2} - \frac{(C_2 - C)Q_2}{V^2} & -\frac{Q_1 + Q_2}{V} \end{bmatrix} \delta x + \begin{bmatrix} 0 & 0 \\ \frac{C_1 - C}{V} & \frac{C_2 - C}{V} \end{bmatrix} \delta u$$

Let

$$d_a = \begin{bmatrix} \frac{-k}{\sqrt{A_c}}\left(\frac{1}{2\sqrt{V}}\right) & 0 \\ -\frac{(C_1-C)Q_1}{V^2} - \frac{(C_2-C)Q_2}{V^2} & -\frac{Q_1+Q_2}{V} \end{bmatrix}$$

and

$$d_b = \begin{bmatrix} 0 & 0 \\ \frac{C_1-C}{V} & \frac{C_2-C}{V} \end{bmatrix}.$$

Now,

$$\| d_a \| = \left(\left(\frac{-k}{\sqrt{A_c}} \left(\frac{1}{2\sqrt{V}} \right) \right)^2 + \left(-\frac{(C_1 - C)Q_1}{V^2} - \frac{(C_2 - C)Q_2}{V^2} \right)^2 + \left(-\frac{Q_1 + Q_2}{V} \right) \right)^{\frac{1}{2}}$$

and

$$\| d_b \| = \left(\left(\frac{C_1 - C}{V} \right)^2 + \left(\frac{C_2 - C}{V} \right)^2 \right)^{\frac{1}{2}}.$$

Since, $\| d_a \|$ and $\| d_b \|$ are finite for $V \neq 0$, δf is bounded. By our assumption, in the mixing tank process V(t) is nonzero for all time t, therefore bounded δf implies that the nonlinear function f in the system (4.2.5) is Lipschitz continuous w.r.t. both the arguments.

The equation (4.2.5) represents the dynamics of state for the phenomenon. It is a continuous time-invarient semilinear dynamical system. Here, we are concentrating only on the state equation, where as in general, the system may also have the associated output equation.

When the volume and the concentration are at equilibrium state, the system runs in the stable mode. If the equilibrium state is slightly perturbed we want to change the inflow rates Q_1, Q_2 so as to bring the state to the equilibrium. If this is possible, we say that the system is controllable to the equilibrium state. Since, the system (4.2.5) is highly nonlinear, we first linearize it about the equilibrium (x_0, u_0) and investigate the controllability of the linearized system.

4.3 Controllability of Linearized Model

The linearization of (4.2.5) at the equilibrium point (x_0, u_0) is given by

$$\dot{x}(t) = ax(t) + (B + b)u(t) \tag{4.3.7}$$

where, $a = \frac{\partial f}{\partial x}\Big|_{(x_0,u_0)}$ $b = \frac{\partial f}{\partial u}\Big|_{(x_0,u_0)}$.

In our case, for system (4.2.5),

$$\frac{\partial f}{\partial x}\Big|_{(x_0,u_0)} = \begin{bmatrix} \frac{-k}{\sqrt{A_c}}\left(\frac{1}{2\sqrt{V(t)}}\right) & 0 \\ -\frac{(C_1-C(t))Q_1(t)}{V^2(t)} - \frac{(C_2-C(t))Q_2(t)}{V^2(t)} & -\frac{Q_1(t)+Q_2(t)}{V(t)} \end{bmatrix}_{(x_0,u_0)}$$

Let $a_{11} = \frac{-k}{\sqrt{A_c}}\left(\frac{1}{2\sqrt{V}}\right)$, then at equilibrium $-\frac{Q_1+Q_2}{V} = 2a_{11}$ and $-\frac{(C_1-C)Q_1}{V^2} - \frac{(C_2-C)Q_2}{V^2} = 0$.

Hence,

$$\frac{\partial f}{\partial x}\Big|_{(x_0,u_0)} = \begin{bmatrix} a_{11} & 0 \\ 0 & 2a_{11} \end{bmatrix}.$$

Also,

$$\frac{\partial f}{\partial u}\Big|_{(x_0,u_0)} = \begin{bmatrix} 0 & 0 \\ \frac{C_1-C(t)}{V(t)} & \frac{C_2-C(t)}{V(t)} \end{bmatrix}_{(x_0,u_0)} = \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \end{bmatrix}$$

where, $b1 = \frac{C_1-C}{V}$ and $b2 = \frac{C_2-C}{V}$.

Define,

$$A_L = a = \begin{bmatrix} a_{11} & 0 \\ 0 & 2a_{11} \end{bmatrix}$$

and

$$B_L = B + b = \begin{bmatrix} 1 & 1 \\ b_1 & b_2 \end{bmatrix}.$$

Thus, the linearized system is of the form

$$\dot{x}(t) = A_L x(t) + B_L u(t) \tag{4.3.8}$$
$$x(t_0) = x_0 = \begin{bmatrix} V_0 \\ C_0 \end{bmatrix}$$

The controllability matrix for the linearized system (4.3.8) is given by:

$$U_L = [B_L|A_L B_L] = \begin{bmatrix} 1 & 1 & a_{11} & a_{11} \\ b_1 & b_2 & 2a_{11}b_1 & 2a_{11}b_2 \end{bmatrix}$$

The above given controllability matrix is of full rank when C_1 and C_2 are different. That is, when the concentrations of the inflows are different. Hence, the linearized system is controllable.

Let $\Phi_L(t-t_0)$ be the state transition matrix of the system (4.3.8) given by

$$\Phi_L(t-t_0) = e^{A_L(t-t_0)}$$

Then the solution of the system (4.3.8) is given by

$$x(t) = \Phi_L(t-t_0)x_0 + \int_{t_0}^{t} \Phi_L(t-\tau)B_L u(\tau)d\tau.$$

The controllability of linearized system indicates the local controllability for our semi-linear system (4.2.5) (refer Kalamka [40]).

Numerical Experiment :

We consider the following values for the parameters in our model:

$k = 1$
$A_c = 4$
$Q_{10} = 10$ liters/sec
$Q_{20} = 20$ liters/sec
$C_1 = 9$ g-moles/litre
$C_2 = 18$ g-moles/litre

where, Q_{10}, Q_{20} are the inflow rates at time t_0, C_1 and C_2 are the respective concentrations at time t_0 and V_0 is the volume of the mixture in the tank at time t_0.

For these values of parameters, the matrices A_L and B_L defined in (4.3.8) are computed as

$$A_L = a = \begin{bmatrix} \frac{-1}{\sqrt{4}}\frac{1}{2\sqrt{3600}} & 0 \\ -\frac{(9-15)10}{3600^2} - \frac{(18-15)20}{3600^2} & -\frac{10+20}{3600} \end{bmatrix} = \begin{bmatrix} -0.0042 & 0 \\ 0 & -0.0083 \end{bmatrix}$$

and

$$B_L = B + b = \begin{bmatrix} 1.0000 & 1.0000 \\ -0.0017 & 0.0008 \end{bmatrix}$$

Thus, the linearized system (4.3.8) becomes,

$$\dot{x}(t) = \begin{bmatrix} -0.0042 & 0 \\ 0 & -0.0083 \end{bmatrix} x(t) + \begin{bmatrix} 1.0000 & 1.0000 \\ -0.0017 & 0.0008 \end{bmatrix} u(t) \tag{4.3.9}$$

The system represented by equation (4.3.9) is linear time invariant continuous dynamical system in R^2. The equilibrium state for (4.3.9) is (3600, 15).

The optimal control for the system (4.3.9) (refer chapter 2) is given by

$$u(t) = -B_L^* e^{A_L^*(T-t)} W^{-1}(t_0, T)(x_0 - e^{A_L(t_0-T)} x(T)) \tag{4.3.10}$$

where, the controllability Grammian $W(t_0, T)$ for the system (4.3.9) is given by

$$W(t_0, T) = \int_{t_0}^{T} e^{A_L(t_0-\tau)} B_L B_L^* e^{A_L^*(t_0-\tau)} d\tau$$

The controller given by (4.3.10) steers the initial state x_0 to the desired final state x_f in finite time T. The steered state for all time $t \in [0, T]$ is given by

$$x(t) = e^{A_L(t-t_0)} x_0 + \int_{t_0}^{t} e^{A_L(t-\tau)} B u(\tau) d\tau$$

Using these definitions for the state $x(t)$ and the steering control $u(t)$, we proceed to implement the controller using Artificial Neural Networks for the system (4.3.9).

Neural Network Controller :

In general, the following steps are used for computation of the controller using Neural Networks:

1. First, generate the *input-output* pairs. Input : [Initial state (perturbed state), final state (Equilibrium state)]; Output : Optimal steering control, using the theoretical definition of the optimal controller given by equation (4.3.10).

2. Define suitable multiple layer Neural Network NN_u, which will act as controller.

3. Randomly select 80 % of the data for the training of the network NN_u. The training of the Neural Network is done using the Back-propagation algorithm

4. Test the remaining 20 % of the data by comparing the output generated by NN_u with the computed desired output u, if the error is large goto step (2) with the changes in architecture.

5. Once the Neural Network is trained with the acceptable error. It generalizes very efficiently, i.e. for any given small deviated state the Neural Network controller steers the state to the desired equilibrium state in few steps.

The Multi layered Neural Network as Steering Control for the Mixtank process

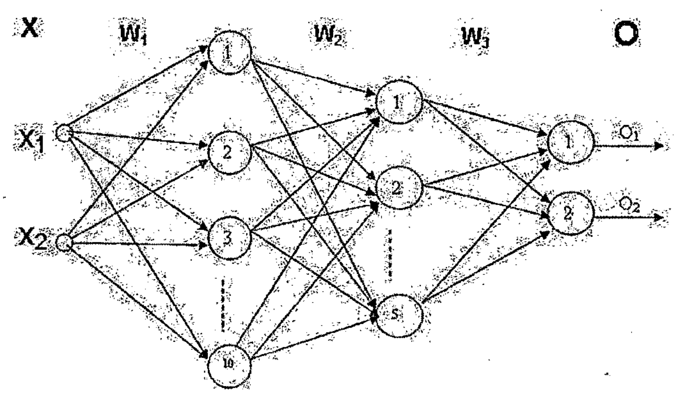


Figure 4.2: The ANN steering control with two hidden layers h_1 and h_2 having 10 and 5 nodes respectively and an output layer O with 2 nodes.

For our system (4.3.9) we designed the ANN steering control NN_u using the architecture $NN^3_{2,10,5,2}$ as shown in the Figure 4.2. The training patterns are generated by randomly varying the state components in the pre-specified range, volume $V(t)$ between $(3550, 3650)$ and concentration $C(t)$ between $(10, 20)$. Remember that the equilibrium is $(3600, 15)$. The perturbed initial states lie in the neighborhood of the equilibrium.

Since, presently the system is in the continuous form to verify the performance of control signal produced by the Neural Network we design another Neural Network NN_x with the architecture $NN^3_{2,10,5,2}$ to depict the state dynamics.

NN_x is trained using the input as (Initial perturbed state ,Control) and output as the final state. Once both the networks are trained, they are placed in series. We give initial perturbed state to NN_u which produces the steering control signal which in turn is given as input to NN_x to produce the final state as equilibrium state.

This simulation strengthens the following facts:

1. The control and state are inversely related dynamics and they can be placed in a closed loop form for implementing the adaptive controller in the automated plant.
2. The Neural Networks can learn from the I/O pairs and thus, the mathematical modeling of the phenomenon can be avoided.
3. Observe that the computation of the control signal using formula requires computation of the inverse of Grammian, which is itself a costly affair in terms of time. The Neural Network controller would require spending such this time only once, that is, while training the network.

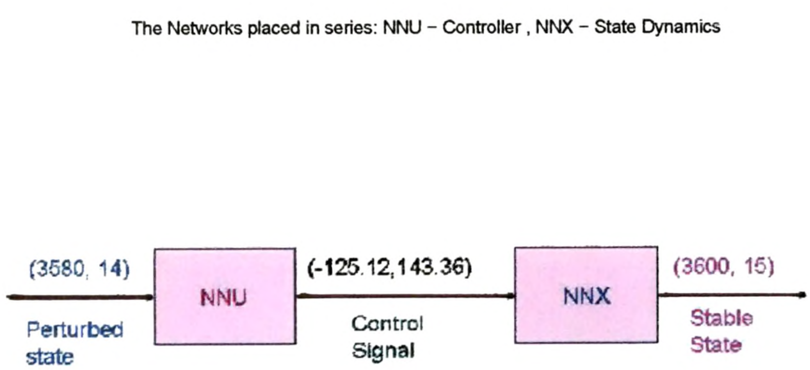


Figure 4.3: The ANN steering control and state network placed in series to validate the ANN control signal.

The working of the NNs in series is as shown in the Figure 4.3 for the initial perturbed state (3580, 14). For details see Program nnctrlc.m in Appendix-A. In the following section, we derive the steering control for the semilinear system (4.2.5).

4.4 Controller for continuous Semilinear Dynamical System

Now, we consider a general time-invariant semilinear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t) + f(x(t), u(t))$$

$$x(0) = x_0 \tag{4.4.11}$$

where, $A_{n\times n}$ is the evolution matrix, $B_{n\times m}$ is called the control matrix. The state $x(t) \in R^n$, $u(t) \in R^m$ is the control input to the system and $f : R^n \times R^m \to R^n$ is the nonlinear function dependent on state as well as control. For the system (4.4.11) we investigate the global controllability of the system and use Neural Network to implement the steering control.

The solution of (4.4.11) can be given as

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + \int_0^t e^{A(t-\tau)}f(x(\tau), u(\tau))d\tau \tag{4.4.12}$$

where, e^{At} is the state transition matrix for the linear part of the system. We first prove the existence and uniqueness of solution of (4.4.12).

LEMMA 4.4.1 *If the nonlinear function f in the system (4.4.11) satisfies Lipshitz condition*

$$\| f(x_1) - f(x_2) \| \leq \alpha_x \| x_1 - x_2 \| \quad \forall x_1, x_2 \in R^n, u \in R^m$$

then, for each fixed $u \in L^2([0,T]; R^m$ there exists unique solution given by (4.4.12) for the system (4.4.11).

Proof: Let the solution of the semilinear system (4.4.11) be as given before,

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + \int_0^t e^{A(t-\tau)}f(x(\tau), u(\tau))d\tau \tag{4.4.13}$$

Define an operator $Q : C([0,T]; R^n) \to C([0,T]; R^n)$ as

$$(Qx)(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + \int_0^t e^{A(t-\tau)}f(x(\tau), u(\tau))d\tau \tag{4.4.14}$$

Thus, for fixed u we get

$$\begin{aligned}\| (Qx_1)(t) - (Qx_2)(t) \| &= \| \int_0^t e^{A(t-\tau)} \{f(x_1(\tau), u(\tau)) - f(x_2(\tau), u(\tau))\} \, d\tau \| \\ &\leq \int_0^t \| e^{A(t-\tau)} \| \| f(x_1(\tau), u(\tau)) - f(x_2(\tau), u(\tau)) \| \, d\tau \\ &\leq \alpha_A \int_0^t \| f(x_1(\tau), u(\tau)) - f(x_2(\tau), u(\tau)) \| \, d\tau\end{aligned}$$

where,

$$\alpha_A = \max_{t \in [0,T]} \| e^{At} \|$$

Thus,

$$\begin{aligned}\| (Qx_1)(t) - (Qx_2)(t) \| &\leq \alpha_A \alpha_x \int_0^t \| x_1(\tau) - x_1(\tau) \|_{R^n} \, d\tau \\ &\leq \alpha_A \alpha_x t \| x_1 - x_1 \|_{C([0,T];R^n)}\end{aligned}$$

Hence,

$$\begin{aligned}\| (Qx_1) - (Qx_2) \|_{C([0,T];R^n)} &= \sup_{t \in [0,T]} \| (Qx_1)(t) - (Qx_2)(t) \|_{R^n} \\ &\leq \alpha_A \alpha_x T \| x_1 - x_1 \|_{C([0,T];R^n)}\end{aligned}$$

Similarly,

$$\begin{aligned}\| (Q^2 x_1 - Q^2 x_2)(t) \| &= \| Q(Qx_1)(t) - Q(Qx_2)(t) \|_{R^n} \\ &\leq \alpha_A \alpha_x \int_0^t \| (Qx_1)(\tau) - (Qx_2)(\tau) \| \, d\tau \\ &\leq \alpha_A^2 \alpha_x^2 \int_0^t \tau \| x_1 - x_2 \|_{C([0,T];R^n)} \, d\tau \\ &\leq \alpha_A^2 \alpha_x^2 \frac{T^2}{2!} \| x_1 - x_2 \|_{C([0,T];R^n)}\end{aligned}$$

$$\| (Q^3 x_1 - Q^2 x_3)(t) \| \leq \alpha_A^3 \alpha_x^3 \frac{T^3}{3!} \| x_1 - x_2 \|_{C([0,T];R^n)}$$

In general,

$$\| (Q^r x_1 - Q^r x_3)(t) \| \leq \frac{(\alpha_A \alpha_x T)^r}{r!} \| x_1 - x_2 \|_{C([0,T];R^n)} .$$

As r tends to infinity the quantity $\frac{(\alpha_A \alpha_x T)^r}{r!}$ tends to zero. That is, there exists some r such that the Lipschitz constant $\frac{(\alpha_A \alpha_x T)^r}{r!}$ of Q^r is less than 1 and hence Q^r is a contraction mapping for some $r \geq 1$.

Therefore, by the generalized Banach contraction principle there exists a unique solution to the system (4.4.11).

We want to prove the existence of steering controller for the system (4.4.11). We say that the system (4.4.11), is controllable if there exists the control function $u(t)$ defined in $L^2([0,T];R^m)$ such that for any given initial state, x_0 at time $t = 0$ and any desired final state x_f, the solution of the system (4.4.12) satisfying $x(0) = x_0$ also satisfies $x(T) = x_f$, with this $u(t)$.

We assume that the linear system is controllable and hence the Grammian matrix $W(0,T)$ is invertible. Therefore, we define a control given by

$$u(t) = B^* e^{A^*(T-t)} W^{-1}(0,T) \left(x_f - e^{AT} x_0 - \int_0^T e^{A(T-s)} f(x(s), u(s)) ds \right) \quad (4.4.15)$$

Substituting this control in (4.4.11), it follows easily that $x(t_0) = x_0$ and $x(T) = x_f$.

From Lemma 4.4.1 for every $u \in L^2([0,T];R^m)$ there exists a unique solution x for (4.4.12).

We define an operator

$$P : L^2([0,T];R^m) \to C([0,T];R^m)$$

given by

$$(Pu)(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau + \int_0^t e^{A(t-\tau)} f(x(\tau), u(\tau)) d\tau. \quad (4.4.16)$$

The operator P assigns the unique solution x, corresponding to the control $u \in L^2([0,T];R^m)$. We now prove that the operator P is Lipschitz.

LEMMA 4.4.2 *Suppose that the nonlinear function f satisfies the Lipschitz condition*

$$\| f(x_1, u_1) - f(x_2, u_2) \| \leq \alpha_x \| x_1 - x_2 \| + \alpha_u \| u_1 - u_2 \| \quad (4.4.17)$$

then, the operator P defined in (4.4.16) satisfies the following condition

$$\| Pu_1 - Pu_2 \| \leq \alpha_p \| u_1 - u_2 \|_{L^2}$$

where, α_p is a constant defined in the proof.

Proof: We consider the operator P as given before

$$(Pu)(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + \int_0^t e^{A(t-\tau)}f(x(\tau), u(\tau))d\tau$$

Let u_1, u_2 be two control functions and x_1, x_2 be its corresponding unique solutions of (4.4.11).

Then, we have $Pu_1 = x_1$ and $Pu_2 = x_2$.

Therefore, $\| (Pu_1)(t) - (Pu_2)(t) \| = \| x_1(t) - x_2(t) \|$

$$= \| \int_0^t e^{A(t-\tau)}B(u_1(\tau) - u_2(\tau)) + \int_0^t e^{A(t-\tau)} \{f(x_1(\tau), u_1(\tau)) - f(x_2(\tau), u_2(\tau))\} d\tau \|$$

$$\leq \int_0^t \| e^{A(t-\tau)} \| \| B \| \| u_1(\tau) - u_2(\tau) \| + \int_0^t \| e^{A(t-\tau)} \| \| f(x_1(\tau), u_1(\tau)) - f(x_2(\tau), u_2(\tau)) \| d\tau$$

$$\leq \alpha_A b \int_0^t \| u_1(\tau) - u_2(\tau) \| + \alpha_A \int_0^t \| f(x_1(\tau), u_1(\tau)) - f(x_2(\tau), u_2(\tau)) \| d\tau$$

where,

$$\alpha_A = \max_{t\in[0,T]} e^{At} \text{ and } b = \| B \| .$$

Therefore,

$$\| x_1(t) - x_2(t) \| \leq (\alpha_A b + \alpha_u) \int_0^t \| u_1(\tau) - u_2(\tau) \| + \alpha_A \alpha_x \int_0^t \| x_1(\tau) - x_2(\tau) \| d\tau$$

Using Gronwal's inequality we get,

$$\| (x_1)(t) - (x_2)(t) \| \leq e^{\alpha_A \alpha_x T}(\alpha_A b + \alpha_u) \int_0^t \| u_1(\tau) - u_2(\tau) \| d\tau \qquad (4.4.18)$$

Using Cauchy Schwartz inequality we get,

$$\begin{aligned} \int_0^t \| u_1(\tau) - u_2(\tau) \| d\tau &\leq \left(\int_0^T 1 d\tau\right)^2 \left(\int_0^T \| u_1(\tau) - u_2(\tau) \| d\tau \right)^2 \\ &\leq \sqrt{(T)} \| u_1 - u_2 \| \end{aligned}$$

Therefore, putting it in (4.4.18) we get,

$$\| x_1(t) - x_2(t) \| \leq e^{\alpha_A \alpha_x T}(\alpha_A b + \alpha_u)\sqrt{(T)} \| u_1 - u_2 \|_{L^2}$$

$$\sup_t \| (x_1)(t) - (x_2)(t) \| \leq e^{\alpha_A \alpha_x T}(\alpha_A b + \alpha_u)\sqrt{(T)} \| u_1 - u_2 \|_{L^2}$$

Thus,

$$\| Pu_1 - Pu_2 \| \leq e^{\alpha_A \alpha_x T}(\alpha_A b + \alpha_u)\sqrt{(T)} \| u_1 - u_2 \|_{L^2} \tag{4.4.19}$$

Putting, $\alpha_p = e^{\alpha_A \alpha_x T}(\alpha_A b + \alpha_u)\sqrt{(T)}$ we get

$$\| Pu_1 - Pu_2 \| \leq \alpha_p \| u_1 - u_2 \|_{L^2}$$

Thus, P is an operator from $L^2([0,T];R^m)$ to $C([0,T];R^m)$ satisfying the condition given by equation (4.4.19).

Let $X = L^2([0,T];R^m)$ and $U = L^2([0,T];R^n)$.

Define the following operators

1. $K_1 : X \to X$ as $(K_1x)(t) = \int_0^t e^{A(t-\tau)}x(\tau)d\tau$
2. $N : U \to U$ as $(Nu)(t) = f(x(t), u(t)) = f((Pu)(t), u(t))$
3. $K_2 : U \to X$ as $(K_2u)(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$
4. $L_c : U \to X$ as $(L_cu)(t) = B^*e^{A^*(T-t)}W^{-1}(0,T)$
5. $K_3 : X \to X$ as $(K_1x)(t) = \int_0^T e^{A(t-\tau)}x(\tau)d\tau$

The operators K_1, K_2, K_3 and L_c are bounded linear operators and N is the nonlinear Nymytski operator (refer Joshi and Bose [35]).

For $t \in [0,T]$, the bounds of the operators can be found as:

$\| K_1 \| \leq \alpha_A T \| x \| = k_1 \| x \|$ where, $k_1 = \alpha_A T$.

$\| K_2 \| \leq \alpha_A T b \| x \| = k_2 \| x \|$ where, $k_2 = k_1 b$.

$\| L_c \| \leq b\alpha_A w = \alpha_l$ (say) where, $w = \| W^{-1} \|$ and $\| B \| = \| B^* \|$, $\| e^{AT} \| = \| e^{A^*T} \|$.

$$\| K_3 \| \leq \alpha_A T \| x \| = k_1 \| x \| .$$

The operator N is Lipschitz as shown in the following:

$$\begin{aligned}\| (Nu_1)(t) - (Nu_2)(t) \| &= \| f((Pu_1)(t), u_1(t)) - f((Pu_2)(t), u_2(t)) \| \\ &\leq \alpha_x \| Pu_1 - Pu_2 \| + \alpha_u \| u_1 - u_2 \| \\ &\leq \alpha_p \alpha_x \| u_1 - u_2 \| + \alpha_u \| u_1 - u_2 \| \} \\ &\leq \alpha_N \| u_1 - u_2 \|\end{aligned}$$

where, $\alpha_N = \alpha_p \alpha_x + \alpha_u$.

With the operators defined by (1), (2), (3), (4) and (5) the pair (x, u) given by equations (4.4.12) and (4.4.15) can be written as follows:

$$x(t) = e^{At} x_0 + (K_2 u)(t) + (K_1 N u)(t) \tag{4.4.20}$$

$$u(t) = L_c(x_f - e^{AT} x_0)(t) - (L_c K_3 N u)(t) \tag{4.4.21}$$

Define an operator $M : U \to U$ as,

$$Mu = L_c(x_f - e^{AT} x_0) - L_c K_3 N u \tag{4.4.22}$$

Before, proving our main result for controllability of system (4.4.11), we prove that the operator M is Lipschitz.

LEMMA 4.4.3 *If f satisfies Lipschitz condition*

$$\| f(x_1, u_1) - f(x_2, u_2) \| \leq \alpha_x \| x_1 - x_2 \| + \alpha_u \| u_1 - u_2 \|$$

for all $x_1, x_2 \in R^n$ and $u_1, u_2 \in R^m$ then, the operator M defined by (4.4.22) is Lipschitz.

Proof: We know that

$$Mu = L_c(x_f - e^{AT} x_0) - L_c K_3 N u$$

Therefore,

$$\begin{aligned} \| Mu_1 - Mu_2 \| &= \| L_c K_3 N u_1 - L_c K_3 N u_2 \| \\ &\leq \alpha_l k_1 \| Nu_1 - Nu_2 \| \\ &\leq \alpha_l k_1 \alpha_N \| u_1 - u_2 \| \\ &= \alpha_m \| u_1 - u_2 \| \end{aligned}$$

where, $\alpha_m = \alpha_l k_1 \alpha_N$.

Thus, M is Lipschitz with the constant α_m.

Now, we give our main result regarding the controllability for the system (4.4.11) in terms of the solvability of the pair (x, u) given by the equations (4.4.20) and (4.4.21).

THEOREM 4.4.4 *The nonlinear system (4.4.11) is controllable under the following assumptions:*

1. *Linear system (4.3.10) is controllable.*
2. *The nonlinear function f in the equation (4.4.11) is Lipschitz continuous.*
3. *The Lipschitz constant α_m of operator M less than 1.*

Further, the control can be computed using the iterative scheme:

$$u^{n+1} = Mu^n$$

starting with arbitrary u^0.

Proof: Under the hypothesis the operator M is contraction and hence by Banach Contraction Principle it has unique fixed point which can be iteratively computed by

$$u^{n+1} = Mu^n = L_c(x_f - e^{AT}x_0) - L_c K_3 N((Pu)^n(t), u^n(t))$$

starting with arbitrary u^0.

The sequence of control converges to the required control which steers the initial state x_0 to x_f.

Along with the controller the steered state can be computed iteratively as

$$x^{n+1}(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu^n(\tau)d\tau + \int_0^t e^{A(t-\tau)}f(x^n(\tau), u^n(\tau))d\tau$$

Hence, the proof. •

Thus, for computation of the steering control for the semilinear dynamical system we solve the coupled iterative equations starting with arbitrary $(x^0(t), u^0(t))$

$$u^{n+1}(t) = B^* e^{A^*(t_f - t)} W^{-1}(0, t_f) \left(x_f - e^{At_f}x_0 - \int_0^{t_f} e^{A(t_f - s)} f(x^n(s), u^n(s))ds \right) \tag{4.4.23}$$

and

$$x^{n+1}(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu^n(\tau)d\tau + \int_0^t e^{A(t-\tau)}f(x^n(\tau), u^n(\tau))d\tau \tag{4.4.24}$$

The process is repeated until the prescribed accuracy is reached.

REMARK 4.4.5 *The condition 3. in the above theorem can be relaxed if the nonlinear function f is uniformly bounded. In this case we can use Schauder's fixed point Theorem instead of Banach Contraction Principle to prove the theorem. However, we do not have iterative procedure for the computation of control.*

In the following, we revisit the Mixing tank problem now in the semilinear form.

Mixtank Process represented in Semilinear form :

The mathematical representation of the Mixtank problem in the semilinear form is

$$\dot{x}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix} + \begin{bmatrix} -k\sqrt{\frac{V(t)}{A_c}} \\ \frac{(C_1 - C(t))Q_1(t)}{V(t)} + \frac{(C_2 - C(t))Q_2(t)}{V(t)} \end{bmatrix}$$

$$x(t_0) = x_0 \tag{4.4.25}$$

The nonlinear function f, in the system (4.4.25) is Lipschitz except at the zero volume which is avoided as per our initial assumption. But observe that the linear part of the system given by (4.4.25) is not controllable as A is zero matrix, therefore the mixing tank problem is not globally controllable. Thus, the system (4.4.25) is only locally controllable around the equilibrium (3600, 15), as shown before with the local steering control implemented using Artificial Neural Network.

However, in the latter chapters we will see the systems which can be steered to the desired state by the ANN controller defined for the system in the semilinear form.

4.5 Summary

In this chapter, we obtain mathematical model for a Chemical the subprocess: mixing tank of a chemical plant that synthesizes Ethyl Acetate. The mathematical model give rise to nonlinear time invariant dynamical system, for which local controllability is established. The optimal local controller for the mixtank process is implemented using Multilayered feedforward Neural Network, trained using Backpropagation algorithm. The simulation results are implemented and tested using MATLAB and are found to be well acceptable. After, developing the results for the controllability of the system in the semilinear we investigate the mixing tank problem for the global controllability and find that the system is not controllable in the semilinear form.

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