Appendix B

Matrix Elements of Surface Delta Interaction and Pairing Plus Quadrupole-Quadrupole Interaction

B.1 Surface Delta Interaction (SDI)

The mathematical form of the two-body surface delta interaction (SDI) is given by

$$V_{SDI}(1,2) = -4\pi G'_T \delta(\vec{r}(1) - \vec{r}(2))\, \delta(r(1) - R_0) \tag{B.1}$$

where G'_T is the strength parameter of the interaction, T denotes isospin, 1,2 denote the particle index, $\vec{r}(1)$ and $\vec{r}(2)$ are the radius vectors of the position of the particles 1 and 2 respectively and R_0 is the nuclear radius. The matrix elements of the SDI in isospin formalism can be written as [Br-77]

$$\left\langle j_a j_b | V^{SDI} | j_c j_d \right\rangle_{JT} = V^{JT}_{abcd} = (-1)^{n_a+n_b+n_c+n_d} \times$$

$$\frac{G_T}{2(2J+1)} \left[\frac{(2j_a+1)(2j_b+1)(2j_c+1)(2j_d+1)}{(1+\delta_{ab})(1+\delta_{cd})} \right]^{1/2} \times$$

$$\Big\{ (-1)^{j_b+j_d+\ell_b+\ell_d} \langle j_b \; -\tfrac{1}{2} \; j_a \; \tfrac{1}{2} | J\, 0 \rangle \times$$

$$\langle j_d \; -\tfrac{1}{2} \; j_c \; \tfrac{1}{2} | J 0 \rangle \left[1 - (-1)^{l_a+l_b+J+T} \right] -$$

$$\langle j_b \; \tfrac{1}{2} \; j_a \; \tfrac{1}{2} | J\, 1 \rangle \; \langle j_d \; \tfrac{1}{2} \; j_c \; \tfrac{1}{2} | J\, 1 \rangle \left[1 + (-1)^T \right] \Big\} \tag{B.2}$$

where $G_T = G'_T C(R_0)$, $C(R_0)$ is a radial integral. In pn formalism the pp(nn) and pn matrix elements follow from $V^{J,pp}_{abcd} = V^{J,\,T=1}_{abcd}$ and $V^{J,pn}_{abcd} = \frac{1}{2}\left[(1+\delta_{ab})(1+\delta_{cd})\right]^{1/2} \left[V^{J,T=0}_{abcd} + V^{J,T=1}_{abcd} \right]$. Taking $G_0 = G_1 = G$ in (B.2), the explicit formula for the pp matrix elements of SDI is,

$$V^{J,pp}_{abcd} = \langle j_a \; j_b \; J | V(pp) | j_c \; j_d \; J \rangle$$

$$= (-1)^{n_a+n_b+n_c+n_d} \times \frac{G}{2(2J+1)} \left[\frac{(2j_a+1)(2j_b+1)(2j_c+1)(2j_d+1)}{(1+\delta_{ab})(1+\delta_{cd})} \right]^{1/2} \times$$

$$\Big\{ (-1)^{j_b+j_d+\ell_b+\ell_d} \langle j_b \; -\tfrac{1}{2} \; j_a \; \tfrac{1}{2} | J\, 0 \rangle \times$$

$$\langle j_d \; -\tfrac{1}{2} \; j_c \; \tfrac{1}{2} | J\, 1 \rangle \left[1 + (-1)^{\ell_a+\ell_b+J} \right] \Big\} \tag{B.3}$$

Similarly $V^{J,nn}_{abcd}$ are defined with strength G. The pn matrix elements are,

$$V^{J,pn}_{abcd} = \langle j_a \; j_b \; J | V(pn) | j_c \; j_d \; J \rangle$$

$$= (-1)^{n_a+n_b+n_c+n_d} \times \frac{G}{2(2J+1)} \left[(2j_a+1)(2j_b+1)(2j_c+1)(2j_d+1) \right]^{1/2} \times$$

$$\Big\{ (-1)^{j_b+j_d+\ell_b+\ell_d} \langle j_b \; -\tfrac{1}{2} \; j_a \; -\tfrac{1}{2} | J\, 0 \rangle \langle j_d \; -\tfrac{1}{2} \; j_c \; \tfrac{1}{2} | J\, 0 \rangle -$$

$$\langle j_b \; \tfrac{1}{2} \; j_a \; \tfrac{1}{2} | J\, 1 \rangle \langle j_d \; \tfrac{1}{2} \; j_c \; \tfrac{1}{2} | J\, 1 \rangle \Big\} \tag{B.4}$$

It should be pointed out that low energy spectroscopy determines the value of G to be approximately $25/A \; MeV$ [Br-77]. Some useful properties of SDI are:

(i) $V_{\alpha\beta}(SDI-pp) = -\frac{G}{2}(N_\alpha/(N_\alpha - \delta_{\alpha\beta}))$; (ii) $V_{\alpha\beta}(SDI-nn) = -\frac{G}{2}N_\alpha/(N_\alpha - \delta_{\alpha\beta}))$; (iii) $V_{\alpha\beta}(SDI-pn) = -G$; (iv) $\zeta^1_\alpha([\mathrm{m}]) = 0$ for all values of α and [m]'s; (v) $V_{\alpha\beta}(SDI-pp) = \frac{G}{2}(N_\alpha/(N_\alpha - \delta_{\alpha\beta}))$; (vi) $V_{\alpha\beta}(SDI-nn) = \frac{G}{2}(N_\alpha/(N_\alpha - \delta_{\alpha\beta}))$; (vii) $V_{\alpha\beta}(SDI-pn) = -G$. Note that SDI-pp stands for pp part of SDI interaction and similarly SDI-nn, SDI-pn are defined. From (iv) it follows that SDI do not produce any SPE renormalizations.

B.2 Pairing plus quadrupole-quadrupole interaction (P + Q.Q)

The P + Q.Q hamiltonian is defined in terms of the pp and nn pairing hamiltonians H^P_p and H^P_n and the pp, nn and pn quadrupole-quadrupole hamiltonians $\bar{Q}_p.\bar{Q}_p$, $\bar{Q}_n.\bar{Q}_n$ and $\bar{Q}_p.\bar{Q}_n$,

$$H_{P+Q.Q} = G_p H^P_p + G_n H^P_n + G^Q_{pp}\bar{Q}_p.\bar{Q}_p + G^Q_{nn}\bar{Q}_n.\bar{Q}_n + G^Q_{pn}\bar{Q}_p.\bar{Q}_n \qquad \text{(B.5)}$$

$$\left\langle (j_a j_b)J|H^P_\rho|(j_c j_d)J\right\rangle^{\text{a.s.m}} = \delta_{ab}\delta_{cd}\delta_{J=0} \times \sqrt{(2j_a+1)(2j_c+1)}(-1)^{\ell_a-\ell_c};\ \rho = p \text{ or } n \qquad \text{(B.6)}$$

$$\bar{Q}_{\mu;p} = \sqrt{\frac{16\pi}{5}}\frac{r_p^2}{b_p^2}Y^2_\mu(\theta_p,\phi_p),\ \bar{Q}_{\mu;n} = \sqrt{\frac{16\pi}{5}}\frac{r_n^2}{b_n^2}Y^2_\mu(\theta_n,\phi_n) \qquad \text{(B.7)}$$

As can be seen from (B.6), the pairing hamiltonians are defined in terms of their antisymmetric matrix elements (a.s.m.). The matrix elements of $Q.Q$ hamiltonians are given by,

$$\langle (j_a j_b)J|\bar{Q}.\bar{Q}|(j_c j_d)J\rangle^{\text{n.a.s.m}} = (-1)^{j_a+j_b+J}\left\{\begin{array}{ccc} j_a & j_b & J \\ j_d & j_c & 2 \end{array}\right\} \times \langle j_a||\bar{Q}||j_c\rangle\langle j_b||\bar{Q}||j_d\rangle; \qquad \text{(B.8)}$$

$$\langle N_a \ell_a \tfrac{1}{2} j_a || \bar{Q} || N_c \ell_c \tfrac{1}{2} j_c \rangle = \langle N_a \ell_a || r^2/b^2 || N_c \ell_c \rangle \times$$
$$\sqrt{2j_a+1} \langle j_b \tfrac{1}{2} 20 | j_c \tfrac{1}{2} \rangle \left\{ 1 + (-1)^{\ell_a+\ell_c} \right\} \tag{B.9}$$

$$\begin{aligned}
\left\langle N, \ell || r^2/b^2 || N, \ell \right\rangle &= N + 3/2 \\
\left\langle N, \ell+2 || r^2/b^2 || N, \ell \right\rangle &= -[(N-\ell)(N+\ell+3)]^{1/2} \\
\left\langle N+2, \ell || r^2/b^2 || N, \ell \right\rangle &= -\frac{1}{2}[(N-\ell+2)(N+\ell+3)]^{1/2} \\
\left\langle N+2, \ell+2 || r^2/b^2 || N, \ell \right\rangle &= \frac{1}{2}[(N+\ell+3)(N+\ell+5)]^{1/2} \\
\left\langle N+2, \ell-2 || r^2/b^2 || N, \ell \right\rangle &= \frac{1}{2}[(N-\ell+2)(N-\ell+4)]^{1/2}
\end{aligned} \tag{B.10}$$

$$\langle (j_a j_b) J | \bar{Q}_\rho . \bar{Q}_\rho | (j_c j_d) J \rangle^{\text{a.s.m}} = [(1+\delta_{ab})(1+\delta_{cd})]^{-1/2} \times$$
$$\left\{ \left\langle (j_a j_b) J | \bar{Q}_\rho . \bar{Q}_\rho | (j_c j_d) J \right\rangle^{\text{n.a.s.m}} - \right.$$
$$\left. (-1)^{j_c+j_d+J} \left\langle (j_a j_b) J | \bar{Q}_\rho . \bar{Q}_\rho | (j_d j_c) J \right\rangle^{\text{n.a.s.m}} \right\} \; ; \; \rho = p, n \tag{B.11}$$

$$\langle (j_a j_b) J | \bar{Q}_p . \bar{Q}_n | (j_c j_d) J \rangle^{\text{a.s.m}} = \langle (j_a j_b) J | \bar{Q}_p . \bar{Q}_n | (j_c j_d) J \rangle^{\text{n.a.s.m}} \tag{B.12}$$

In (B.8) - (B.12) n.a.s.m. stands for non antisymmetric matrix elements. The relationship between the strengths (G_p, G_n) and $(G^Q_{pp}, G^Q_{nn}, G^Q_{pn})$ with the standard parameters [Ba-68, Be-69, Br-77] $(\bar{G}_p,\ \bar{G}_n,\ \chi')$ are

$$\begin{aligned}
G_p = \bar{G}_p/4; &\qquad \bar{G}_p = 27/A \\
G_n = \bar{G}_n/4; &\qquad \bar{G}_n = 22/A
\end{aligned} \tag{B.13}$$

$$\begin{aligned}
G^Q_{pp} &= -\frac{\chi'}{2} x_0 \left(\frac{5}{16\pi} \right) \alpha_p^2 \\
G^Q_{nn} &= -\frac{\chi'}{2} x_0 \left(\frac{5}{16\pi} \right) \alpha_n^2 \\
G^Q_{pn} &= -\chi' x_0 \left(\frac{5}{16\pi} \right) \alpha_p \alpha_n
\end{aligned}$$

$$\alpha_p = (2Z/A)^{1/3};\ \alpha_n = (2N/A)^{1/3};\ x_0 \simeq 2$$
$$\chi' \simeq 242 A^{-5/3}\ MeV$$
$$\text{or} \simeq \frac{242}{A^{5/3}}\left\{1 + \left(\frac{3A}{2}\right)^{-1/3}\right\}\ MeV \simeq 280\, A^{-5/3} MeV \qquad \text{(B.14)}$$

The definition of Q.Q interaction given in (B.5 - B.12) is somewhat different from the form given in [Ba-68, Be-69]. The value of x_0 in (B.14) is verified by calculating variances of Q.Q interaction in *ds* - shell examples and comparing them with the variances produced by CW [Va-77] interaction. Similar checks are made for a heavy nucleus by comparing with variances obtained with SDI. It is useful to mention that for the pairing interaction $V_{\alpha\beta}(pn) = 0$ and $\zeta^1_\alpha([\mathrm{m}]) = 0$ for all α's and [m]'s. Similarly $V_{\alpha\beta}(Q.Q - \mathrm{pn}) = 0$ for all (α, β).