Appendix I

CERTAIN MATRIX THEOREMS

(A.1.1) If the elements of a matrix $X{:}p\times n$ $(n \geqslant p)$ have density function $f(X)$, then XX' is symmetric (sy.) positive definite (p.d.).

(A.1.2) If $X{:}p\times n$ is a matrix of rank t $(t \leq p \leq n)$, then XX' is sy. positive semi-definite (p.s.d.) and if $t=p$, XX' is sy. p.d. [See (1), (18), (83).]

(A.1.3) If $A{:}n\times n$ is a non-singular matrix such that its leading principal minor of order $k=1,2,\ldots,p$ is nonzero, then

$$a_{ij} = \sum_{k=0}^{r-1} \frac{\left|A^{k+2,\ldots\ldots,n}_{k+1,\ldots,j-1,j+1,\ldots,n}\right| \cdot \left|A^{k+1,\ldots,i-1,i+1,\ldots,n}_{k+2,\ldots\ldots,n}\right|}{\left|A^{k+2,\ldots,n}_{k+2,\ldots,n}\right| \cdot \left|A^{k+1,\ldots,n}_{k+1,\ldots,n}\right|} + \left|A^{r+1,\ldots,i-1,i+1,\ldots,n}_{r+1,\ldots,j-1,j+1,\ldots,n}\right| \Big/ \left|A^{r+1,\ldots,n}_{r+1,\ldots,n}\right| ,$$

for $i>r$, $j>r$ and $r=0,1,2,\ldots,p$. [See (1)]

(A.1.4)(a) If $A{:}p\times p$ is sy. p.d., then $A=\tilde{T}\tilde{T}'$ where $t_{ii}>0$

$$\& \; t_{ji} = \left|A^{i+1,\ldots\ldots,p}_{1,\ldots,j-1,j+1,\ldots,p}\right| \Big/ \left\{\left|A^{i,\ldots,p}_{1,\ldots,p}\right| \cdot \left|A^{i+1,\ldots,p}_{i+1,\ldots,p}\right|\right\}^{\frac{1}{2}}$$

for $i=1,2,\ldots,j$ & $j=1,2,\ldots,p$. Similarly $A=\tilde{T}'\tilde{T}$ where $t_{ii}>0$ and the values of t_{ij}'s can be established.

(b) If
$$A=\begin{array}{cc} \begin{pmatrix} A_{11} & A_{12} \\ A'_{12} & A_{22} \end{pmatrix} & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} p & \quad q \end{array} & \end{array} = \begin{array}{cc} \begin{array}{c} p \\ q \end{array} & \begin{pmatrix} \tilde{T}_1 & 0 \\ T_2 & \tilde{T}_3 \end{pmatrix} \\ & \begin{array}{cc} p & \quad q \end{array} \end{array} \begin{pmatrix} \tilde{T}'_1 & T'_2 \\ 0 & \tilde{T}'_3 \end{pmatrix},$$

then $A_{11}=\tilde{T}_1\tilde{T}'_1$, $T_2\tilde{T}'^{-1}_1=A'_{12}A^{-1}_{11}$, $\tilde{T}_3\tilde{T}'_3=A_{22}-A'_{12}A^{-1}_{11}A_{12}$. Similarly

if $A=\begin{pmatrix} \tilde{T}_1 & 0 \\ T_2 & \tilde{T}_3 \end{pmatrix}' \begin{pmatrix} \tilde{T}_1 & 0 \\ T_2 & \tilde{T}_3 \end{pmatrix}$, then $\tilde{T}'_1\tilde{T}_1=A_{11}-A_{12}A^{-1}_{22}A'_{12}$,

$T'_2\tilde{T}'^{-1}_3=A_{12}A^{-1}_{22}$ & $A_{22}=\tilde{T}'_3\tilde{T}_3$. [See (79,80)].

Proof:- Proof of this result is given in the references, but a different proof based on (A.1.3) is indicated here.

(a) In (A.1.3), let r=i-1 and n=p &

$$t_{j,k+1} = \left|A^{k+2,\ldots\ldots\ldots\ldots\ldots,p}_{k+1,\ldots,j-1,j+1,\ldots,p}\right| \Big/ \left\{\left|A^{k+1,\ldots,p}_{k+1,\ldots,p}\right| \cdot \left|A^{k+2,\ldots,p}_{k+2,\ldots,p}\right|\right\}^{\frac{1}{2}}$$

and since A is sy. p.d., we have

$a_{ij}=\sum_{k=1}^{i} t_{jk}\,t_{ik}$ for $j \geq i$, and so $A=\tilde{T}\tilde{T}'$ where $\tilde{T}$ is a triangular matrix, $t_{ii}>0$, $t_{ji}=0$ for $j<i$ & $\neq 0$ for $j \geq i$.

In a similar manner, we can prove $A=\tilde{T}'\tilde{T}$

(b) Under the given conditions, we have obviously

$A_{11}=\tilde{T}_1\tilde{T}'_1$, $A_{12}=\tilde{T}_1T'_2$ and $A_{22}=T_2T'_2+\tilde{T}_3\tilde{T}'_3$. Hence

$A^{-1}_{11}A_{12}=(T_2\tilde{T}^{-1}_1)'$ and $\tilde{T}_3\tilde{T}'_3=A_{22}-A'_{12}A^{-1}_{11}A_{12}$. Similarly the other part of (A.1.4b) can be proved.

(A.1.5) If A is sy. p.s.d. of rank r, $A=\begin{array}{cc} \begin{pmatrix} A_1 & A_2' \\ A_2 & A_3 \end{pmatrix} & \begin{array}{l} r \\ p-r \end{array} \\ \begin{array}{cc} r & p-r \end{array} & \end{array}$

and rank $A_1=r$, then $A=\begin{array}{c} r \\ p-r \end{array}\begin{pmatrix} \tilde{T}_1 \\ T_2 \end{pmatrix}(\tilde{T}_1' \ \ T_2')$ where

$T_2=A_2\tilde{T}_1'^{-1}$. [See (80),(59)] .

This can also be proved from (A.1.3) noting that all the principal minors of order greater than r are zero and all the principal minors of order less than or equal to r are positive or zero with at least one of them as positive.

(A.1.6)(a) If A:pxp is a sy. matrix of rank r, then there exists a semi-orthogonal matrix Δ:pxr=$(\underset{1}{\underline{\delta}}_1,\ldots,\underset{1}{\underline{\delta}}_r)$p such that $A=\Delta D_\alpha \Delta' = \sum_{i=1}^{r} \alpha_i \underline{\delta}_i \underline{\delta}_i'$, $A^s=\sum_{i=1}^{r} \alpha_i^s \underline{\delta}_i \underline{\delta}_i'$ for any s positive integer and D_α:rxr is a diagonal matrix with diagonal elements α_i's as the nonzero roots of A.

[See(1),(18),(83)] .

(b) tr $A^s=\sum_{i=1}^{r}\alpha_i^s$ for any s positive integer and A to be sy. matrix, α_i's nonzero roots of A.

(c) If A is sy. p.s.d., then nonzero roots of A are all positive. [See (1),(18),(83)] .

(A.1.7) If A:pxp & B:pxp are two sy. matrices of ranks r and s respectively and $AB=BA=0$, then there exists a semi-orthogonal matrix $\Delta: p\times(r+s)$ such that

$$A=\Delta\begin{pmatrix} D_\alpha & 0 \\ 0 & 0 \end{pmatrix}\Delta' \ \& \ B=\Delta\begin{pmatrix} 0 & 0 \\ 0 & D_\beta \end{pmatrix}\Delta' \text{ where } D_\alpha: r\times r \ \& \ D_\beta: s\times s$$

are diagonal matrices, α_i's are the nonzero roots of A & β_i's are the nonzero roots of B.

Proof:- By (A.1.6a), we have $A=\Delta_1 D_\alpha \Delta_1'$ & $B=\Delta_2 D_\beta \Delta_2'$

where $\Delta_1: p\times r$ & $\Delta_2: p\times s$ are two semi-orthogonal matrices, and since $AB=0$, we have $\Delta_1 D_\alpha \Delta_1' \Delta_2 D_\beta \Delta_2'=0$ i.e. $\Delta_1'\Delta_2=0$.

Hence $\Delta=(\Delta_1\ \Delta_2): p\times(r+s)$ is semi-orthogonal, and so we can write A & B as given in (A.1.7).

(A.1.8) If $A: p\times n$ is a matrix of rank r and $\alpha_i^2, i=1,2,\ldots,r$ are the nonzero roots of AA', then there exist two semi-orthogonal matrices $\Delta: p\times r$ and $\Sigma: n\times r$ such that $A=\Delta D_\alpha \Sigma'$ where D_α is a diagonal matrix.

Proof:- By (A.1.2), AA' is sy.p.s.d. of rank r and so by (A.1.6c) there exists a semi-orthogonal matrix $\Delta: p\times r$ such that $AA'=\Delta D_\alpha^2 \Delta'$ and $(\Delta_1'A)(\Delta_1'A)'=0$ where $\Delta_1: p\times(p-r)$ of rank(p-r) is semi-orthogonal and $\Delta_1'\Delta=0$ i.e. $\Delta_1'A=0$.

$$\text{Also } (D_\alpha^{-1}\Delta'A)(D_\alpha^{-1}\Delta'A)'=I_r \text{ i.e. } D_\alpha^{-1}\Delta'A=\Sigma' \text{(say)}$$

: $r\times n$ is semi-orthogonal and so $\Delta'A=D_\alpha\Sigma'$.

Hence $(\Delta \quad \Delta_1)' A = (D_\alpha \quad 0)' \Sigma'$ and so $A = \Delta D_\alpha \Sigma'$.

(A.1.9) If A:nxn & B:nxn are sy. matrices and L:pxn and M:pxn are any two matrices such that AB=0, LB=MA=0 and LM'=0, then there exists a semi-orthogonal matrix Δ: nx (t+u) such that $A=\Delta\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}\Delta'$, $B=\Delta\begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix}\Delta'$, $L=(T \quad 0)\Delta'$ and $M=(0 \quad U)\Delta'$ where C:txt and E:uxu are two sy.matrices, T:pxt and U:pxu are two matrices, t=rank $(A \quad L')$ and u= rank $(B \quad M')$.

Proof:- Since A and B are sy. matrices of ranks r and s respectively and AB=0, then by (A.1.7), there are semi-orthogonal matrices Q_1:nxr and Q_2:nxs such that

(A.1.9.1) $\quad Q_1Q_2'=0, \quad A=Q_1D_\alpha Q_1'$ and $B=Q_2D_\beta Q_2'$ where D_α:rxr and D_β:sxs are non-singular diagonal matrices, and

(A.1.9.2) $$\text{rank}\,(A \quad L') = \text{rank}\,(Q_1 \quad L')\begin{pmatrix} D_\alpha Q_1' & 0 \\ 0 & I \end{pmatrix}$$
$$= \text{rank}\,(Q_1 \quad L') = t,$$

for $$\text{rank}\begin{array}{cc} \begin{pmatrix} D_\alpha Q_1' & 0 \\ 0 & I \end{pmatrix} & \begin{matrix} r \\ p \end{matrix} \\ \begin{matrix} n \quad\quad & p \end{matrix} & \end{array} = (r+p)\left[\leq(n+p)\right].$$ [See (79)].

(A.1.9.3) Also rank $(B \quad M')$ = rank $(Q_2 \quad M')$ = u.

Hence the given conditions are written as

(A.1.9.4) $$\begin{pmatrix} Q_1' \\ L \end{pmatrix} (Q_2 \quad M') = 0.$$

Now applying (A.1.8), we write

(A.1.9.5) $(Q_1 \quad L')=\Delta_1 D_\lambda(P_{11} \quad P_{12})$ and $(Q_2 \quad M')=$

$\Delta_2 D_\delta(P_{21} \quad P_{22})$ where

Δ_1:nxt, $(\underset{r}{P_{11}} \quad \underset{p}{P_{12}})t$, Δ_2:nxu, $(\underset{s}{P_{21}} \quad \underset{p}{P_{22}})u$ are all semi-orthogonal matrices and D_λ:txt and D_δ:uxu are non-singular diagonal matrices.

Hence by using the conditions (A.1.9.4), we have

(A.1.9.6) $$\Delta_1' \Delta_2 = 0 \quad \text{or} \quad \Delta_2' \Delta_1 = 0.$$

i.e. $n(\underset{t}{\Delta_1} \quad \underset{u}{\Delta_2})=\Delta$ is a semi-orthogonal matrix and substituting the values of Q_1, Q_2 in (A.1.9.1), we can write A, B, L and M as stated in (A.1.9).

(A.1.10) If X:nxm, Y:nxp are two matrices such that XX'=YY', then there exists an orthogonal matrix (i) Δ :mxm such that $X\Delta=(\underset{p}{Y} \quad \underset{m-p}{0})n$ if $m \geq p$ or (ii) Δ:pxp such that $Y\Delta=(\underset{m}{X} \quad \underset{p-m}{0})n$ if $p \geq m$.

Proof:- We shall prove (A.1.10) for only one case, namely $m \geq p$. The other case can similarly be proved.

By using (A.1.8), we can write

$$X=Q\begin{pmatrix} D_\lambda & 0 \\ 0 & 0 \end{pmatrix} P \quad \text{and} \quad Y=Q\begin{pmatrix} D_\lambda & 0 \\ 0 & 0 \end{pmatrix} R \quad \text{where}$$

$Q{:}n\times n$, $P{:}m\times m$ and $R{:}p\times p$ are orthogonal matrices and $D_\lambda{:}r\times r$ is a diagonal matrix, $r=\text{rank } XX'$ and λ_i's are the nonzero roots of $(XX'=YY')$.

Let $m \geqslant p$. Then

$$\begin{array}{c c} \begin{array}{c} r \\ m \end{array} & \begin{pmatrix} D_\lambda & 0 \\ 0 & 0 \end{pmatrix} \\ & \begin{array}{cc} r & m-r \end{array} \end{array} = Q'XP'=Q'(Y \quad 0)\begin{array}{c c} \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix} & \begin{array}{c} p \\ m-p \end{array} \\ \begin{array}{cc} p & m-p \end{array} & \end{array}, \text{ and so}$$

$$\text{if} \quad P'\begin{pmatrix} R' & 0 \\ 0 & I \end{pmatrix} = \Delta, \text{ then } X\Delta = (Y \quad 0).$$

(A.1.11) If $X{:}p\times n$ is a matrix of rank r such that

$$XX' = \begin{array}{c c} \begin{array}{c} r \\ p-r \end{array} & \begin{pmatrix} \tilde{T}_1 \\ T_2 \end{pmatrix} \\ & r \end{array} (\tilde{T}_1' \quad T_2'), \text{ then there exists a semi-orthogonal}$$

matrix $\Delta{:}n\times r$ such that $X=\begin{pmatrix} \tilde{T}_1 \\ T_2 \end{pmatrix}\Delta'$. [See (79),(80)].

This follows from (A.1.10).

(A.1.12) If $A{:}n\times n$ is sy. and idempotent of rank r, then $A=\Delta\Delta'$ where $\Delta{:}n\times r$ is a semi-orthogonal matrix.

Proof:- Since A is idempotent, then nonzero roots of A are all unity and using (A.1.6), $A=\Delta D_\alpha \Delta'$ where $\Delta{:}n\times r$ is a semi-orthogonal matrix and α's are the nonzero roots of A, for A is sy. Hence it follows that $A=\Delta\Delta'$, by using the property of idempotent matrix.

(A.1.13) Let $A_1, A_2, \ldots, A_m$ be a collection of nxn sy. matrices, rank of $A_i = q_i$ $i=1,2,\ldots,m$ and rank $(A=\sum_{i=1}^{m} A_i)$ $=q$. Then from the four conditions:

c_1: each A_i is an idempotent matrix;

c_2: $A_i A_j = 0$ for all $i \neq j$;

c_3:- A is idempotent and c_4: $q = \sum_{i=1}^{m} q_i$;

(a) if any two of the conditions c_1, c_2, c_3 are satisfied, then remaining conditions are satisfied and (b) if c_3 and c_4 conditions are satisfied, then c_1 and c_2 are also satisfied. [See (25)].

Proof:- We shall give here a different proof.

(a) (i) Let c_1 and c_2 be given. Then by (A.1.12),

$A_i = \Delta_i \Delta_i'$ where $\Delta_i : nxq_i$ is semi-orthogonal and $\Delta_i' \Delta_j = 0$ for $i \neq j$ (due to condition c_2). Hence

$\Delta = (\Delta_1, \ldots, \Delta_m) : nx\sum_{i=1}^{m} q_i$ is semi-orthogonal and $A = \Delta\Delta'$.

Hence c_3 and c_4 follow immediately.

(ii) Let c_1 and c_3 be given. Then by (A.1.12),

$A_i = \Delta_i \Delta_i'$ and $A = \Delta\Delta'$ where $\Delta : nxq$ and $\Delta_i : nxq_i$ $i=1,2,\ldots,m$ are semi-orthogonal. Let $\Sigma = (\Delta_1, \ldots, \Delta_m)$ and so $A = \Sigma\Sigma' = \Delta\Delta'$. Also rank $A = q \leq \sum_i q_i$. Therefore by (A.1.10), $\Sigma Q = (\underset{q}{\Delta} \quad \underset{\sum_i q_i - q}{0})n$ where $Q: \sum_i q_i x \sum_i q_i$ is an orthogonal matrix.

Let $Q': \sum_i q_i \times \sum_i q_i = (Q'_1, \ldots, Q'_m) \sum_i q_i$ and
$ \qquad\qquad q_1 \qquad q_m$

$Q_i: q_i \times \sum_i q_i = (Q_{1i} \quad Q_{2i})\ q_i$. Then $\sum Q = (\Delta \quad 0)$ gives
$\qquad\qquad\quad q \quad \sum_i q_i - q$

$\Delta_i = \Delta Q'_{1i} = (\Delta \quad 0) Q'_i$ for $i=1,2,\ldots,m$.

Hence $I_{q_i} = \Delta'_i \Delta_i = Q_{1i} Q'_{1i}$. Also since Q is an orthogonal matrix, Q_i is semi-orthogonal and so $Q_i Q'_i = Q_{1i} Q'_{1i} + Q_{2i} Q'_{2i} = I_{q_i}$, but $Q_{1i} Q'_{1i} = I_{q_i}$. Hence $Q_{2i} Q'_{2i} = 0$ i.e. $Q_{2i} = 0$ for $i=1,2,\ldots,m$. This is impossible, unless $q = \sum_i q_i$. Hence c_4 is proved and c_2 is proved from $\Delta_i = \Delta Q'_i$ or $\Delta'_i \Delta_j = Q_i Q'_j = 0$ for $i \neq j$.

(iii) Let c_2 and c_3 be given. Then by (A.1.7), $A_i = \Delta_i D_{\alpha_i} \Delta'_i$ where $\Delta = (\Delta_1, \ldots, \Delta_m)$ is semi-orthogonal and $D_{\alpha_i}: q_i \times q_i$ is a non-singular diagonal matrix. Also $A = \sum_i A_i = \Delta D \Delta'$ where $D = \text{diagonal}\ (D_{\alpha_1}, \ldots, D_{\alpha_m})$, but A is idempotent, hence $D^2 = D$ i.e. $D = I$ for D is non-singular. Hence c_1 and c_4 are satisfied.

(b) Let c_3 and c_4 be given. Since A_i is symmetric and A is sy.idempotent, then by (A.1.12) and (A.1.6), we have $A_i = \Delta_i D_{\alpha_i} \Delta'_i$ where $\Delta_i: n \times q_i$ $i=1,2,\ldots,m$ is semi-orthogonal and $D_{\alpha_i}: q_i \times q_i$ is non-singular diagonal and $A=\Delta\Delta'$ where $\Delta: n\times q$ is semi-orthogonal. Letting $\Sigma=(\Delta_1,\dots,\Delta_m)$ and $D=\text{diag.}(D_{\alpha_1},\dots,D_{\alpha_m})$. Then by definition of A, $A=\Sigma D\Sigma'$ and also $A=\Delta\Delta'$. Now by c_4, we have rank $\Sigma=$ rank $\Sigma D\Sigma'=q=\sum q_i$ i.e. Σ is a non-singular matrix.

Now $\Sigma D\Sigma' = \Delta\Delta'$ i.e.

$D=\{(\Sigma'\Sigma)^{-1}\Sigma'\Delta\}\{(\Sigma'\Sigma)^{-1}\Sigma'\Delta\}'$ is positive definite by (A.1.2). Hence let $D=D_1^2$ and so using (A.1.10), $\Sigma D_1 Q=\Delta$ where $Q: q\times q$ is an orthogonal matrix, or $\Sigma D_1=\Delta Q'=$a semi-orthogonal matrix, i.e. $\Delta_i D_{1_{\alpha_i}}$ is semi-orthogonal where $D_1=\text{diag}(D_{1_{\alpha_1}},\dots,D_{1_{\alpha_m}})$,

But $\Delta_i'\Delta_i=I_{q_i}$ and by above $D_{1\alpha_i}\Delta_i'\Delta_i D_{1\alpha_i}=I_{q_i}$ i.e. $D_{\alpha_i}=I_{q_i}$ for $D^2_{1_{\alpha_i}}=D_{\alpha_i}$. i.e. $D=I$ and so Σ is semi-orthogonal. Hence c_1 and c_2 must hold good.

(A.1.14) $$\begin{array}{c} \\ m \\ n \\ \end{array}\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A| \cdot |D-CA^{-1}B| \text{ if } A \text{ is non-singular}$$
$$\quad\; m \quad n \qquad = |D| \cdot |A-BD^{-1}C| \text{ if } D \text{ is non-singular.}$$

[See (79)].

(A.1.15) $|A+\theta I| = \sum_{i=1}^{p} \theta^i \operatorname{tr}_{p-i} A$ where $A: p\times p$ is a matrix and $\operatorname{tr}_j A=$ the sum of principal minors of order j in A, θ

is any value. [See (2),(18),(83)].

(A.1.16) If $A:m\times n$ $(n \geq m)$, $B:n\times m$ are two matrices, then

(a) $tr_i\, AB = tr_i\, BA$ for $i=1,2,\ldots,m$ and

$tr_i BA = 0$ for $i=m+1,\ldots,n$;

(b) the roots of AB are the roots of BA ex-cept for some zero roots. [See (79),(83)].

Proof:- For any nonzero θ, $\begin{vmatrix} I_m\,\theta & -A \\ B & I_n \end{vmatrix} = |AB+\theta I_m|$

$= \theta^{n-m}|BA+\theta I_n|$

by using (A.1.14); and then using (A.1.15) and equating the coefficients of θ^{i}'s for $i \geq 1$, we have the result as stated in (a). Then (b) follows from (a) by using (A.1.15).

(A.1.17) If $R:m\times m$ is any matrix, then

(a) $|I-\theta R|^{-p} = \sum_{i=0}^{\infty} b_i\theta^i = \exp(p\sum_{j=1}^{\infty}\theta^j trR^j/j)$ if θ is such that expansion is valid and $b_0=1$,

$$b_i = \sum_{j=1}^{i}(pj+i-j)(-1)^{j-1}b_{i-j}tr_jR/i \quad \text{for } i \geq 1;$$

(b) $tr_iR = \sum_{t=1}^{i}(-1)^{t-1}(trR^t)(tr_{i-t}R)/i$ & $tr_0R=1$;

(c) if in (b), $trR^t=p$ for $t=1,2,\ldots,m$, then

$tr_i R = p(p-1)\ldots(p-i+1)/i!$ for $i=1,2,\ldots,m$.

Proof:- (a) Let b_t be the coefficient of θ^t in the expansion of $(1-\sum_{i=1}^{\infty} a_i\theta^i)^{-p}$. Then

$$b_t=\sum_{i=1}^{t} \sum_{\pi's} \Gamma(p+i) \prod_{j=1}^{\infty} (a_j^{\pi_j}/ \pi_j!)/ \Gamma(p) \text{ for } t \geq 1 \ \& \ b_o=1,$$

where $\sum_{\pi's}$ denotes the summation over $\pi_1, \pi_2,\ldots$ such that $\sum_{j=1}^{\infty} \pi_j=i$ and $\sum_{j=1}^{\infty} j\,\pi_j=t$. Now it is easy to verify that

$$b_1=pa_1, b_2=\tfrac{1}{2}(p+1)a_1b_1+pa_2, b_3=\tfrac{1}{3}(p+2)b_2a_1+\tfrac{1}{3}(2p+1)a_2b_1+pa_3$$

and so on. In general we can write

$$\text{(A.1.17.1)} \quad b_t=\sum_{i=1}^{t}(pi+t-i)a_ib_{t-i}/t \quad \text{for } t\geq 1 \ \& \ b_o=1.$$

Now by (A.1.15), we have

$$|I-\theta R| = 1- \sum_{i=1}^{\infty} a_i\theta^i \quad \text{where} \quad a_i=(-1)^{i-1}tr_iR.$$

Hence by using (A.1.17.1), we have the expansion of $|I-\theta R|^{-p}$ as stated in (a) for its first part. Also

$$\log|I-\theta R|^{-p} =-p \sum_{i=1}^{r} \log(1-\alpha_i\theta) \text{ where } \alpha_i\text{'s are the}$$

nonzero roots of R. Using $\sum_{i=1}^{r}\alpha_i^j = tr\, R^j$, we have

$$\log |I-\theta R|^{-p} =p \sum_{j=1}^{\infty} \theta^j\, tr\, R^j/j.$$

Hence the part (a) is completely proved.

(b) We have

$$|I-\theta R|=1+\sum_{i=1}^{\infty}(-1)^{i}(tr_{i}R)\theta^{i}=\exp\left(-\sum_{j=1}^{\infty}\theta^{j}trR^{j}/j\right),$$

for some value of θ .

Using the result similar to (5.4.17) and equating the coefficients of θ^{t}, we have the result as stated in (b).

The result (c) follows immediately from (b).

(A.1.18) (a) If $R:m\times m$ is any matrix such that $\sum_{i=0}^{\infty} R^{i}$ is convergent, taking $R^{0}=I$, then $(I-R)^{-1} = \sum_{i=0}^{\infty} R^{i}$.

(b) If $A:p\times p$ is a non-singular matrix, and $B: p\times p$, $Y:p\times n$ and $X:p\times n$ are any matrices, then

$$Y'(A\pm XBY')^{-1}X=Y'A^{-1}X(I\pm BY'A^{-1}X)^{-1}=(I\pm Y'A^{-1}XB)^{-1}Y'A^{-1}X.$$

__Proof:-__ (a) If $\sum_{i=0}^{\infty} R^{i}=P$(say) is convergent, then it is easy to verify that $P(I-R)=(I-R)P=I$.

Hence (a) is proved.

(b) We have $Y'(A\pm XBY')^{-1}X=Y'A^{-1}(I\pm XBY'A^{-1})^{-1}X$

$$=Y'A^{-1}\sum_{i=0}^{\infty}(\pm XBY'A^{-1})^{i}X(-1)^{i} \text{ by using (a)}$$

$$=Y'A^{-1}X\sum_{i=0}^{\infty}(\pm BY'A^{-1}X)^{i}(-1)^{i} .$$

Hence $Y'(A\pm XBY')^{-1}X=Y'A^{-1}X(I\pm BY'A^{-1}X)^{-1}$.

Similarly the other part can be proved.

(A.1.19) (a) If $A=\begin{array}{cc|c} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} & & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} p & q \end{array} & & \end{array}$ is a non-singular

matrix, then $A^{-1}=\begin{pmatrix} A_{11}^{-1}+A_{11}^{-1}A_{12}A_{2.1}^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{2.1}^{-1} \\ -A_{2.1}^{-1}A_{21}A_{11}^{-1} & A_{2.1}^{-1} \end{pmatrix}$

if A_{11} is non-singular and

$A^{-1}=\begin{pmatrix} A_{1.2}^{-1} & -A_{1.2}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}A_{1.2}^{-1} & A_{22}^{-1}+A_{22}^{-1}A_{21}A_{1.2}^{-1}A_{12}A_{22}^{-1} \end{pmatrix}$ if A_{22} is non-

singular where $A_{1.2}=A_{11}-A_{12}A_{22}^{-1}A_{21}$ & $A_{2.1}=A_{22}-A_{21}A_{11}^{-1}A_{12}$.

[See (80),(1)].

(b) If $T=\begin{array}{c|c} \begin{array}{c} p \\ q \end{array} & \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \\ & \begin{array}{cc} p & q \end{array} \end{array}$ is non-singular, then

$T^{-1}=\begin{pmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{pmatrix}$. [See (83),(80)].

(c) If L:nxn is a non-singular matrix, and N:rxn, M:nxr are matrices, then

$(L+MN)^{-1}=L^{-1}-L^{-1}M(I+NL^{-1}M)^{-1}NL^{-1}$. [See (80)].

(d) If D:nxn is a diagonal matrix, $\underline{x}$:nx1 & $\underline{y}$:nx1 are two column vectors & a is any nonzero constant, then

$$(D+a\underline{x}\underline{y}')^{-1}=D^{-1}(D-b\underline{x}\underline{y}')D^{-1} \text{ where } b=a/(1+\underline{y}'D^{-1}\underline{x}a) .$$

[See (81,80)] .

<u>Proof:-</u> (a) This can be proved by verifying $AA^{-1}=A^{-1}A=I$.

(b) This follows from (a) by putting $A_{12}=0$.

(c) $(L+MN)^{-1}=(I+L^{-1}MN)^{-1}L^{-1}=\sum_{i=0}^{\infty}(-1)^i(L^{-1}MN)^iL^{-1}$

by using (A.1.18a), and so

$$(L+MN)^{-1}=L^{-1}-\sum_{i=0}^{\infty} L^{-1}M(NL^{-1}M)^iNL^{-1}(-1)^i$$

$$= L^{-1}-L^{-1}M(I+NL^{-1}M)^{-1} NL^{-1} .$$

(d) This can be proved from (c) by putting $L=D$, $M=a\underline{x}$, $N=\underline{y}'$.

(A.1.20) If $S=\begin{array}{cc} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} p & \;\; q \end{array} & \end{array}$ is a non-singular matrix and

$Y'= \underset{p}{(Y_1'}\ \underset{q}{Y_2')}r$, then

$Y'S^{-1}Y-Y_2'S_{22}^{-1}Y_2=(Y_1-S_{21}'S_{22}^{-1}Y_2)'S_{1.2}^{-1}(Y_1-S_{12}S_{22}^{-1}Y_2)$ if S_{22} is non-singular and $Y'S^{-1}Y-Y_1'S_{11}^{-1}Y_1=(Y_2'-Y_1'S_{11}^{-1}S_{12})S_{2.1}^{-1}(Y_2-S_{21}S_{11}^{-1}Y_1)$ if S_{11} is non-singular and $S_{1.2}=S_{11}-S_{12}S_{22}^{-1}S_{21}$, $S_{2.1}=S_{22}-S_{21}S_{11}^{-1}S_{12}$.

Proof:- By (A.1.19a), we write if S_{22} is non-singular,

$$S^{-1}-\begin{pmatrix} 0 & 0 \\ 0 & S_{22}^{-1} \end{pmatrix} = \begin{pmatrix} I \\ -S_{22}^{-1} S_{21} \end{pmatrix} S_{1.2}^{-1} \quad (I \quad -S_{12}S_{22}^{-1}).$$

Hence $Y'S^{-1}Y-Y_2'S_{22}^{-1} Y_2=(Y_1'-Y_2'S_{22}^{-1} S_{21}) S_{1.2}^{-1} (Y_1-S_{12}S_{22}^{-1} Y_2)$.

Similarly the other part can be proved.

(A.1.21) If $S=\begin{array}{cc} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} p & q \end{array} & \end{array}$ & $W=\begin{array}{cc} \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} p & q \end{array} & \end{array}$ are non-singular matrices, $Q=W-S=\begin{array}{c} p \\ q \end{array}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_r (Y_1' \quad Y_2')=\begin{array}{cc} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{pmatrix} & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} p & q \end{array} & \end{array}$,

$W_{i.j}=W_{ii}-W_{ij}W_{jj}^{-1}W_{ji}$ and $S_{i.j}=S_{ii}-S_{ij}S_{jj}^{-1} S_{ji}$ for $i\neq j$, $i,j=1,2$,

then $W_{i.j}-S_{i.j}=(Y_i-S_{ij}S_{jj}^{-1} Y_j)(I+Y_j'S_{jj}^{-1}Y_j)^{-1}(Y_i'-Y_j'S_{jj}^{-1} S_{ji})$

for $i\neq j$, $i,j=1,2$, and S_{jj}, W_{jj} are non-singular.

Proof:- We shall prove (A.1.21) only for $i=1$ & $j=2$. Now,

we have $P=(Y_1-S_{12}S_{22}^{-1}Y_2) \; (I+Y_2'S_{22}^{-1} Y_2)^{-1}(Y_1'-Y_2'S_{22}^{-1} S_{21})$

$$=(I \quad -S_{12}S_{22}^{-1})Y\left[I+Y'\begin{pmatrix} 0 & 0 \\ 0 & S_{22}^{-1} \end{pmatrix} Y\right]^{-1} Y'(I \quad -S_{21}'S_{22}'^{-1})' .$$

By using (A.1.18b), we have

$$P=(I_p \quad -S_{12}S_{22}^{-1})\begin{pmatrix} I_p & Q_{12}S_{22}^{-1} \\ 0 & I_q+Q_{22}S_{22}^{-1} \end{pmatrix}^{-1} (W-S) \begin{pmatrix} I_p \\ -S_{22}^{-1}S_{21} \end{pmatrix} .$$

Now using (A.1.19b),

$$P=(I_p \quad -S_{12}S_{22}^{-1})\begin{pmatrix} I_p & -Q_{12}W_{22}^{-1} \\ 0 & S_{22}W_{22}^{-1}\end{pmatrix}\begin{pmatrix} W_{11}-W_{12}S_{22}^{-1}S_{21}-S_{1.2} \\ W_{21}-W_{22}S_{22}^{-1}S_{21}\end{pmatrix}$$

i.e. $$P=(I \quad -S_{12}S_{22}^{-1})\begin{pmatrix} W_{1.2}-S_{1.2}+S_{12}W_{22}^{-1}W_{21}-S_{12}S_{22}^{-1}S_{21} \\ S_{22}W_{22}^{-1}W_{21}-S_{21}\end{pmatrix}$$

$$P=W_{1.2}-S_{1.2} \quad .$$

Similarly the other part can be proved.

(A.1.22) In the same notation as (A.1.21), we have

(a) the roots of $(I_r+Y_j'S_{jj}^{-1}Y_j)^{-1}(Y'S^{-1}Y-Y_j'S_{jj}^{-1}Y_j)$ or the roots of $(I_r+Y_j'S_{jj}^{-1}Y_j)^{-1}(Y_i'-Y_j'S_{jj}^{-1}S_{ji})S_{i.j}^{-1}(Y_i-S_{ij}S_{jj}^{-1}Y_j)$ are the roots of $(W_{i.j}S_{i.j}^{-1}-I_p)$ except for some zero roots if S_{jj},W_{jj} are non-singular, $i\neq j$, $i,j=1,2$, and

(b) the roots of $(I+Y'S^{-1}Y)^{-1}(Y'S^{-1}Y-Y_j'S_{jj}^{-1}Y_j)$ are the roots of $(I-S_{i.j}W_{i.j}^{-1})$ except for some zero roots if S_{jj},W_{jj} are non-singular and $i\neq j$, $i,j=1,2$. [See (37)].

Proof:- (a) This follows immediately by considering (A.1.21), (A.1.20) and (A.1.16b).

(b) We shall prove this by taking i=1 & j=2.

By applying (A.1.20) and (A.1.16b), we have the roots of $P=(I+Y'S^{-1}Y)^{-1}(Y'S^{-1}Y-Y_2'S_{22}^{-1}Y_2)$ are the roots of $R=(Y_1-S_{12}S_{22}^{-1}Y_2)(I+Y'S^{-1}Y)^{-1}(Y_1'-Y_2'S_{22}^{-1}S_{21})S_{1.2}^{-1}$ and applying (A.1.18b), we have

$$R=(I \quad -S_{12}S_{22}^{-1})S\,W^{-1}(W-S)(I \quad -S_{21}'S_{22}'^{-1})' .$$

$$=(S_{1.2} \quad 0)\,W^{-1}\begin{pmatrix} W_{11}-W_{12}S_{22}^{-1}S_{21}-S_{1.2} \\ W_{21}-W_{22}S_{22}^{-1}S_{21} \end{pmatrix}$$

$$=(S_{1.2}W_{1.2}^{-1} \quad -S_{1.2}W_{1.2}^{-1}W_{12}W_{22}^{-1})\begin{pmatrix} W_{11}-W_{12}S_{22}^{-1}S_{21}-S_{1.2} \\ W_{21}-W_{22}S_{22}^{-1}S_{21} \end{pmatrix}$$

$$= S_{1.2}(I-W_{1.2}^{-1}S_{1.2}) .$$

Similarly, the other part for i=2 & j=1 can be proved.

Hence, we have the result (A.1.22).

(A.1.23) If A:pxp is sy.p.d. and B:pxp is sy.and at least p.s.d., then for all non-null $\underline{d}$:px1, (i) $\underline{d}'B\underline{d}/\underline{d}'A\underline{d}$ is non-negative, (ii) the stationary values of $\underline{d}'B\underline{d}/\underline{d}'A\underline{d}$ (under variation of $\underline{d}$) are the roots of the determinantal equation in θ: $|B-A\theta|=0$ and (iii) in particular, the largest and the smallest (nonzero) values of $\underline{d}'B\underline{d}/\underline{d}'A\underline{d}$ are respective the largest and the smallest (nonzero) root of the determinantal equation. [See (79)].

(A.1.24) If A:pxp is sy.p.d. and B:pxp is sy.and at least p.s.d., the statement, "$g_1 \leq \underline{d}'B\underline{d}/\underline{d}'A\underline{d} \leq g_2$ for all non-null $\underline{d}$:px1" is exactly equivalent to "$g_1 \leq \theta_1 \leq \theta_p \leq g_2$" where θ_1 and θ_p stand for ~~all~~ the smallest and the largest roots of the equation in θ: (all positive) $|B-A\theta|=0$. [See (79)].

(A.1.25) Let $B=\underline{x}\,\underline{x}'$ ($\underline{x}$:px1) and A:pxp is sy.p.d., then the statement "$(\underline{x}'\underline{d})^2 \leq g_2$ $(\underline{d}'A\underline{d})$ for all non-null $\underline{d}$:px1" is exactly equivalent to "$\underline{x}'A^{-1}\underline{x} \leq g_2$".

This follows from (A.1.24).

(A.1.26) If A:pxp is sy.p.s.d. and B:pxp is sy.p.s.d., then

(i) $\max.\left[\theta_{max}^A\ \theta_{min}^B,\ \theta_{min}^A\ \theta_{max}^B\right] \leq \theta_{max}^{AB} \leq \theta_{max}^A\ \theta_{max}^B$ and

(ii) $\theta_{min}^A\ \theta_{min}^B \leq \theta_{min}^{AB} \leq \min.\left[\theta_{max}^A\ \theta_{min}^B,\ \theta_{min}^A\ \theta_{max}^B\right]$

where θ_{max}^M and θ_{min}^M stand respectively for the largest and the smallest roots of M. [See (79)].

Proof:- Since B:pxp & A:pxp are sy.p.s.d. of rank r (say), and s (say) respectively, then by (A.1.6c), we can write $B=YY'$ where Y:pxr is of rank r and it is obvious from (A.1.23) that $\underline{x}'A\underline{x} \leq \theta_{max}^A\ \underline{x}'\underline{x}$, for any $\underline{x}$: px1.

Now for any nonzero $\underline{e}$:rx1,

$$\underline{e}'Y'AY\underline{e} \,/\, \underline{e}'\underline{e} \leq \theta_{max}A \;\; \underline{e}'Y'Y\underline{e}/\underline{e}'\underline{e} \leq \theta_{max}A \;\; \theta_{max}Y'Y \; .$$

Now applying (A.1.24) and (A.1.16b), we have

(A.1.26.1) $\quad$ all $\theta(AB) \leq \theta_{max}A \;\; \theta_{max}B.$

Now if C:pxp is any sy.p.d., then C=ZZ' where Z:pxp is non-singular and using (A.1.16b), we have

$$\text{all } \theta(AB) = \text{all } \theta(Z^{-1}AZ'^{-1}Z'BZ) \leq \theta_{max}Z^{-1}AZ'^{-1} \;\; \theta_{max}Z'BZ.$$

(A.1.26.2) $\quad$ i.e. all $\theta(AB) \leq \theta_{max}\, AC^{-1} \;\; \theta_{max}\, CB.$

(a) If A & B are sy.p.s.d., then the result is proved from (A.1.26.1).

(b) If A is sy.p.d. & B is sy.p.s.d., then from (A.1.26.1) & (A.1.26.2), we have

$$\theta_{max}(AB) \leq \theta_{max}A \;\; \theta_{max}B \text{ and } \theta_{max}B \leq \theta_{max}A^{-1} \;\; \theta_{max}AB \; .$$

Hence the result (A.1.26) is proved.

Similarly if A is sy.p.s.d. and B is sy.p.d., then result (A.1.26) can be proved.

(c) If A & B are both p.d., the first part follows from (b) and taking the inverses, the second part can be proved. Hence

(A.1.26) is completely proved.

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