Part I

MULTICOLLINEARITY OF MEANS

Chapter 1

ON CONDITIONS FOR THE FORMS OF THE TYPE : $XAX' + (LX' + XL')/2 + C$ TO BE DISTRIBUTED INDEPENDENTLY OR TO OBEY WISHART DISTRIBUTION, WITH APPLICATIONS.

1.1: Introduction and summary:-

In the definition of the form of the type: $XAX' + (LX' + XL')/2 + C$, C:pxp and A:nxn are symmetric matrices, L:pxn is a matrix and the elements of the matrix X:pxn obey the distribution,

$$(1.1.1) \quad MN(\mu, \Sigma) = (2\pi)^{-pn/2} \, |\Sigma|^{-n/2} \exp\left\{-\mathrm{tr}\, \Sigma^{-1}(X-\mu)(X-\mu)'\tfrac{1}{2}\right\}$$

where μ:pxn is a matrix and Σ:pxp is a positive definite matrix. The case of $p=1$ has been considered by R.G.Laha(45) and the case when $C=0, L=0$ has been considered by the author (39). The two main theorems established here are :
(a) "The necessary and sufficient conditions for the form: $XAX' + (LX' + XL')/2 + C$ to be distributed as non-central Wishart if rank $A \geq p$, are (i) A is idempotent, (ii) $L=LA$ and (iii) $C = LAL'/4$", and (b) "The necessary and sufficient conditions for the independence of the forms: $XAX' + (LX' + XL')/2 + C$ and $XBX' + (MX' + XM')/2 + N$ are $AB=0$, $LB=MA=0$ & $LM'=0$". The various particular cases of these theorems are studied.

The above results are applied to (i) testing the regression like parameters, (ii) testing the equality of mean-vectors and (iii) testing the multicollinearity of means of first kind for multivariate normal populations. Also it has been shown that these tests are invariant under certain types of non-singular linear transformations of the variates.

1.2: Conditions that $XAX' + (LX'+XL')/2 + C$ is distributed as Wishart:-

We shall first establish the following lemma:

Lemma: If $X : p \times n$ has the density function given in (1.1.1), the cumulant generating function of the elements of $XAX' + (LX'+XL')/2 + C$ is

$$\sum_{s=1}^{\infty} 2^{s-1} s^{-1} \operatorname{tr} A^s \operatorname{tr}(\Sigma\Theta)^s + \operatorname{tr}\Theta(C + L\mu') + \tfrac{1}{2} \operatorname{tr}\Theta\Sigma\Theta LL' + \sum_{s=0}^{\infty} 2^s \operatorname{tr}(\mu+\Sigma\Theta L) A^{s+1} (\mu+\Sigma\Theta L)' \Theta(\Sigma\Theta)^s$$

where $\Theta : p \times p$ is a symmetric matrix.

Proof: The moment generating function of the elements of $XAX' + (LX' + XL')/2 + C$ is

$$(1.2.1)\quad \phi(\Theta) = b \int\cdots\int \exp\left\{-\operatorname{tr}\left(\tfrac{1}{2}\Sigma^{-1}XX' - \Theta XAX' - \Theta LX' - \Sigma^{-1}\mu X'\right)\right\} dX$$

where $b = (2\pi)^{-pn/2} |\Sigma|^{-n/2} \exp(\operatorname{tr}\Theta C - \tfrac{1}{2}\operatorname{tr}\Sigma^{-1}\mu\mu')$ and $\Theta : p \times p$ is any symmetric matrix. Here we may note that $\operatorname{tr}PQ = \operatorname{tr}QP$.

By (A.1.6a), we can write $A=QD_{\alpha}Q'$ where $Q:n\times n = (\underline{q}_1, \underline{q}_2, \ldots, \underline{q}_n)$ is an orthogonal matrix and $D_{\alpha}:n\times n$ is a diagonal matrix, and by (A.1.4a) $\Sigma=\tilde{T}\tilde{T}'$ where $\tilde{T}:p\times p$ is a triangular matrix.

Transform the elements of $X:p\times n$ by the relation $X=\tilde{T}YQ'$. The jacobian of the transformation is $|\tilde{T}|^n \cdot |Q'|^p = |\Sigma|^{n/2}$ [See (A.2.1)].

Let $W=\tilde{T}'\Theta\tilde{T}$, $\nu=\tilde{T}^{-1}\mu Q=(\underline{\nu}_1,\underline{\nu}_2,\ldots,\underline{\nu}_n)$ and $V=\tilde{T}^{-1}LQ=(\underline{v}_1,\underline{v}_2,\ldots,\underline{v}_n)$ i.e. $\underline{\nu}_i=\tilde{T}^{-1}\mu\underline{q}_i$ & $\underline{v}_i=\tilde{T}^{-1}L\underline{q}_i$. Also let $Y=(\underline{y}_1,\underline{y}_2,\ldots,\underline{y}_n)$. Then equation (1.2.1) is

$$(1.2.2)\quad \phi(\Theta)=b\,|\Sigma|^{n/2}\int\!\!\cdots\!\!\int dY\,\exp\Big\{-\tfrac{1}{2}\Big[\sum_{i=1}^{n}\underline{y}_i'(I-2\alpha_i W)\underline{y}_i-2(\underline{\nu}_i+W\underline{v}_i)'\underline{y}_i\Big]\Big\}.$$

$$\text{i.e. } \phi(\Theta)=\prod_{i=1}^{n}\left|I-2\alpha_i W\right|^{-\frac{1}{2}}\exp(\mathrm{tr}\Theta C-\tfrac{1}{2}\mathrm{tr}\Sigma^{-1}\mu\mu').$$

$$\exp\Big\{\tfrac{1}{2}\sum_{i=1}^{n}(\underline{\nu}_i+W\underline{v}_i)'(I-2\alpha_i W)^{-1}(\underline{\nu}_i+W\underline{v}_i)\Big\}.$$

Substituting the values of $\underline{\nu}_i$'s, $\underline{v}_i$'s, W and using $(I-2\alpha_i W)^{-1}=I+2\alpha_i W(I-2\alpha_i W)^{-1}$ & $|I-PS|=|I-SP|$, we can write (1.2.2) as

$$(1.2.3)\quad \phi(\Theta)=\prod_{i=1}^{n}\left|I-2\alpha_i\Sigma\Theta\right|^{-\frac{1}{2}}\exp\{\mathrm{tr}\Theta(C+L\mu')+\tfrac{1}{2}\mathrm{tr}\Theta\Sigma\Theta LL'\}.$$

$$\cdot\exp\Big\{\sum_{i=1}^{n}\alpha_i\,\mathrm{tr}(\mu+\Sigma\Theta L)\underline{q}_i\underline{q}_i'(\mu+\Sigma\Theta L)'\Theta(I-2\alpha_i\Sigma\Theta)^{-1}\Big\}.$$

Hence using (A.1.17a) & (A.1.18a) and then using $\sum_{i=1}^{n}\alpha_i^s = \mathrm{tr}A^s$ and $\sum_{i=1}^{n}\underline{q}_i\underline{q}_i'\alpha_i^s = A^s$, we can simplify (1.2.3) and write it as mentioned in the lemma.

Thus the lemma is established.

Corollary 1: If in the lemma L=0 & C=0, then the cumulant generating function of the elements of XAX' is

$$\sum_{s=1}^{\infty} 2^{s-1} s^{-1} \operatorname{tr}A^s \operatorname{tr}(\Sigma\Theta)^s + \sum_{s=0}^{\infty} \operatorname{tr}\mu A^{s+1}\mu'\Theta(\Sigma\Theta)^s .. \left[\text{See}(39)\right].$$

Corollary 2: If in the lemma, A is idempotent, then the moment generating function of the elements of XAX'+ (LX'+XL')/2 +C can be easily written down from(1.2.3) as

$$(1.2.4)\quad \phi(\Theta)=|I-2\Sigma\Theta|^{-r/2} \exp\{\operatorname{tr}\Theta(C-\tfrac{1}{4}LAL' +L\mu'-LA\mu')\}.$$
$$.\exp\left\{\tfrac{1}{2}\operatorname{tr}\Theta\Sigma\Theta(LL'-LAL')+\operatorname{tr}\Theta(I-2\Sigma\Theta)^{-1}(\mu+L/2)A(\mu+L/2)'\right\}.$$

Now we shall establish the following theorem:

Theorem I: The necessary and sufficient conditions for XAX'+(LX'+XL')/2 +C to be distributed as non-central Wishart if rank $A \geq p$, are (i) A is idempotent, (ii) L=LA & (iii) $C=\frac{1}{4}LAL'$.

Proof: Let the conditions be satisfied. Then the moment generating function of the elements of XAX'+(LX'+XL')/2+C is easily written down as

$$(1.2.5)\quad \phi(\Theta)=|I-2\Sigma\Theta|^{-r/2} \exp\{\operatorname{tr}\Theta(I-2\Sigma\Theta)^{-1} (\mu+L/2)A(\mu+L/2)'\}$$

which is the moment generating function of non-central Wishart distribution with degrees of freedom r(if $r \geq p$) and non-central parameters as the nonzero roots of

$\Sigma^{-1}(\mu+L/2)A(\mu+L/2)'$. [See (6)] .

Conversely if $XAX'+(LX'+XL')/2+C$ is distributed as non-central Wishart, the moment generating functions given in (1.2.3) and (1.2.5) must be equal, and so taking cumulant generating functions, we must have the relation for any Θ:pxp given by

(1.2.6) $$\sum_{s=1}^{\infty} 2^{s-1} s^{-1} (\mathrm{tr}A^s-r)\ \mathrm{tr}(\Sigma\Theta)^s + \mathrm{tr}\Theta(C+L\mu') + \tfrac{1}{2}\mathrm{tr}\Theta\Sigma\Theta LL' + \sum_{s=0}^{\infty} 2^s \mathrm{tr}\left\{(\mu+\Sigma\Theta L)A^{s+1}(\mu+\Sigma\Theta L)' - \mu A\mu'\right\}\Theta(\Sigma\Theta)^s - \sum_{s=0}^{\infty} 2^s \mathrm{tr}(LA\mu'+\tfrac{1}{4}LAL')\Theta(\Sigma\Theta)^s = 0 .$$

Equating the coefficients of $\mathrm{tr}(\Sigma\Theta)^s$, we have $\mathrm{tr}A^s = r$ for $s=1,2,3,\ldots$. Using (A.1.17c) we have

(1.2.7) $\mathrm{tr}_i A = \binom{r}{i}$ and so A is symmetrical idempotent.

Putting this value in (1.2.6), we have

(1.2.8) $\mathrm{tr}\Theta(C-\tfrac{1}{4}LAL'+L\mu'-LA\mu')+\tfrac{1}{2}\mathrm{tr}\Theta\Sigma\Theta(LL'-LAL')=0$.

Equating the coefficients of the elements of $\Theta\Sigma\Theta$, we have $L(I-A)L'=0$. Since A is symmetrical idempotent, then $(I-A)$ is also symmetrical idempotent. Then using (A.1.12), we can write $I-A = RR'$ where R:nxt is semi-orthogonal and $t=\mathrm{rank}\ (I-A)$. Hence $L(I-A)L'=0$ means $(LR)(LR)' = 0$ and so $LR=0$ or $LRR'=0$ or

(1.2.9) $$L = LA .$$

Putting L=LA in (1.2.8) and then equating the coefficients of the elements of Θ, we have

(1.2.10) $$C = LAL'/4 .$$

Hence, we have established theorem I.

§1.3: <u>Conditions for the independence of $XAX'+(LX'+XL')/2+C$ and $XBX'+(MX'+XM')/2+N$</u>:-

<u>Theorem II</u>:- If X:pxn has the density function as given in (1.1.1), the necessary and sufficient conditions for the forms $XAX'+(LX'+XL')/2 + C$ and $XBX'+(MX'+XM')/2+N$ to be independently distributed are $AB=0$, $LB=MA=0$, $LM'=0$.

<u>Proof</u>:- Let $\phi(\Theta_1,\Theta_2)$ be the joint moment generating function of $XAX'+(LX'+XL')/2+C$ and $XBX'+(MX'+XM')/2+N$ where Θ_1:pxp and Θ_2:pxp are any two symmetric matrices.

The necessary and sufficient condition for their independence is

(1.3.1) $$\phi(\Theta_1,\Theta_2) = \phi(0,\Theta_2)\ \phi(\Theta_1,0) .$$

This result (1.3.1) is true for any Θ_1 & Θ_2. Now we shall first consider a particular case $\Theta_1=\Theta$ & $\Theta_2=\rho\Theta$ for any nonzero ρ and any Θ:pxp and establish the conditions for (1.3.1) to be true. Then utilizing the conditions thus obtained, we can easily obtain conditions for (1.3.1) to be true for any values of Θ_1 & Θ_2. Thus we shall establish any further conditions which might have been lost or modified in the first particular case. We shall thus use the above mentioned procedure to establish the conditions for (1.3.1) to be true for any values of θ_1 and θ_2 .

For the particular case $\theta_1=\theta$ and $\theta_2=\rho\theta$, we use the results of the lemma & rewrite (1.3.1) as

$$(1.3.2)\quad \sum_{s=1}^{\infty} 2^{s-1} s^{-1}\left\{tr(A+\rho B)^{s}-trA^{s}-tr(\rho B)^{s}\right\}tr(\Sigma\theta)^{s} + \sum_{s=0}^{\infty} 2^{s}tr(\mu+\Sigma\theta L+\rho\Sigma\theta M)(A+\rho B)^{s+1}(\mu+\Sigma\theta L+\rho\Sigma\theta M)'\theta(\Sigma\theta)^{s}$$
$$-\sum_{s=0}^{\infty} 2^{s}\, tr(\mu+\Sigma\theta L)A^{s+1}(\mu+\Sigma\theta L)'\theta(\Sigma\theta)^{s}-\tfrac{1}{2}tr(LL'+\rho^{2}MM')\theta\Sigma\theta$$
$$-\sum_{s=0}^{\infty} 2^{s}\, tr(\mu+\rho\Sigma\theta M)B^{s+1}(\mu+\rho\Sigma\theta M)'\rho^{s+1}\theta(\Sigma\theta)^{s}$$
$$+\tfrac{1}{2}tr(L+\rho M)(L+\rho M)'\theta\Sigma\theta = 0 .$$

Equating the coefficients of $tr(\Sigma\theta)^{s}$, we have

$$(1.3.3)\quad tr(A+\rho B)^{s} = trA^{s}+\rho^{s}\, trB^{s} \text{ for } s=1,2,\ldots\ldots\infty .$$

Let in (1.3.3), s=4 and equating the coefficients of ρ^2, we have

$$(1.3.4)\quad 2trPP' + trP^{2} = 0 \text{ where } P=AB.$$
$$\text{i.e. } tr(P+P')^{2} = trP^{2} .$$

Since $P+P'$ is a symmetric matrix, $tr(P+P')^2$ is positive and so trP^2 is positive.

Therefore Equation (1.3.4) gives $trPP'=0$ & $trP^2=0$ for PP' is positive semidefinite. Hence $PP'=0$ and so

$$(1.3.5)\quad P=0 \quad \text{i.e.} \quad AB=0 .$$

Using this result in (1.3.2) and then equating the coefficients of the elements of $\rho^2\theta(\Sigma\theta)^3$, we find that

$$(1.3.6)\quad MA^{2}M' + LB^{2}L' = 0 .$$

Since MA^2M' & LB^2L' are positive semi-definite matrices, (1.3.6) gives $MA^2M'=0=LB^2L'$.

(1.3.7) i.e. $(MA)(MA)'=0=(LB)(LB)'$ i.e. $MA=LB=0$.

Using (1.3.5) & (1.3.7) in (1.3.2), we find that $\mathrm{tr}LM'\Theta\Sigma\Theta=0$ i.e. equating the coefficients of elements of $\Theta\Sigma\Theta$, we have $LM'=-(LM')'$ i.e.

(1.3.8) LM' is a skew-symmetric matrix .

We can conjecture that the result (1.3.8) is not true for the general case. So, we shall first use the conditions (1.3.5) and (1.3.7) and then we use (A.1.7) for the transformation of X to Y by an orthogonal matrix. After certain modifications as in the lemma, we find

(1.3.9)
$$\begin{aligned}\mathrm{Log}\,\phi(\theta_1,\theta_2) &= \sum_{s=1}^{\infty} 2^{s-1} s^{-1}\left\{ \mathrm{tr}A^s\mathrm{tr}(\Sigma\theta_1)^s+\mathrm{tr}B^s\mathrm{tr}(\Sigma\theta_2)^s\right\}\\ &+\sum_{s=0}^{\infty} 2^s\mathrm{tr}(\mu+\Sigma\theta_1L)A^{s+1}(\mu+\Sigma\theta_1L)'\theta_1(\Sigma\theta_1)^s+\mathrm{tr}(\theta_1C+\theta_2N)\\ &+\mathrm{tr}(\theta_1L\mu'+\theta_2M\mu')+\mathrm{tr}(\theta_1L+\theta_2M)(\theta_1L+\theta_2M)'\Sigma/2 +\\ &\sum_{s=0}^{\infty} 2^s\mathrm{tr}(\mu+\Sigma\theta_2M)B^{s+1}(\mu+\Sigma\theta_2M)'\theta_2(\Sigma\theta_2)^s .\end{aligned}$$

Using this expression in (1.3.1), we find that $\mathrm{tr}LM'(\theta_2\Sigma\theta_1)=0$ for any θ_1 & θ_2 . Hence equating the coefficients of the elements of $\theta_2\Sigma\theta_1$, we have

(1.3.10) $LM' = 0$.

Hence, in general, we have the necessary and sufficient conditions as given in (1.3.5), (1.3.7) & (1.3.10), which proves the theorem II.

Corollary 3:- If the two forms $P(X)=XAX'+(LX'+XL')/2+C$ and $Q(X)=XBX'+(MX'+XM')/2+N$ are independently distributed, then there exists an orthogonal transformation $Y=X\Delta$ (Δ: nxn is an orthogonal matrix) such that $P(X)=P_1(Y_1: pxt)$ and $Q(X)=Q_1(Y_2: pxs)$ where $Y=(\underset{t}{Y_1},\underset{s}{Y_2},\underset{n-s-t}{Y_3})n$, $\Delta=(\underset{t}{\Delta_1},\underset{s}{\Delta_2},\underset{n-s-t}{\Delta_3})n$,

$Y_i=X\Delta_i$ i=1,2,3 and t=rank $(A\ L')$ and s=rank $(B\ M')$.

This follows immediately by using (A.1.9) and theorem II.

Corollary 4:- The necessary and sufficient conditions for the independence of $XAX'+(LX'+XL')/2+C$ and MX' (or $\frac{1}{2}MX'+\frac{1}{2}XM'$) are $AM'=0$ & $LM'=0$.

This follows immediately by putting B=0 & N=0 in theorem II or we can independently derive from (1.3.1), for $\phi(\theta_1,\theta_2)$ is easy to write down in this case.

Corollary 5:- The necessary and sufficient condition for the forms $(X+L_i)A_i(X+L_i)'$ for i=1,2,....,m to be independently distributed is $A_iA_j=0$ for $i\neq j$.

This is easily seen to follow from theorem II.

1.4: Conditions that a number of forms: $(X+L)A(X+L)'$ are independently distributed as Wishart:-

Theorem III: Let $A=\sum_{i=1}^{m} A_i$ (A & A_i's are symmetric matrices), rank $A=r$ and rank $A_i=r_i$ ($r_i\geqslant p$). Then from the following conditions,

a_1: $(X+L_i)A_i(X+L_i)'$ $i=1,2,\dots,m$ are distributed as Wishart,

a_2: $(X+L_i)A_i(X+L_i)'$ and $(X+L_j)A_j(X+L_j)'$ are independently distributed for $i\neq j$,

a_3: $(X+L)A(X+L)'$ is distributed as Wishart,

b_1: A_i $i=1,2,\dots,m$ are idempotent,

b_2: $A_iA_j=0$ for all $i\neq j$,

b_3: A is idempotent,

e_1: $\sum_{i=1}^{m} r_i=r$,

(a) if any two of the first three a_1,a_2,a_3 are satisfied, all the remaining conditions are satisfied,

(b) if any two of the three conditions b_1,b_2,b_3 are satisfied, all the remaining conditions are satisfied,

(c) if any two conditions of a_i and b_j $(i\neq j)$ one from each are satisfied, all the remaining conditions are satisfied and

(d) if (e_1 and a_3) or (e_1 and b_3) are satisfied, all the remaining conditions are satisfied.

<u>Proof</u> : (a) If any two of the conditions a_1,a_2 and a_3 are satisfied, then by theorems I and II, any two of b_1,b_2 and b_3 are satisfied. Now by (A.1.13), any two of b_1,b_2 and b_3 imply the rest of b_i and e_1. Now by theorems I or II, b_i implies a_i . Thus (a) is proved.

(b) If any two of the conditions b_1, b_2 and b_3 are satisfied, by (A.1.13) the remaining b_i and e_1 are satisfied. Hence by Theorems I and II, b_1, b_2 & b_3 imply a_1, a_2 & a_3. Thus (b) is proved.

(c) If a_i and b_j $i \neq j$ are satisfied, then by theorem I or II, b_i is satisfied. Hence by (A.1.13) all b_i's and e_1 are satisfied and so by theorems I and II, all a_i's are satisfied. Thus (c) is proved.

(d) By theorem I, a_3 implies b_3 and by (A.1.13) e_1 and b_3 imply b_1 & b_2, and then by theorems I and II, b_1, b_2, b_3 imply a_1, a_2 and a_3. Thus (d) is proved.

1.5 : Applications:-

(i) <u>To test the equality of mean-vectors for k-multivariate populations:-</u>

(1.5.1) Let $X_i : pxn_i$ be distributed as $MN(\boldsymbol{\mu}_i \mathbf{1}_i', \boldsymbol{\Sigma})$ for $i=1,2,\ldots,k$ and let them be independent, $\boldsymbol{\mu}_i' : 1xp$, $\mathbf{1}_i' : 1xn_i = (1,1,\ldots,1)$. Let $X : pxn = (X_1, X_2, \ldots, X_k)$ where $n = \sum_{i=1}^{k} n_i$,

$\boldsymbol{\mu} : pxn = (\boldsymbol{\mu}_1 \mathbf{1}_1', \boldsymbol{\mu}_2 \mathbf{1}_2', \ldots, \boldsymbol{\mu}_k \mathbf{1}_k')$.

(1.5.2) Let us define $E_i = \mathbf{1}_i \mathbf{1}_i' / n_i : n_i x n_i$ for $i=1,2,\ldots,k$, $B = \text{diag.}(E_1, E_2, \ldots, E_k) : nxn$, and $A : nxn = \mathbf{1}\mathbf{1}'/n$ where $\mathbf{1}' : 1xn = (1,1,\ldots,1)$. Then it is easy to see that A and E_i are symmetrical idempotent matrices. Now it can be verified easily that B is also symmetrical idempotent and AB=BA=A. So B-A is found to be also symmetrical idempotent and we further get the relations $(B-A)A=-(I-B)=0$ and $(B-A)(I-B)=0$.

Hence the forms connected with these matrices (namely $A, B-A, I-B$) are independently distributed and obey Wishart distributions if $(k-1=\text{rank}(B-A)) \geq p$, $(n-k=\text{rank}(I-B)) \geq p$. Also

$$(1.5.3) \qquad (X-\mu)(X-\mu)' = (X-\mu)A(X-\mu)' + (X-\mu)(B-A)(X-\mu)' + (X-\mu)(I-B)(X-\mu)' .$$
$$= S_1 + S_2 + S_3 \text{ (say)}.$$

Now it is easy to verify that

$$(1.5.4) \qquad S_3=(X-\mu)(I-B)(X-\mu)' = \sum_{i=1}^{k} (X_i-\bar{x}_i \mathbf{1}_i')X_i',$$

$$S_2=(X-\mu)(B-A)(X-\mu)' = \sum_{i=1}^{k} n_i(\bar{x}_i-\bar{x}-\mu_i+\bar{\mu})(\bar{x}_i-\bar{x}-\mu_i+\bar{\mu})'$$

and $S_1=(X-\mu)A(X-\mu)' = n(\bar{x}-\bar{\mu})(\bar{x}-\bar{\mu})'$

where $\bar{x}_i=X_i \mathbf{1}_i/n_i$, $\bar{x} = \sum_{i=1}^{k} X_i \mathbf{1}_i/n$, $\bar{\mu} = \sum_{i=1}^{k} n_i \mu_i/n$.

Hence by using theorem III and in view of (1.5.2) and (1.5.4), we have the result that:

(1.5.5) $\sqrt{n}(\bar{x}-\bar{\mu})$, S_3 and S_2 are mutually independently distributed, and the distributions connected with them are:

$\sqrt{n}(\bar{x}-\bar{\mu})$ is $N(\underline{o},\Sigma)$, S_3 is Wishart $(n-k,p;\Sigma)$ if $n-k \geq p$, and if $S_2=(X-\mu)\Delta\Delta'(X-\mu)'$ where $\Delta: n\times(k-1)$ is semi-orthogonal such that $\Delta\Delta'=B-A$ and $\Delta'\Delta=I_{k-1}$ [See (A.1.8a)], then $(X-\mu)\Delta$ is $MN(0,\Sigma)$.

Under null hypothesis $S_2=S=\sum_{i=1}^{k} n_i(\bar{x}_i-\bar{x})(\bar{x}_i-\bar{x})'$ is the S.P.M. due to hypothesis $(\mu_1=\mu_2=\ldots=\mu_k=\bar{\mu})$ and so the statistics required for the test procedure are the proper functions of the elements of SS_3^{-1}.

Let $V=\sum_{i=1}^{k} n_i(\mu_i-\bar{\mu})(\mu_i-\bar{\mu})'$. The null-hypothesis is $V=0$. Note that if the following functions of the elements of $\Sigma^{-1}V$ are zero, then $V=0$. The functions are

(1.5.6) (i) maximum characteristic root of $\Sigma^{-1}V$,

(ii) $\mathrm{tr}\,\Sigma^{-1}V$, (iii) $\mathrm{tr}\,\Sigma(V+\Sigma)^{-1}-p$ and

(iv) $\mathrm{tr}_p\,\Sigma(V+\Sigma)^{-1}-1$.

(And many other such functions can be stated).

Hence we can at once write down the corresponding sample statistics for the test procedures; namely

(1.5.7) (i) maximum root of SS_3^{-1} [S.N.Roy's criterion (75,13)],

(ii) $\mathrm{tr}\,SS_3^{-1}$ [Hotelling's T-criterion (29)],

(iii) $\mathrm{tr}\,S_3(S+S_3)^{-1}-p$, [Pillai's H-criterion (62)]

and (iv) $\mathrm{tr}_p S_3(S+S_3)^{-1}$ [Wilk's Λ-criterion (85)].

(1.5.8) Henceforth, when we say that the statistics are the proper functions of the elements of (S.P.M. due to hypothesis) (S.P.M. due to error)$^{-1}$, we mean the above four statistics for the test procedure.

(1.5.9) Similarly, we can easily obtain the statistics for the test procedure $H_0[\mu_1=\mu_2=\ldots=\mu_k=\mu_0\text{(given)}]$. The statistics under the null hypothesis are the proper functions of the elements of $\{(X-\mu_0\mathbf{1}')B(X-\mu_0\mathbf{1}')'\}\,S_3^{-1}$.

(ii) <u>To obtain the statistics for testing the regression-like parameters</u> $\Sigma_{12}\Sigma_{22}^{-1}$:-

Let $\begin{array}{c}\begin{pmatrix} Y_i \\ X_i \end{pmatrix} \begin{array}{l} p \\ q \end{array} \\ n_i \end{array}$ be distributed as $MN\left[\begin{array}{c}\begin{array}{l} p \\ q \end{array}\begin{pmatrix} \mu_i \\ \nu_i \end{pmatrix} \underline{1}'_i \\ n_i \end{array}\right.$,

$\left.\begin{array}{c}\begin{array}{l} p \\ q \end{array}\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma'_{12} & \Sigma_{22} \end{pmatrix} \\ \quad p \qquad q \end{array}\right]$ for $i=1,2,\ldots k$ and they are independently

distributed. Let $\xi_i = \mu_i - \beta\nu_i$, $\beta = \Sigma_{12}\Sigma_{22}^{-1}$, $\xi = (\xi_1\underline{1}'_1, \ldots, \xi_k\underline{1}'_k)$

$Y=(Y_1, Y_2, \ldots, Y_k)$: pxn and $X=(X_1, \ldots, X_k)$: qxn, $\sum_{i=1}^{k} n_i = n$.

(1.5.10) Then with the same notations as (1.5.2),

we have $T_1=(Y-\xi-\beta X)(I-B)X'[X(I-B)X']^{-1}X(I-B)(Y-\xi-\beta X)'$

and $T_2=(Y-\xi-\beta X)(I-B)\{I-X'[X(I-B)X']^{-1}X\}(I-B)(Y-\xi-\beta X)'$

are independently distributed (by using theorem II) for

$$(I-B)\{I-X'[X(I-B)X']^{-1}X\}(I-B)X' = 0 \text{ .}$$

Also note that

(1.5.11) rank $(I-B)\{I-X'[X(I-B)X']^{-1}X\}(I-B)=n-q-k$,

and $T_2=Y(I-B)\{I-X'[X(I-B)X']^{-1}X\}(I-B)Y'$

$$=S_{11}-S_{12}S_{22}^{-1}S'_{12}=S_{1.2} \text{ if } \begin{pmatrix} Y \\ X \end{pmatrix}(I-B)(Y' \; X')=\begin{array}{c}\begin{pmatrix} S_{11} & S_{12} \\ S'_{12} & S_{22} \end{pmatrix}\begin{array}{l} p \\ q \end{array} \\ p \qquad q \end{array}$$

and $T_1=S_{12}S_{22}^{-1}S'_{12}-\beta S'_{12}-S_{12}\beta'+\beta S_{22}\beta'$.

(1.5.12) Hence under null-hypothesis $(\beta=0)$, we have the required statistics as the proper functions of the elements of $S_{12}S_{22}^{-1}S'_{12}S_{1.2}^{-1}$ if $n-k-q \geqslant p$. Also note that $S_{1.2}$ is distributed as Wishart $(n-k-q, p; \Sigma_{1.2}=\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma'_{12})$.

(iii) <u>To test the multicollinearity of means of first kind</u>, namely $H_o(\xi_1=\xi_2=\ldots=\xi_k)$ or $H_o[\xi_1=\xi_2=\ldots=\xi_k=\xi_o(\text{given})]$.

With the same notations as (1.5.2) and (1.5.10) we can write,

(1.5.13) $T_3=(Y-\xi-\beta X)(I-A)\{I-X'[X(I-A)X']^{-1}X\}(I-A)(Y-\xi-\beta X)'$

$$=(Y-\xi)(I-A)\{I-X'[X(I-A)X']^{-1}X\}(I-A)(Y-\xi)$$

where $(Y-\xi)(I-A)=Y-\xi-\bar{y}\,\underline{1}'+\bar{\xi}\,\underline{1}'$ and

$$(I-A)\{I-X'[X(I-A)X']^{-1}X\}(I-A)(I-B)\{I-X'[X(I-B)X']^{-1}X\}(I-B)$$
$$=(I-B)\{I-X'[X(I-B)X']^{-1}X\}(I-B) .$$

Hence (T_3-T_2) and T_2 [as defined in (1.5.10)] are independently distributed and T_2 is Wishart $(n-k-q,p;\Sigma_{1.2})$ if $n-k-q \geqslant p$. Also in null hypothesis

$T_3-T_2=W_{1.2}-S_{1.2}$ where $\begin{pmatrix} Y \\ X \end{pmatrix}(I-A)(Y' \;\; X') = \begin{array}{cc} \begin{pmatrix} W_{11} & W_{12} \\ W'_{12} & W_{22} \end{pmatrix} & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} p & q \end{array} & \end{array}$

and $W_{1.2}=W_{11}-W_{12}W_{22}^{-1}W'_{12}$ & $S_{1.2}=S_{11}-S_{12}S_{22}^{-1}S'_{12}$.

(1.5.14) Hence for testing $H_o(\xi_1=\ldots=\xi_k)$, the required statistics are the proper functions of the elements of $(W_{1.2}S_{1.2}^{-1}-I)$.

(1.5.15) Similarly for testing $H_o[\xi_1=\ldots=\xi_k=\xi_o(\text{given})]$, we have the required statistics as the proper functions of the elements of $(Q_{1.2}S_{1.2}^{-1}-I)$ where $Q_{1.2}=Q_{11}-Q_{12}Q_{22}^{-1}Q'_{12}$, and $$Q=\begin{array}{cc} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{pmatrix} & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} p & q \end{array} & \end{array} = \begin{array}{c} p \\ q \end{array} \begin{pmatrix} Y-\xi_0 \mathbf{1}' \\ X \end{pmatrix}_{n} \begin{pmatrix} Y-\xi_0 \mathbf{1}' \\ X \end{pmatrix}' .$$

1.6: Invariance of the test criteria given in (1.5):

We have to consider the invariance of the roots of the form $(XQX')(XRX')^{-1}$ where $QR=0$, $Q^2=Q$ & $R^2=R$.

Let $Q:n\times n$ & $R:n\times n$ be of ranks r and $t(t \geq p)$ respectively & $X:p\times n$. Then by (A.1.7) and (A.1.12), there exists on orthogonal matrix $\Delta=(\underset{r}{\Delta_1},\underset{t}{\Delta_2},\underset{n-r-t}{\Delta_3})\,n$ such that

$Q=\Delta_1\Delta_1'$, $R=\Delta_2\Delta_2'$. With this, it is easy to see that $Z_1=X\Delta_1$, $Z_2=X\Delta_2$ and $Z_3=X\Delta_3$ are independently distributed as $MN(\mu\Delta_1,\Sigma)$, $MN(\mu\Delta_2,\Sigma)$ and $MN(\mu\Delta_3,\Sigma)$ respectively. Also the form $(XQX')(XRX')^{-1}$ is equal to $(Z_1Z_1')(Z_2Z_2')^{-1}$.

(i) The roots of $(Z_1Z_1')(Z_2Z_2')^{-1}$ are invariant under the transformation $P_1=GZ_1\eta_1$ & $P_2=GZ_2\eta_2$ where $\eta_1:r\times r$ & $\eta_2:t\times t$ are orthogonal matrices and $G:p\times p$ is a non-singular matrix, for

$$\text{roots of } P_1P_1'(P_2P_2')^{-1} = \text{roots of } G(Z_1Z_1')G'G'^{-1}(Z_2Z_2')^{-1}G^{-1}$$
$$= \text{roots of } (Z_1Z_1')(Z_2Z_2')^{-1} \text{ by using (A.1.16b)}.$$

This gives the invariance of the statistics derived in (1.5.8) & (1.5.9) by the transformation given above.

(ii) Let A and B be the same matrices as defined in (1.5.2). Then since I-B and B-A are idempotent matrices of ranks **(n-k)** & **(k-1)** respectively, and $(I-B)(B-A)=0$, then by (A.1.7) & (A.1.12), we can write $(I-B)=\Delta_2\Delta_2'$ and $B-A=\Delta_1\Delta_1'$ such that $\Delta_1'\Delta_2=0$ & $\Delta_1: n\times(k-1)$, $\Delta_2: n\times(n-k)$ are semi-orthogonal matrices.

(1.6.1) Let $\begin{pmatrix} Y \\ X \end{pmatrix} (\Delta_1 \quad \Delta_2) = \begin{array}{cc} \begin{pmatrix} Z_1 & Z_3 \\ Z_2 & Z_4 \end{pmatrix} & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} k-1 & n-k \end{array} & \end{array}$ where the distribution of $\begin{pmatrix} Y \\ X \end{pmatrix}$ is given in [(ii) of (1.5)].

This gives $S=\begin{pmatrix} Y \\ X \end{pmatrix}(I-B)(Y' \; X')=\begin{pmatrix} Z_3 \\ Z_4 \end{pmatrix}(Z_3' \; Z_4')$.

(1.6.2): Then the statistics given in (1.5.12) are invariant under the transformation $V_3=G_1Z_3\eta$ & $V_4=G_2Z_4\eta$ where $G_1: p\times p$ & $G_2: q\times q$ are non-singular matrices and $\eta:(n-k)(n-k)$ is an orthogonal matrix, for roots of

$V_3V_4'(V_4V_4')^{-1}V_4V_3'\{V_3V_3'-V_3V_4'(V_4V_4')^{-1}V_4V_3'\}^{-1}$ = roots of

$G_1Z_3Z_4'(Z_4Z_4')^{-1}Z_4Z_3'\{Z_3Z_3'-Z_3Z_4'(Z_4Z_4')^{-1}Z_4Z_3'\}^{-1}G_1^{-1}$ = roots of

$S_{12}S_{22}^{-1}S_{12}'S_{1.2}^{-1}$ by using (A.1.16b) and $S_{1.2}=S_{11}-S_{12}S_{22}^{-1}S_{12}'$,

$$S=\begin{array}{cc} \begin{pmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{pmatrix} & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} p & q \end{array} & \end{array} = \begin{array}{cc} \begin{array}{c} p \\ q \end{array} & \begin{pmatrix} Z_3 \\ Z_4 \end{pmatrix} \\ & n-k \end{array} (Z_3' \; Z_4') .$$

(iii) With the same notations as (1.6.1) i.e.

$$W=S+\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}(Z_1' \quad Z_2') \quad \& \quad S=\begin{pmatrix} Z_3 \\ Z_4 \end{pmatrix}(Z_3' \quad Z_4') ,$$

the statistics given in (1.5.14) are invariant under the transformation given by

$$G\begin{pmatrix} Z_1 & Z_3 \\ Z_2 & Z_4 \end{pmatrix}\eta = \begin{pmatrix} V_1 & V_3 \\ V_2 & V_4 \end{pmatrix} \text{ where } \eta:(n-1)(n-1)= \begin{array}{cc|c} \multicolumn{2}{c}{\begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}} & \begin{matrix} k-1 \\ n-k \end{matrix} \\ k-1 & n-k & \end{array}$$

is an orthogonal matrix and

$$G = \begin{array}{cc} \begin{pmatrix} G_1 & G_2 \\ 0 & G_3 \end{pmatrix} & \begin{matrix} p \\ q \end{matrix} \\ \begin{matrix} p & \quad q \end{matrix} & \end{array}$$

is a non-singular matrix.

<u>Proof:</u> Let

$$E = \begin{array}{cc} \begin{pmatrix} E_{11} & E_{12} \\ E_{12}' & E_{22} \end{pmatrix} & \begin{matrix} p \\ q \end{matrix} \\ \begin{matrix} p & \quad q \end{matrix} & \end{array} = \begin{pmatrix} V_3 \\ V_4 \end{pmatrix} (V_3' \quad V_4') \text{ and}$$

$$F = \begin{array}{cc} \begin{pmatrix} F_{11} & F_{12} \\ F_{12}' & F_{22} \end{pmatrix} & \begin{matrix} p \\ q \end{matrix} \\ \begin{matrix} p & \quad q \end{matrix} & \end{array} = E + \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} (V_1' \quad V_2') \text{ . Then}$$

(1.6.3) $\quad E=GSG'$ and $F=GWG'$.

Now $E_{11}=(G_1 \quad G_2)\, S(G_1 \quad G_2)'$, $E_{12}=(G_1 \quad G_2)S(0 \quad G_3)'$

and $E_{22}=G_3\, S_{22}\, G_3'$. Hence

$$E_{12}E_{22}^{-1}\,E_{12}'=(G_1 \quad G_2)\begin{pmatrix} S_{12} \\ S_{22} \end{pmatrix} S_{22}^{-1}\,(S_{12}' \quad S_{22})\begin{pmatrix} G_1' \\ G_2' \end{pmatrix} \text{. Hence}$$

(1.6.4) $\quad E_{1.2}=E_{11}-E_{12}E_{22}^{-1}\,E_{12}' = G_1\, S_{1.2}G_1'$ where $S_{1.2}=S_{11}-S_{12}S_{22}^{-1}S_{12}'$.

Similarly $F_{1.2}=F_{11}-F_{12}F_{22}^{-1}F'_{12} = G_1W_{1.2}G'_1$ where $W_{1.2}=W_{11}-W_{12}W_{22}^{-1}W'_{12}$. Hence

$$\text{roots of } (S_{1.2}^{-1}W_{1.2}-I)=\text{roots of } (G'_1E_{1.2}^{-1}G_1G_1^{-1}F_{1.2}G_1'^{-1}-I)$$

$$=\text{roots of } G'_1(E_{1.2}^{-1}F_{1.2}-I)G_1'^{-1}.$$

$$=\text{roots of } (E_{1.2}^{-1}F_{1.2}-I) \text{ by using (A.1.16b).}$$

similarly, we can prove that the statistics given in(1.5.15) are invariant under the similar transformations as given above in [(iii) of (1.6)] .

1.7 : Remarks:-

If we consider the form $X'AX+\frac{1}{2}(L'X+X'L)+C$ where $A:pxp$ & $C:nxn$ are symmetric matrices, $L:pxn$ and $X:pxn$ obey the density function as given in (1.1.1), then the following results can be easily established independently or from the results of (1.2, 1.3, 1.4):

(1) Cumulant generating function of $X'AX+\frac{1}{2}(L'X+X'L)+C$ is $\sum_{s=1}^{\infty} 2^{s-1}s^{-1}tr(\Sigma A)^s tr\theta^s + tr\theta(C+L'\mu)+\frac{1}{2}tr\theta^2L'\Sigma L+$ $\sum_{s=0}^{\infty} 2^s tr(\mu+\Sigma L\theta)'A(\Sigma A)^s(\mu+\Sigma L\theta)\theta^{s+1}$ where $\theta:nxn$ is any symmetric matrix.

(2) The necessary and sufficient conditions for $X'AX+\frac{1}{2}(L'X+X'L)+C$ to be distributed as non-central Wishart are (i) $A\Sigma A=A$, (ii) $L'=L'\Sigma A$ and (iii) $C=\frac{1}{4}L'\Sigma A\Sigma L$ if rank $A \geq n$.

(3) The necessary and sufficient conditions for $X'AX+\frac{1}{2}(L'X+X'L)+C$ and $X'BX+\frac{1}{2}(M'X+X'M)+N$ to be independently distributed are $A\Sigma B=0$, $L'\Sigma B=M'\Sigma A=0$ & $L'\Sigma M=0$.

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