Appendix III

CERTAIN INTEGRALS

(A.3.1) If the elements of X:pxn ($n \geq p$) are random variables,

$$\underset{S \leq XX' \leq S+S^*}{\int \cdots \int} dX = \pi^{\{2pn-p(p-1)\}/4} \prod_{i=1}^{p} \left\{ \Gamma\left(\frac{n-i+1}{2}\right) \right\}^{-1} |S|^{(n-p-1)/2} dS.$$

[See (38),(59),(60),(79)].

(A.3.2) If the elements of $\underline{y}$:nx1 are random variables & $\underline{m}$:nx1 is fixed, then

$$\underset{x \leq \underline{y}'\underline{y} \leq x+dx}{\int \cdots \int} e^{2\,\underline{m}'\underline{y}} d\underline{y} = \pi^{n/2} \sum_{r=0}^{\infty} (\underline{m}'\underline{m})^{r} x^{r-1+n/2} dx/r!\, \Gamma(r+n/2).$$

[See (68)].

(A.3.3) Let S:pxp be a symmetric matrix having p(p+1)/2 random variables and X:rxp of rank r($\leq$ p). Then

(a) $$I_1 = \int \cdots \int |XSX'|^{t} |S|^{(n-p-1)/2} e^{-tr\Sigma^{-1}S} dS$$

$$= 2^{p} \int \cdots \int |X\tilde{T}\tilde{T}'X'|^{t} \prod_{i=1}^{p} t_{ii}^{n-i} \exp(-tr\, \Sigma^{-1} \tilde{T}\tilde{T}') d\tilde{T}$$

$$= 2^{p} \int \cdots \int |X\tilde{T}'\tilde{T}X'|^{t} \prod_{i=1}^{p} t_{ii}^{n-p-1+i} \exp(-tr\Sigma^{-1}\tilde{T}'\tilde{T}) d\tilde{T}$$

$$= \pi^{p(p-1)/4} \prod_{i=r+1}^{p} \Gamma\left(\frac{n-i+1}{2}\right) \prod_{j=1}^{r} \Gamma\left(\frac{n-j+1}{2} + t\right) |\Sigma|^{n/2} |X\Sigma X'|^{t}.$$

(b) $$I_2 = \int \cdots \int |XS^{-1}X'|^{t} |S|^{(n-p-1)/2} \exp(-tr\Sigma^{-1}S)\, dS$$

$$= \pi^{p(p-1)/4} |\Sigma|^{n/2} |X\Sigma^{-1}X'|^{t} \prod_{j=p-r+1}^{p} \Gamma\left(\frac{n-j+1}{2} - t\right) \prod_{i=1}^{p-r} \Gamma\left(\frac{n-i+1}{2}\right)$$

if $n+1 \geqslant p+2t$.

<u>Proof:</u>- Since $X:rxp$ is of rank r $(r \leq p)$, then there exists a matrix $Y:(p-r)xp$ of rank $(p-r)$ such that $(X' \;\; Y')':pxp$ is non-singular. Since $\Sigma:pxp$ is sy.p.d., then $\begin{pmatrix} X \\ Y \end{pmatrix} \Sigma (X' \;\; Y')$ is sy.p.d. and so $\begin{pmatrix} X \\ Y \end{pmatrix} \Sigma (X' \;\; Y') = \tilde{T}\tilde{T}'$ where $T = \begin{array}{cc} \begin{pmatrix} \tilde{T}_{11} & 0 \\ T_{12} & \tilde{T}_{22} \end{pmatrix} & \begin{array}{c} r \\ p-r \end{array} \\ \begin{array}{cc} r & \;\; p-r \end{array} & \end{array}$

i.e. $\tilde{T}_{11} \tilde{T}'_{11} = X\Sigma X'$.

Transform S to R such that $S = \begin{pmatrix} X \\ Y \end{pmatrix}^{-1} \tilde{T} R \tilde{T}' (X' \;\; Y')^{-1}$,

and hence the jacobian of the transformation is $\left| \begin{pmatrix} X \\ Y \end{pmatrix}^{-1} \tilde{T} \right|^{p+1}$ $= |\Sigma|^{(p+1)/2}$ [See (A.2.6)]. Also let $R = \begin{array}{cc} \begin{pmatrix} R_{11} & R_{12} \\ R'_{12} & R_{22} \end{pmatrix} & \begin{array}{c} r \\ p-r \end{array} \\ \begin{array}{cc} r & \;\; p-r \end{array} & \end{array}$ and

so $XSX' = \tilde{T}_{11} R_{11} \tilde{T}'_{11}$ for $\begin{pmatrix} X \\ Y \end{pmatrix} S (X' \;\; Y') = \tilde{T} R \tilde{T}'$.

Hence the integral given in (A.3.3a) is

$$I_1 = |\Sigma|^{n/2} |X\Sigma X'|^{t} \int \cdots \int |R_{11}|^{t} \, |R|^{(n-p-1)/2} \exp(-\mathrm{tr}\, R) dR.$$

Let $R_{11} = VV'$ where $V:rxr$ is non-singular, because R_{11} is sy.p.d. Transform $V^{-1}R_{12} = W_{12}$ & $W_{22} = R_{22} - W'_{12} W_{12}$.

Then the jacobian of the transformation is $|V|^{p-r} = |R_{11}|^{(p-r)/2}$.

Hence

$$I_1 = |\Sigma|^{n/2} |X\Sigma X'|^{t} \int \cdots \int |R_{11}|^{t+(n-r-1)/2} |W_{22}|^{(n-p-1)/2} \cdot$$

$$\exp(-\mathrm{tr}\, R_{11} - \mathrm{tr}\, W_{22} - \mathrm{tr}\, W_{12} W'_{12})\, dR_{11}\, dW_{22}\, dW_{12} \,.$$

i.e. $I_1 = |\Sigma|^{n/2} |X\Sigma X'|^{t} \pi^{p(p-1)/4} \prod_{j=1}^{r} \Gamma\left(\frac{n-j+1}{2}+t\right) \prod_{i=r+1}^{p} \Gamma\left(\frac{n-i+1}{2}\right) \cdot$

We can make the transformations of S similar to (A.2.4) & (A.2.5), and shall thus prove the part (a).

Similarly part (b) can be proved.

(A.3.4) If $I_{0,t} = \int \cdots \int (\mathrm{tr}\, XSX')^{t} |S|^{(n-p-1)/2} \exp(-\mathrm{tr}\Sigma^{-1} S)dS$,

then $I_{0,t} = (t-1)! \sum_{i=1}^{t} (\frac{1}{2}ni+t-i)(-1)^{i-1}(\mathrm{tr}_i X\Sigma X') I_{0,t-i}/(t-i)!$

$$= \tfrac{1}{2}n[(t-1)!] \sum_{i=1}^{t} \mathrm{tr}(X\Sigma X')^{i}\, I_{0,t-i}/(t-i)! \text{ for } t \geq 1$$

and $I_{0,0} = \pi^{p(p-1)/4} |\Sigma|^{n/2} \prod_{i=1}^{p} \Gamma\left(\frac{n-i+1}{2}\right) .$

<u>Proof</u>:- $I_{0,t}$=Coeff. of $\theta^{t}/t!$ in $\int \cdots \int |S|^{(n-p-1)/2} \cdot$

$$\exp\left\{-\mathrm{tr}\Sigma^{-1}(I-\theta X'X)S\right\} dS$$

$$= I_{0,0} \text{ Coeff. of } \theta^{t}/t! \text{ in } |I-\theta X\Sigma X'|^{-n/2} .$$

Then using (A.1.17a), it is easy to see that

$$I_{0,t} = (t-1)! \sum_{i=1}^{t} (\tfrac{1}{2}ni+t-i)(-1)^{i-1} (\mathrm{tr}_i X\Sigma X')\, I_{0,t-i}/(t-i)!$$

and similar to finding the coefficients from (5.4.16), and using (A.1.17a), we have

$$I_{0,t}=(t-1)!\sum_{i=1}^{t}(\tfrac{1}{2}n)\,tr(X\Sigma X')^{i}\,I_{0,t-i}/(t-i)! \quad \text{for } t\geq 1.$$

<u>Corollary:-</u> If in (A.3.4), rank of X is one, then

$$I_{0,t}=|\Sigma|^{n/2}(tr\,X\Sigma X')^{t}\,\pi^{p(p-1)/4}\,\Gamma(t+n/2)\prod_{i=2}^{p}\Gamma\left(\frac{n-i+1}{2}\right),$$

where $I_{0,t}$ is the same as defined in (A.3.4).

$$\text{(A.3.5)}\quad \int_c^1 x^{m-1}(1-x)^{n-1}dx/B(m,n)<\int_c^1 x^{m+j-1}(1-x)^{n-1}dx/B(m+j,n)$$

if $0<c<1$, $m>0$, $n>0$ and j is any positive integer.

<u>Proof</u>:- Take j=1 and any c lying in $0<c<1$, then consider

$$g.B(m,n)=\int_c^1 x^{m-1}(1-x)^{n-1}dx-\frac{m+n}{m}\int_c^1 x^{m}(1-x)^{n-1}dx$$

$$=\int_c^1 x^{m-1}(1-x)^{n}dx-\frac{n}{m}\int_c^1 x^{m}(1-x)^{n-1}dx\;.$$

Hence integrating by parts,

$$g.B(m,n)=\left[\{x^{m}(1-x)^{n}\}/m\right]_c^1+\frac{n}{m}\int_c^1 x^{m}(1-x)^{n-1}dx-\frac{n}{m}\int_c^1 x^{m}(1-x)^{n-1}dx$$

$$=-c^{m}(1-c)^{n}/m<0 \quad \text{for } 0<c<1.$$

Hence $g<0$.

$$\text{i.e.}\int_c^1 x^{m-1}(1-x)^{n-1}dx/B(m,n)<\int_c^1 x^{m}(1-x)^{n-1}dx/B(m+1,n)\;.$$

We can in this manner extend the integral and prove the integral (A.3.5).

(A.3.6) (a) If $S:s\times s$ and $p(S)=\text{const.}|S|^{m}|I_s-S|^{n}$; then the distribution of $t=\text{tr } S$ is approximately given by

$$p(t)=c\ t^{\{s(2m+s+1)-2\}/2}(1-t/s)^{\{(2n+s+1)s-2\}/2} \quad \text{for } 0\leq t\leq s$$

where $c^{-1}=s^{s(2m+s+1)/2}B\left[\tfrac{1}{2}s(2m+s+1),\tfrac{1}{2}s(2n+s+1)\right]$ and if $v=1-t/s$, then $p(v)=B\left[\tfrac{1}{2}s(2m+s+1),\tfrac{1}{2}s(2n+s+1);v\right]$.

The approximation is valid if $m+n>30$ when $s=2$ and when s increases by one unit, $m+n$ must increase by 10 to give satisfactory result.

(b) If $S:s\times s$ and $p(s)=\text{const.}|S|^{m}|I_s-S|^{n}$, then the distribution of $u=\text{tr } S(I_s-S)^{-1}$ is

$$p(u)=c_1\ u^{\{s(2m+s+1)-2\}/2}(1+u/s)^{-\{s(2m+2n+s+1)+2\}/2} \quad \text{for } 0\leq u\leq 00$$

Where $c_1^{-1}=s\ B\left[\tfrac{1}{2}s(2m+s+1),\ sn+1\right]$.

The approximation given above is valid if $(n+s)$ satisfies the conditions stated for $(m+n)$ in (A.3.6a).

[See (62,61)].

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