Appendix II

CERTAIN JACOBIAN THEOREMS

(A.2.1) The jacobian of the transformation $Y=AXB$ where $A: p\times p$, $B: q\times q$, $X: p\times q$, $Y: p\times q$ is $J(Y;X)=|A|^{q}\,|B|^{p}$ [See (16)].

(A.2.2) The jacobian of the transformation $\tilde{Y}=\tilde{A}\tilde{X}$ $(\tilde{Y},\tilde{A},\tilde{X}: p\times p)$ is $J(\tilde{Y};\tilde{X})=\prod_{i=1}^{p} a_{ii}^{i}$. [See (16,58)].

(A.2.3) The jacobian of the transformation $\tilde{Y}=\tilde{X}\tilde{A}(\tilde{Y},\tilde{A},\tilde{X}: p\times p)$ is $J(\tilde{Y};\tilde{X})=\prod_{i=1}^{p} a_{ii}^{p-i+1}$. [See (16,58)].

(A.2.4) The jacobian of the transformation $S=\tilde{T}\tilde{T}'(S,\tilde{T}: p\times p)$, ($S$ is symmetric) is $J(S;\tilde{T})=2^{p}\prod_{i=1}^{p} t_{ii}^{p-i+1}$. [See (16,58)].

(A.2.5) The jacobian of the transformation $S=\tilde{T}'\tilde{T}(\tilde{T},S: p\times p$, S is symmetric) is $J(S;\tilde{T})=2^{p}\prod_{i=1}^{p} t_{ii}^{i}$. [See (16,58)].

(A.2.6) The jacobian of the transformation $S=GRG'(S,R: p\times p$ symmetric and $G: p\times p)$ is $J(S;R)=|G|^{p+1}$. [See (16,58)].

(A.2.7) The jacobian of the transforma-tion $G=(I-S)^{-1}$ ($G,S: p\times p$ are symmetric) is $J(G;S)=|I-S|^{-(p+1)}$. [See (58,59)].

(A.2.8) Let $Y=F(X)$ be a matrix transformation. Then the transformation in the differentials $Y^{*}=\{F(X)\}^{*}$ is linear and $J(Y;X)=J(Y^{*};X^{*})$. [See (16)].

(A.2.9) If the matrix transformation is $Y=F(X,Z,V)$, $Z=\psi(X,W,V)$ and $\Phi(X,W,V)=0$, then the jacobian of the transformation $J(Y,Z;X,W)=\left|\frac{\partial(F,\Psi,\Phi)}{\partial(X,W,V)}\right| / \left|\frac{\partial\Phi}{\partial V}\right|$ where $\frac{\partial F}{\partial W}=0$.

Proof:- Let X and Y, Z and W have variables $x_1,\ldots,x_p$ and $y_1,y_2,\ldots,y_p$; $z_1,\ldots,z_q$ and $w_1,\ldots,w_q$ respectively and V has $v_1,v_2,\ldots,v_r$ other variables subject to r equations given by $\Phi(X,W,V)=0$. The equations of transformations are

(A.2.9.1) $y_i=f_i(x_1,\ldots,x_p,z_1,\ldots,z_q,v_1,\ldots,v_r)$ for $i=1,\ldots,p$;

$z_j=\psi_j(x_1,\ldots,x_p,w_1,\ldots,w_q,v_1,\ldots,v_r)$ for $j=1,\ldots,q$

and $\phi_k(x_1,\ldots,x_p,w_1,\ldots,w_q,v_1,\ldots,v_r)=0$ for $k=1,\ldots,r$.

(A.2.9.2) Let $\underline{x}^{*\prime}:1\text{x}p=(x_1^*,\ldots,x_p^*)$, $\underline{y}^{*\prime}=(y_1^*,\ldots,y_p^*):1\text{x}p$, $\underline{w}^{*\prime}=(w_1^*,\ldots,w_q^*):1\text{x}q$, $\underline{z}^{*\prime}=(z_1^*,\ldots,z_q^*):1\text{x}q$, $\underline{v}^{*\prime}:1\text{x}r=(v_1^*,\ldots,v_r^*)$ and for $i,i'=1,2,\ldots p$; $j,j'=1,2,\ldots,q$ and $k,k'=1,2,\ldots,r$, let $R_{11}:p\text{x}p=\left(\frac{\partial f_i}{\partial x_{i'}}\right)$, $R_{12}:p\text{x}q=\left(\frac{\partial f_i}{\partial z_{j'}}\right)$, $R_{13}:p\text{x}r=\left(\frac{\partial f_i}{\partial v_{k'}}\right)$, $R_{21}:q\text{x}p=\left(\frac{\partial \psi_j}{\partial x_{i'}}\right)$,

$R_{22}:q\text{x}q=\left(\frac{\partial \psi_j}{\partial w_{j'}}\right)$, $R_{23}:q\text{x}r=\left(\frac{\partial \psi_j}{\partial v_{k'}}\right)$, $R_{31}:r\text{x}p=\left(\frac{\partial \phi_k}{\partial x_{i'}}\right)$,

$R_{32}:r\text{x}q=\left(\frac{\partial \phi_k}{\partial w_{j'}}\right)$, $R_{33}:r\text{x}r=\left(\frac{\partial \phi_k}{\partial v_{k'}}\right)$.

Now, taking the differentials of (A.2.9.1), we can write them with the help of (A.2.9.2) as

(A.2.9.3) $\underline{y}^*=R_{11}\underline{x}^* + R_{12}\,\underline{z}^* + R_{13}\,\underline{v}^*$;

(A.2.9.4) $\quad z^* = R_{21}\, x^* + R_{22}\, w^* + R_{23}\, y^*$ and

(A.2.9.5) $\quad R_{31}\, x^* + R_{32}\, w^* + R_{33}\, y^* = 0.$

Considering R_{33} to be non-singular, equation (A.2.9.5) can be written as $y^* = -R_{33}^{-1} R_{31}\, x^* - R_{33}^{-1} R_{32}\, w^*$.

Hence substituting this value in (A.2.9.3) and (A.2.9.4), we finally arrive at

$$z^* = (R_{21} - R_{23}R_{33}^{-1}R_{31})\, x^* + (R_{22} - R_{23}R_{33}^{-1}R_{32})\, w^* \text{ and then}$$

$$y^* = \left\{R_{11} - R_{13}R_{33}^{-1}R_{31} + R_{12}(R_{21} - R_{23}R_{33}^{-1}R_{31})\right\} x^* +$$
$$\left\{R_{12}(R_{22} - R_{23}R_{33}^{-1}R_{32}) - R_{13}R_{33}^{-1}R_{32}\right\} w^* .$$

Now by (A.2.8), $J(y,z;x,w) = J(y^*,z^*;x^*,w^*)$ and by (A.2.1)

$$J(y,z;x,w) = \begin{vmatrix} R_{1.3} + R_{12}(R_{21} - R_{23}R_{33}^{-1}R_{31}) & R_{12}R_{2.3} - R_{13}R_{33}^{-1}R_{32} \\ R_{21} - R_{23}R_{33}^{-1}R_{31} & R_{2.3} \end{vmatrix}$$

where $R_{1.3} = R_{11} - R_{13}R_{33}^{-1}R_{31}$ and $R_{2.3} = R_{22} - R_{23}R_{33}^{-1}R_{32}$. That is

$$J(y,z;x,w) = \begin{vmatrix} R_{1.3} & -R_{13}R_{33}^{-1}R_{32} \\ R_{21} - R_{23}R_{33}^{-1}R_{31} & R_{2.3} \end{vmatrix} = \begin{vmatrix} R_{11} & 0 & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{vmatrix} \Big/ |R_{33}| .$$

This proves the result (A.2.9).

(A.2.10) If we have in (A.2.9), no relation for V, then the jacobian of the transformation $Y = F(X,Z)$ and $Z = \psi(X,W)$

is $J(Y,Z;X,W)=J(Y;X)\,J(Z;W)$.

(A.2.11) If in (A.2.9), we have no transformation of Z, then the jacobian of the transformation $Y=F(X,V)$ and $\Phi(X,V)=0$ is $J(Y;X)=\left|\frac{\partial(F,\Phi)}{\partial(X,V)}\right| \Big/ \left|\frac{\partial\Phi}{\partial V}\right|$. [See (78,79)].

(A.2.12) The jacobian of the transformation $X_j=F_j(X_{j+1},Z_j)$ and $Z_j=\psi_j(X_{j+1},Z_{j+1})$ for $j=1,2,\ldots,k$ is

$$J(X_1,Z_1;X_{k+1},Z_{k+1})=\prod_{j=1}^{k} J(X_j;X_{j+1})\;J(Z_j;Z_{j+1}) .$$

This follows from (A.2.10) .

(A.2.13) If in (A.2.12), we have no transformation for Z, then the jacobian of the transformation $X_j=F_j(X_{j+1})$ $j=1,2,\ldots,k$, then $J(X_1;X_{k+1}) = \prod_{j=1}^{k} J(X_j;X_{j+1})$. [See (16)].

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