CHAPTER III

LONGITUDINAL OSCILLATIONS

In this chapter we apply the CHD equations, derived in the previous chapter, to study the longitudinal oscillations in the QGP. The novel features arising due to non-abelian nature of the gauge fields will be discussed.

3.1 INTRODUCTION : Density fluctuations in the QGP may cause powerful color electric fields which in turn can exert a restoring force on the plasma particles and as in a Coulomb plasma the longitudinal oscillations would be set in the medium. Indeed, using the linear response formalism similar to that in QED the longitudinal oscillations and ordinary Debye screening behavior similar to that Coulomb plasma have already been found in the QGP.[1] Using the linear response formalism one finds for the plasma frequency ω_p (in the one loop order, $O(g^2)$)

$$\omega^2_p = \frac{(N_f + 2N)}{18} g^2 T^2 \tag{3.1}$$

where N_f and N are number of quark flavors and the number of colors of the gluons respectively. g is the strong interaction constant and T is the temperature of the medium.

The study of longitudinal oscillations is important because it is one of the simplest indication of collective behavior of a plasma. Their existence in QGP can be a unique proof of the deconfined color in the plasma. Moreover, they can contribute to energy density and pressure of the plasma and they may influence the emitted particle spectra[2] A classical approach also, in the linearized limit, gives the same values for the plasma frequency and the screening length[3].

For simplicity we have considered that the plasma comprises of two species (particle and antiparticle). The entire volume (infinite in extent) is assumed to be locally color neutral when no perturbation is applied. Longitudinal waves in such a plasma can cause local color density fluctuations, which in turn may cause, as in a Coulomb plasma, a powerful restoring force and the longitudinal oscillations can be sustained in the medium. As mentioned in the introduction our primary interest is to study the non-perturbative aspects of the QGP. Hence we solve non-linear CHD and Yang-Mills equations to study the longitudinal oscillations. It will be shown that even when the non-linearity parameter is small the linear plasma frequency of the oscillations get strongly modified.

3.2 BASIC EQUATIONS : We will study the CHD equations in the non-relativistic limit

$$\frac{\partial n_A}{\partial t} + \nabla\cdot(n_A \mathbf{V}_A) = 0 \tag{3.2a}$$

$$\frac{\partial}{\partial t} \mathbf{V}_A + (\mathbf{V}_A \cdot \nabla)\mathbf{V}_A = \frac{g}{m_A} I_{Aa}\left[\mathbf{E}_a + \mathbf{V}_A \times \mathbf{B}_a \right] \tag{3.2b}$$

$$\frac{\partial}{\partial t} I_a + \mathbf{V}_A \cdot \nabla I_a = -g\, \varepsilon_{abc} \left[A_b^o - \mathbf{V}_A \cdot \mathbf{A}_b \right] I_c \tag{3.2c}$$

These equations together with the Yang-Mills equations [2.46 in chapter II] form a closed set of equations which may be utilized for a description of self consistent collective non-linear oscillations of the QGP. These equations are deficient in that no finite temperature effects in terms of the quark pressure have been retained here. As mentioned in Chapter II this is justified when the phase velocity of the wave is much greater than the thermal velocity of the plasma particles. Thus the thermal dispersion can be neglected for the longitudinal oscillations as they are long wave-length waves.[4]

Equations (3.2a) - (3.2c) (also the Yang-Mills equations) are a set of nonlinear coupled partial differential equations which are quite difficult solve in their generality. Therefore we shall look for special solutions of these equations which are nonlinear plane stationary waves. Thus we assume that all the quantities depend only on the variables t and x_3(say) and that too only through the single variable $\zeta = x_3 + \beta t$. Mathematically, this assumption will convert the partial differential equations to the ordinary differential equations because $\nabla_3 = d/d\zeta$ and $\partial/\partial t = \beta\ d/d\zeta$. Physically, the crucial assumption here is that the nonlinear solutions are stationary in a frame moving with the phase speed β. Such nonlinear stationary plane wave solutions are widely discussed in Coulomb plasma literature and also have been considered for non-abelian fields[5]. The phase velocity β plays the role of a parameter in the final equations.

Since our interest is to study the longitudinal oscillations we may ignore the coupling to color magnetic fields. In fact, it can be shown directly that (from the field equations) that if A^1, A^2 and their derivatives are zero at $\zeta = 0$ then they are zero for all values of ζ. Physically, this means that the symmetry properties of the field and plasma equations ensure that the purely longitudinal disturbance can propagate without coupling with the color magnetic perturbations. In rest of this chapter we will consider only the longitudinal disturbances and therefore we set A^1 and A^2 to zero.

We now make the gauge choice $A^0 = 0$. The only non-vanishing field strength can now be written as $E_a^3 = -\partial_t A_a^3 = -\beta a_a'$ where $A_a^3 \equiv a_a$ and the prime denotes differentiation with respect to ζ. One can also write Ampere's equations (space component of the Yang-Mills equations) as

$$a_a'' = \frac{1}{\beta^2} J_a^{\,3} \tag{3.3}$$

As we show below J_a^3 can be expressed in terms of the Yang-Mills potentials by integrating the hydrodynamic equations [Eq.(3.1)]. Further, we assume that in equilibrium both the species have the same density i.e. $n_{10} = n_{20} = n_0$ and the velocity of both the species are zero. The color neutrality condition in this case will read as $I_{1a0} + I_{2a0} = 0$.

Eqs.(3.2) on integration will yield the following relations:

$$n_A = \beta\, n_0 \big/ (\beta + V_A) \tag{3.4a}$$

$$V_A = -\beta + \left(1 - \frac{2g}{m_A \beta} I_a a_a \right)^{1/2} \tag{3.4b}$$

$$I_{Aa}' = \left(\frac{g}{\beta\, n_0} \right) \varepsilon_{abc}\, a_b\, I_{Ac}\, n_A\, V_A \tag{3.4c}$$

Eq.(3.4c) may be combined with eqs.(3.3) and using the definition of current as given in the previous chapter we get

$$\sum_A (I_{Aa})' = \frac{\beta}{n_o} \varepsilon_{abc}\, a_b\, J_c = \frac{\beta}{n_o} \varepsilon_{abc}\, a_b\, a_c''$$

On integration one finds the conservation law

$$I_{1a} + I_{2a} = \frac{\beta}{n_o} \varepsilon_{abc}\, a_b\, a_c' \tag{3.4d}$$

We now assume for the simplicity of calculation that one specie is much heavier than the other, so that (say) $m_2 >> m_1$. In this case we may set $V_2 = 0$, which in turn implies that I_{2a}= constant = I_{2ao} = $-I_{1ao}$. Note that the same situation would have arisen for a spatially homogeneous plasma if we go to a frame in which the specie 2 is at rest. We may now write

$$J_a^3 = g\, n_1\, V_1$$

$$= g\, \beta\, n_o[\beta/n_o\, \varepsilon_{abc}\, a_b a_c' + I_{1ao}]\left[1 - \left(1 - \frac{2g}{m_1\beta} I_{1ao} a_a\right)^{-1/2}\right]$$

which may be substituted in the Ampere's Law eq.(3.3) to give the final nonlinear equations describing the longitudinal oscillation.

$$a_a'' = [g\, \varepsilon_{abc}\, a_b a_c' + \frac{g\, n_o}{\beta} I_{1ao}]\left[1 - \left(1 - \frac{2g}{m_1\beta} I_{1ao} a_a\right)^{1/2}\right] \quad (3.5)$$

One can observe that that eq.(3.5) contains two types of nonlinear terms, those arising from the non-abelian nature of the gauge potentials (first term in the first bracket) and those arising from the hydrodynamic framework used to describe the plasma (second term in the second braket). It is well-known that latter type of terms are also present for a Coulomb plasma. As our main interest is in the study of non-abelian effects, we expand the square root in eq.(3.5) and retain only the terms linear in the field amplitude. We then obtain the equation

$$a_a'' = - \frac{g^2 n_o}{m\, \beta^2} I_{1ao}(\, I_{1bo} a_b\,) - \frac{g^2}{m\, \beta} (\, \varepsilon_{abc}\, a_b\, a_c')(I_{1do} a_d) \quad (3.6)$$

where we have dropped the specie label on the mass. The assumption of weak plasma nonlinearity corresponds to the condition $(gI_{1ao}/m\beta)a_a <<$ 1 which is equivalent to $|V_1|\beta << 1$ i.e. the directed particle velocity in the wave-fields is much less than wave phase velocity. At the same time the retention of non-abelian term means that we must have $g\ a/|K| > 1$ where $|K|$ measures the magnitude of the of derivative term $|a'/a|$. Thus the assumption is justified if $\frac{\beta\ g}{\omega_p} a_a > 1 > \frac{g I_o a_a}{m\ \beta}$. In order to get a feeling for these inequalities in terms of physical quantities we rewrite the longitudinal electric field $E_a \sim -\partial_t a_a \sim -\omega_p a_a$ in terms of the wave energy density $\varepsilon_w \sim E_a^2$ and use a normalizing energy density $\varepsilon_c \sim nmc^2 \sim$ 2-5GeV/fm^3 (the typical energy density in the plasma needed for a deconfinig transition). We may then write the above inequality as

$$\frac{10^{-2}}{\bar{g}(\beta/c)} < \left(\varepsilon_w/\varepsilon_c\right)^{1/2} < \frac{\beta}{c}\left(n/n_c\right)^{1/2}$$

where we have assumed that the phase velocity $\beta/c \sim 1$ and the typical wavelength $k^{-1} \sim c/\omega_p \sim$ few fermis From the inequality above we may see that our treatment is valid for $\varepsilon_w/\varepsilon_c$ as low as 10^{-4} i.e. for wave energy density of order hundredth of a percent of the typical plasma energy density.

Eq.(3.6) treats the plasma in an approximate manner, which may be given a simple derivation (see Appendix A). The oscillations in this case are interpreted as nonlinear temporal oscillations and have no spatial dependence ($d/dx \to 0$). For finite wavelength perturbations ($d/dx = 0$), the correct interpretation is still in terms of the variable $\zeta = x + \beta t$ or its temporal analog $\tau = t + x/\beta$.

To proceed further we take, for simplicity, $I_{11o} = I_{12o} = I_{13o} = I_o$ and also define $\omega_p^2 = g^2 n_o (I_o)^2/m$. We further write eq.(3.6) in a neat symmetrical form by introducing scaled normal-mode variables, which remove the coupling between a_1, a_2, a_3 arising through the linearized first term. We thus introduce the quantities $a_a^* = a_o a_a$

(where a_o is a normalizing scale-factor for the vector potential), $X_1 = a_1^* + a_2^* + a_3^*$, $X_2 = \sqrt{3/2}(a_1^* - a_3^*)$, $X_3 = \sqrt{1/2}\,(a_1^* - 2a_2^* + a_3^*)$, $\omega_p \frac{\zeta}{\beta} =$ T. The resulting equations are

$$\ddot{X}_1 = -3\,X_1 + (\varepsilon/\sqrt{3})(\,X_2\,\dot{X}_3 - X_3\,\dot{X}_2\,)\,X_1 \tag{3.7a}$$

$$\ddot{X}_2 = (\varepsilon/\sqrt{3})\,(\,X_3\,\dot{X}_1 - X_1\,\dot{X}_3\,)\,X_1 \tag{3.7a}$$

$$\ddot{X}_3 = (\varepsilon/\sqrt{3})\,(\,X_1\,\dot{X}_2 - X_2\,\dot{X}_1\,)\,X_1 \tag{3.7c}$$

In equations (3.7a-c), the dots denote differentiation with respect to the dimensionless variable T and $\varepsilon = g^2 I_o a_o^2 / m\omega_p$ is a parameter characterizing the strength of the non-abelian terms.

Equations (3.7a-c) may be interpreted as the equations of motion of an 'effective particle' with three degrees of freedom ina nonlinear potential field. It can be shown by direct calculation that these equations have the following conservation laws:

$$1/2\,(\,\dot{X}_1^2 + \dot{X}_2^2 + \dot{X}_3^2\,) + 3/2\,X_1^2 = E \tag{3.8}$$

$$\varepsilon/2\sqrt{3}(\,M_1^2 + M_2^2 + M_3^2\,) - 3\,M_1 = M \tag{3.9}$$

where
$$\begin{aligned} M_1 &= X_2\,\dot{X}_3 - X_3\,\dot{X}_2 \\ M_2 &= X_3\,\dot{X}_1 - X_1\,\dot{X}_3 \\ M_3 &= X_1\,\dot{X}_2 - X_2\,\dot{X}_1 \end{aligned} \tag{3.10}$$

Eq.(3.8) describes an energy conservation law. Note that the first three terms on the left hand side (corresponding to the 'kinetic energy' of the effective particle) actually describe the energy in the longitudinal color electric fields. Eq.(3.8) thus has a clear physical interpretation in terms of exchange of energy between the color electric fields and the kinetic energy of plasma particles. Eq.(3.9) is related to the conservation an angular momentum like vector in color space. Note from Eq.(3.4d) that M_1, M_2 and M_3 are related to color fluctuations being carried by the Yang-Mills fields. In the absence of matter we have the conservation law $M_1^2 + M_2^2 + M_3^2$ = constant. The last term in Eq.(3.9) is a consequence of color charge being exchanged between the chromo-fields and the material particles. It should be emphasized that the conserved quantities in Eqs(3.8-9) are gauge invariant. This is explicitly demonstrated in appendix B by making a gauge choice $A^o = \delta A^3$ and demonstrating that E and M are independent of δ. Finally, it is worth pointing out that for the exact equations (Eq.3.5) we have a different material particle kinetic energy term in the energy conservation law, but Eq.(3.9) does not change.

3.3 NUMERICAL CALCULATIONS AND RESULTS : It is in general very difficult to solve Eqs.(3.7a-c) analytically hence a numerical procedure is used to integrate them for different initial conditions and for different values of ε. The procedure used for integrating the equations was the fourth order Runge-Kutta method with variable step size. The results presented in the figures (below) are the representatives of a very wide choice of initial conditions and the parameter choices ε. Hence the qualitative features of the results are independent of any specific choice of the initial conditions.

Moreover, we have chosen to present the results in terms of the scaled normal mode variable X_1. This variable can be shown to be related to the particle velocity. Therefore this variable is just right for displaying characteristic features of the non-abelian physics.

It is obvious that when $\varepsilon = 0$, we have the usual plasma oscillations for X_1, with frequency $\sqrt{3}\,\omega_p$ whereas X_2 and X_3 may increase linearly with T. When ε is not equal to zero and not too large, the X_1 solutions exhibit two periodic modes. These are the cases depicted in Fig.(1a,2a). In both the cases, for small value of ε, the linear plasma mode with frequency $\sqrt{3}\,\omega_p$ and nearly a constant amplitude oscillations is followed by a new 'non-abelian mode' which differs significantly from the plasma mode in both amplitude and frequency. In fact Fig (1b), for the interval $T_i = 250$ to $T_f = 500$ of Fig.(1a), more clearly shows that the plasma mode is followed by the non-abelian mode having frequency nearly four times that of the plasma frequency and significantly reduced amplitude. Although in these solutions both types of behavior occur periodically, it is not clear when the crossover from the plasma mode to the non-abelian mode occurs. It seems to depend on the phases of X_2 and X_3.

For large values of ε, Fig(3.a), we find intermittency or chaos in X_1 motion i.e. the plasma and the non-abelian mode do not occur periodically in X_1 motion and crossover from one mode to another is irregular Such motions may contribute to the thermalization of QGP. In fact to quantify this further the auto-correlation study of X_1 motion is also carried out (Figs 1c,2b,2c). Figs.(1c,2b) depict the auto-correlation of X_1 for small values of ε (i.e. Fig.1a,2a). The auto-correlations show oscillatory behavior with T. As one expects for the oscillations in Fig.(3a) the auto-correlation of X_1 decreases fast and saturates around zero in a time interval of few plasma periods.

This indicates that the energy associated with the particles goes into random motion and the field energy. We should also mention that X_2 and X_3 motions are quite irregular (for all values of ε except zero) and do not show any plasma mode.

It is worth recalling here that, the numerical results shown in this section do not include the effect of the plasma nonlinearity (see section 3.2). However, we have found after extensive calculations that inclusion of such terms does not change any of the qualitative features presented in this chapter.

3.4 SUMMARY AND CONCLUSIONS : We have studied the effect of non-abelian terms on the longitudinal oscillations of a classical quark-gluon plasma. We have found that for a certain strength of the non-abelian parameter there exists a new periodic mode (non-abelian mode) which alternates with the usual Coulomb plasma mode. The transition from the plasma mode to the non-abelian mode and vice versa is sudden and how exactly these transitions occur is not yet properly understood. It is difficult to see how such a novel behavior which our nonlinear analysis reveals can be obtained by the methods of perturbation theory even for small values of the non-abelian parameter ε.

Also we have observed that for large values of ε the longitudinal oscillations show chaos or intermittancy in the transition from the plasma mode to the non-abelian mode and vice versa. Such oscillations might contribute to the thermalization of the plasma. The auto-correlation study of X_1 "time" series shows that for large values of ε the correlations decays very fast i.e. in a time period of a few plasma oscillations, the correlations reduce to zero value.

APPENDIX A : SIMPLE DERIVATION OF EQ.(3.6)

We start with the approximate linearized form of the plasma equations (3.2 b-c), viz,

$$\partial_t \mathbf{V}_A = \frac{g}{m_A} I_{Aa} \mathbf{E}_a \tag{A1}$$

$$\partial_t I_{Aa} = g\, \varepsilon_{abc} \left(\mathbf{V}_A . \mathbf{A}_b \right) I_{Ac} \tag{A2}$$

We have used the gauge condition $A^o \equiv 0$. The neglect of the plasma nonlinearity through $(\mathbf{V}_A \cdot \nabla)$ term is fully justified when $\partial/\partial x \to 0$. Equations (A2) shows that $A_a \frac{\partial}{\partial t} I_a = 0$. Noting that $E_a = - \partial_t A^3_a$, we may now integrate Eq.(A1) to get

$$V_A = - \frac{g}{m_A} I_{Aa}\, a_a \tag{A3}$$

Again, from the definition of the current density (see chapter two, Eqs 2.45), $J^3_a \cong g n_o V_A I_{Aa}$ and Ampere's law $\dot{a}_a = J^3_a$, we may use Eq.(A2) to derive the equation

$$\sum_A I_{Aa} = \frac{g}{n_o} \varepsilon_{abc}\, a_b\, \dot{a}_c$$

where a over dot denotes differentiation with respect to time variable t. As in the main text if we assume that specie 2 is heavy then we get

$$a''_a = - \frac{g^2}{m} I_{1ao} \left(I_{1bo} a_b \right) - \frac{g^2}{m} \left(\varepsilon_{abc}\, a_b\, a'_c \right) \left(I_{1do} a_d \right) \tag{A4}$$

Eq.(A4) describe nonlinear temporal oscillations and they are very similar to Eqs.(3 6) discussed in the text.

APPENDIX B: GAUGE INVARIANCE OF CONSERVED QUANTITIES E AND M

The quantities E and M are defined by the conservation laws, Eqs. (3.8-9) and we expect them to be gauge invariant. To explicitly demonstrate this gauge invariance let us make a general gauge choice $A^o = \delta A^3$. The parameter δ is a constant and was chosen to be zero in the text. With this choice of the gauge we obtain the following equations describing the longitudinal oscillations

$$\ddot{X}_1 + 3 X_1 = \left[\frac{\varepsilon}{\sqrt{3}} \left(1 + \delta/\beta \right)^2 X_1 + g\, \delta/\beta \right] M_1 \tag{B1}$$

$$\ddot{X}_2 = \left[\frac{\varepsilon}{\sqrt{3}} \left(1 + \delta/\beta \right)^2 X_1 + g\, \delta/\beta \right] M_2 \tag{B2}$$

$$\ddot{X}_3 = \left[\frac{\varepsilon}{\sqrt{3}} \left(1 + \delta/\beta \right)^2 X_1 + g\, \delta/\beta \right] M_3 \tag{B3}$$

where M_a (a= 1,2,3) are as defined in the text. For $\delta = 0$, we can recover Eqs.(3.7) derived in the text. It can be directly verified that these equations have the same conservation laws as, Eq.(3.8-9), in the text. Thus we have explicitly demonstrated that E and M are gauge invariant quantities.

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FIGURE CAPTIONS

Fig.1a. Oscillations of the field variable X_1. Initial conditions are given in terms of variable $x_1, x_2, x_3, \dot{x}_1, \dot{x}_2$ and $\dot{x}_3$ which are related to the field variables used in this chapter by $X_1 = x_1$, $X_2 = (3/2)^{1/2} x_1$ and $X_3 = (1/2)^{1/2}$ and their derivatives are defined accordingly.

$\varepsilon/\sqrt{3} = 0.05$ and the intial conditions are

$x_1 = x_2 = x_3 = 0.$ and $\dot{x}_1 = 2$, $\dot{x}_2 = 0.1$ and $\dot{x}_3 = 0.3$

Fig.1b Oscillations of the field variable X_1. The values of the parameters ε and the initial conditions same as Fig.1a. The scale for the variable T is expanded.

Fig.1c Auto-correlation of X_1 of Fig.1a.

Fig.2a Oscillations of the field variable X_1.

$\varepsilon/\sqrt{3} = 0.05$ and the intial conditions are

$x_1 = x_2 = x_3 = 0$ and $\dot{x}_1 = 2$, $\dot{x}_2 = 1$ and $\dot{x}_3 = 3$.

Fig.2b Auto-correlation of X_1 of Fig.2a.

Fig.3a Oscillations of the field variable X_1

$\varepsilon/\sqrt{3} = 0.05$ and the intial conditions are

$x_1 = x_2 = x_3 = 0$ and $\dot{x}_1 = 2$, $\dot{x}_2 = 0.1$ and $\dot{x}_3 = 0.3$

Fig.3b Auto-correlation of X_1 of Fig.3a

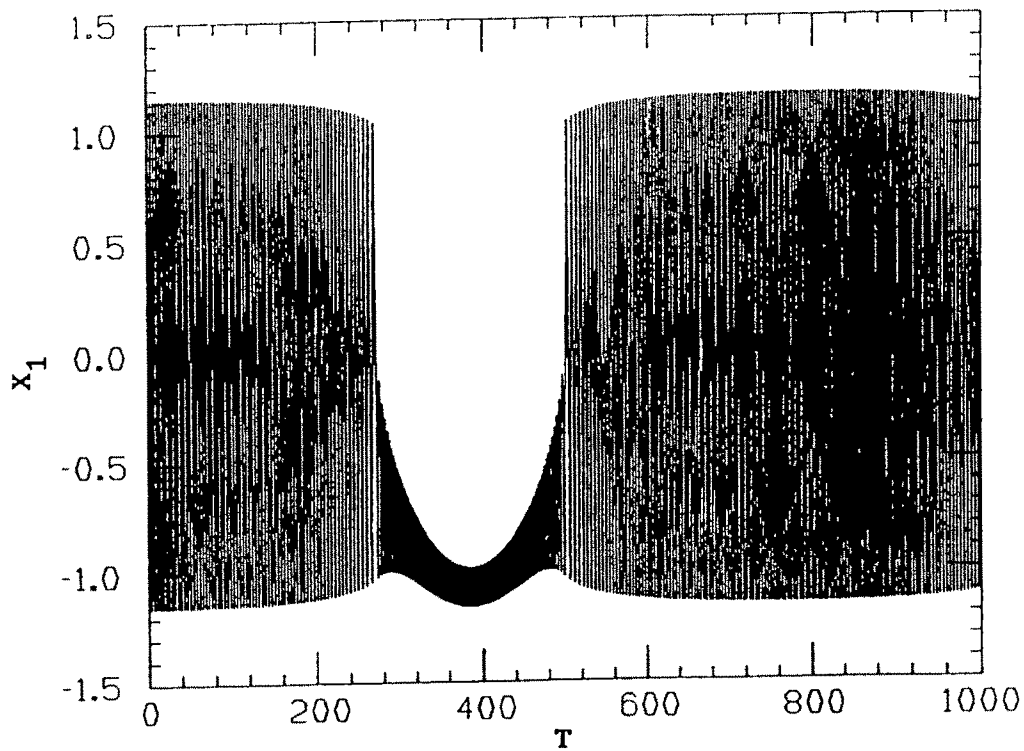

Fig. 1a

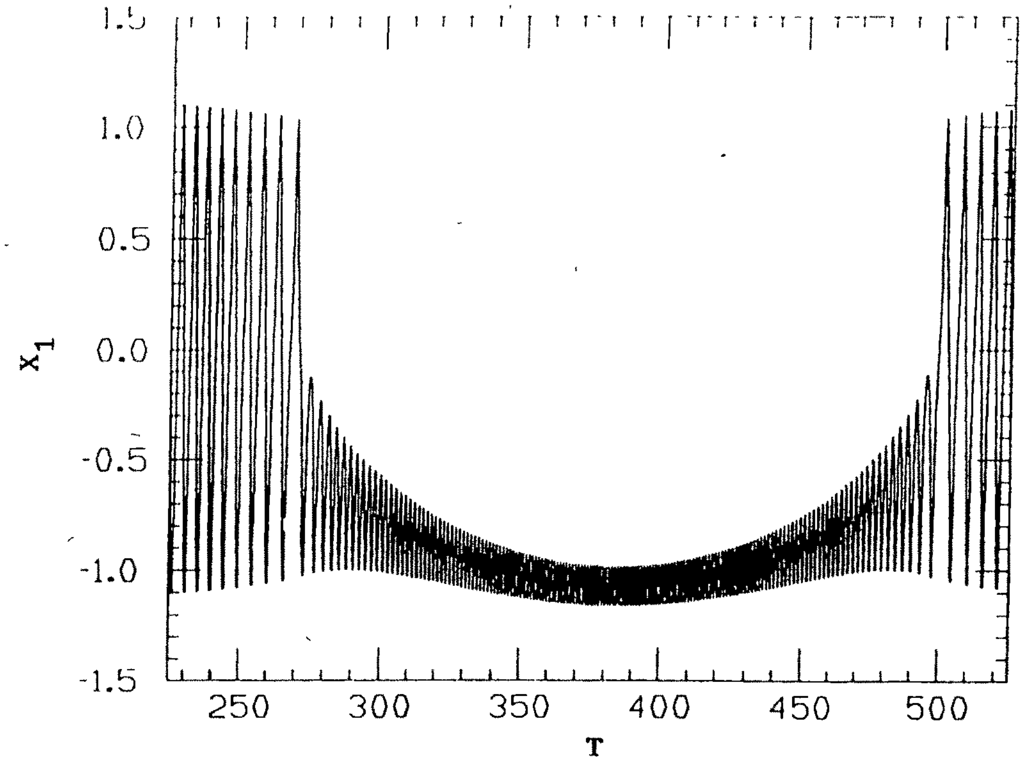

Fig. 1b

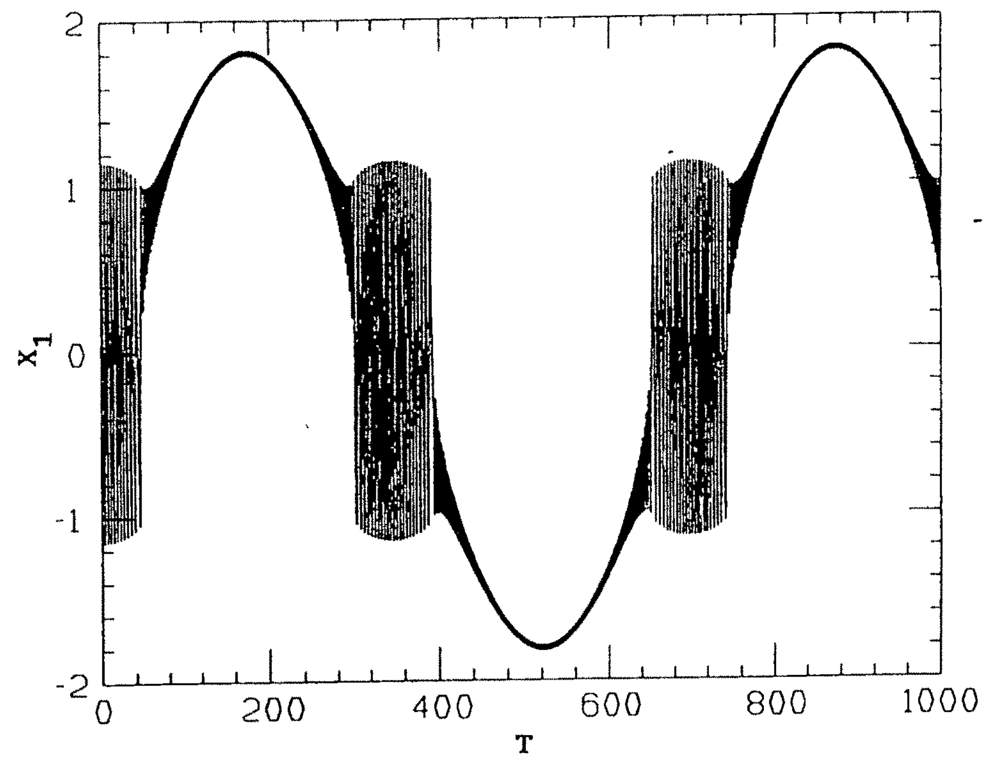

Fig. 2a

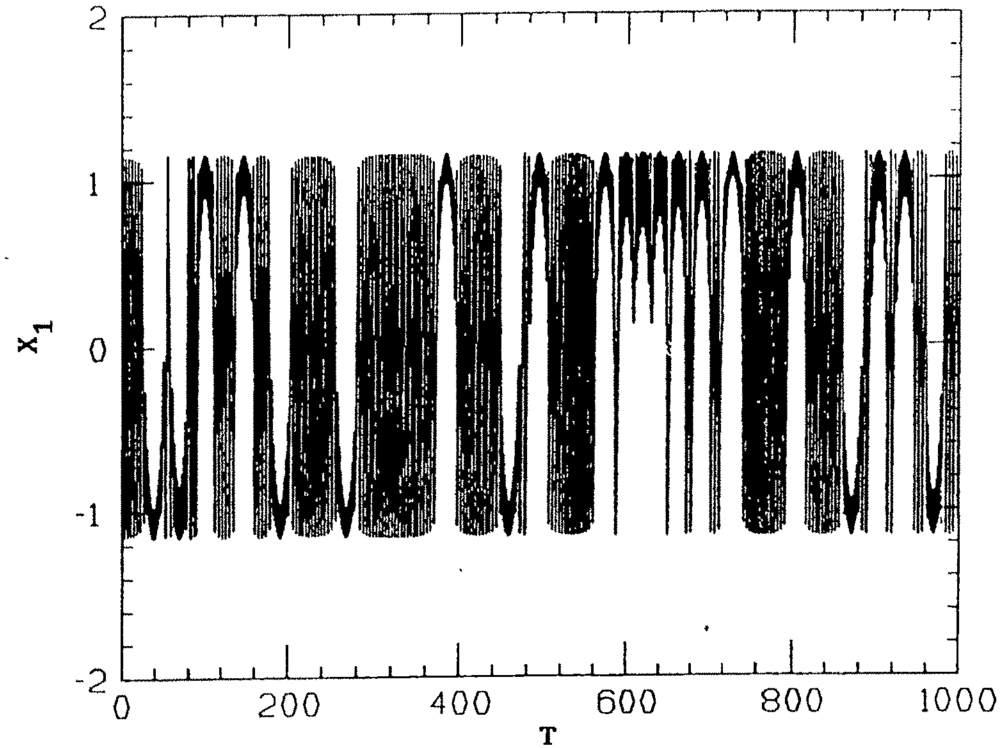

Fig. 3a

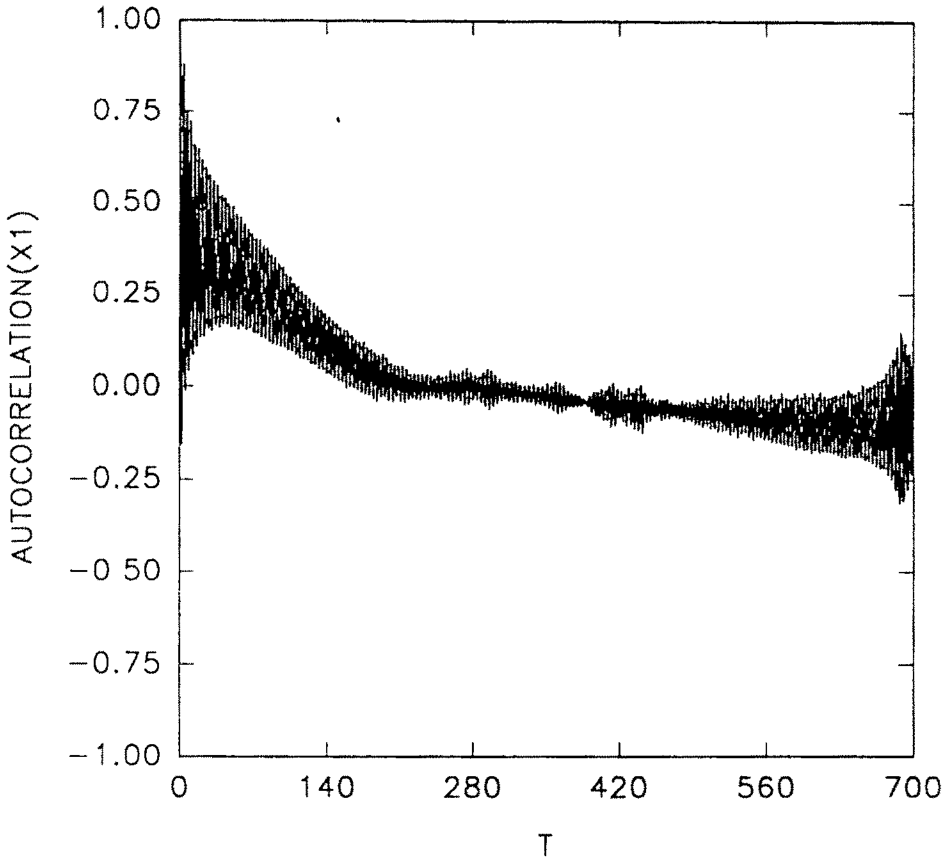

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Fig.1c

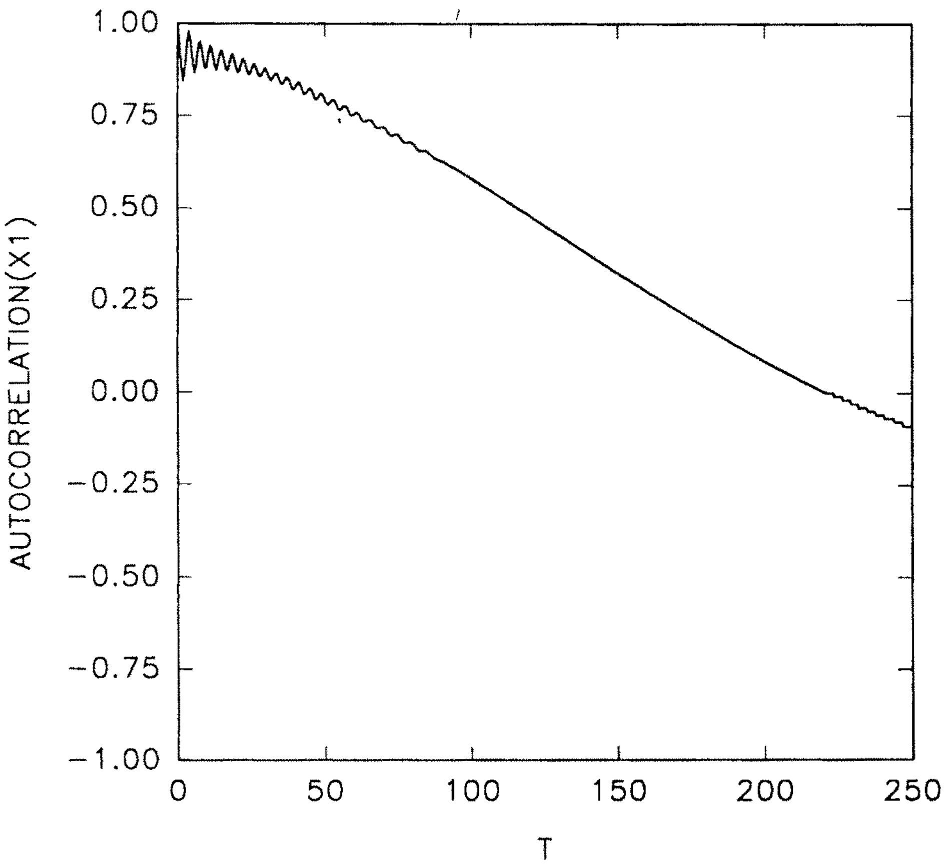
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Fig 2b

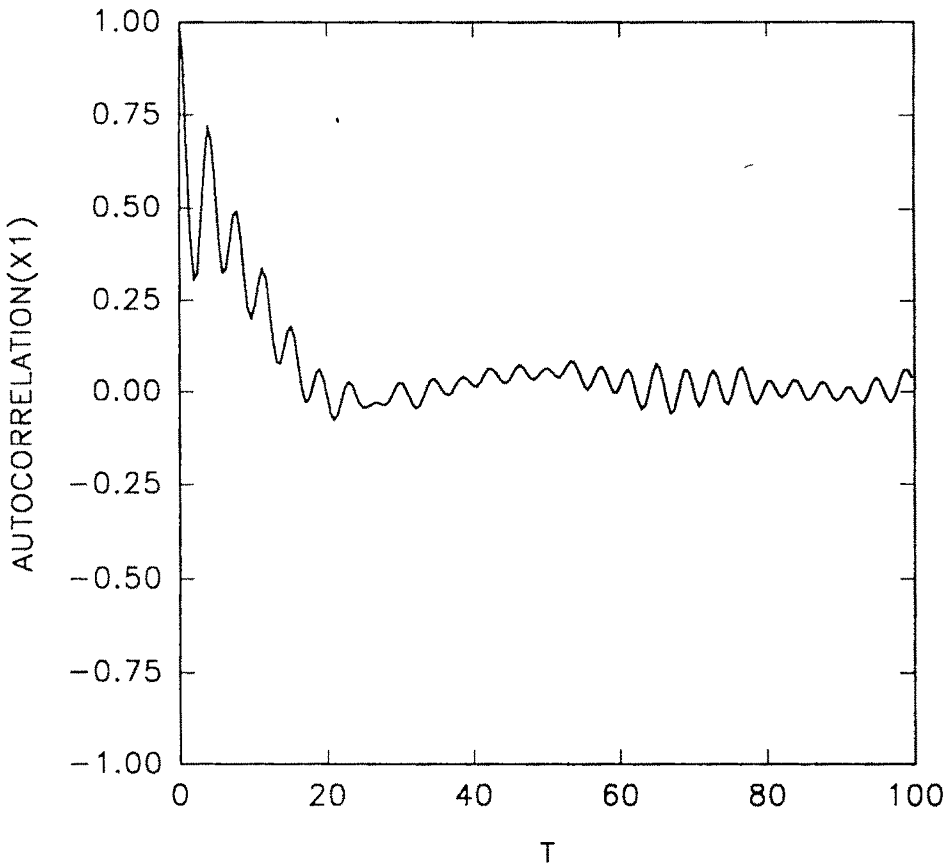
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Fig 3b