Chapter 3

MONOTONIC LOWER BOUND ON THE POWER FUNCTION REGARDING THE TEST PROCEDURE FOR MULTICOLLINEARITY OF MEANS

3.1 : Introduction:-

In this chapter, we shall show that the test procedure (the maximum root criterion) given in chapter I for the multicollinearity of means of the first kind possesses the property of monotonic~~s~~ lower bound (i.e. $1-\beta \geq \alpha$ or the power of the test procedure is always greater than or at least equal to the probability of the error of first kind or size of the test) in the sense given by S.N.Roy (79) and K.V.Ramachandran (66,67). First, we shall prove this for the particular case when r=1 and then for the general case. For the particular case we can also refer to Narain (54).

3.2 : Monotonicity of the test when r=1 :-

The test procedure for testing $H_0(\underline{\delta}_1=\underline{\mu}-\Sigma_{12}\Sigma_{22}^{-1}\underline{\nu}=\underline{0})$ is to accept H_0 if $U \leq c$ and reject otherwise, where c is obtained from $Pr(U \leq c/H_0)= 1-\alpha$, α being the size of the test.

The power function$=1-\beta=Pr(U \geq c/\text{alternative})$

$=Pr(u \geq c/(1+c)=b/\text{alternative})$

where $U=(\underline{y}-S_{12}S_{22}^{-1}\underline{x})(S_{11}-S_{12}S_{22}^{-1}S'_{12})^{-1}(\underline{y}-S_{12}S_{22}^{-1}\underline{x})/(1+\underline{x}'S_{22}^{-1}\underline{x})$

and $1-u=(1+x'S_{22}^{-1}x)\Big/\left\{1+(y'\ \ x')\,S^{-1}\begin{pmatrix}y\\x\end{pmatrix}\right\}$, as given earlier in (2.1.1) or (2.3).

By the use of the power function of u given in (2.3.2), we can write :

$$(3.2.1)\quad 1-\beta=\int_b^1 du\int_0^1\sum_{i,j}\frac{v^{\frac{1}{2}q+i-1}(1-v)^{\frac{1}{2}(m-q+1)+j-1}}{B\{\frac{1}{2}q+i,\frac{1}{2}(m-q+1)\}}\,\cdot$$

$$\frac{(\delta_2'\delta_2/2)^i}{i!}\ \frac{(\delta_1'\delta_1/2)^j}{j!}\ e^{-\{\delta_2'\delta_2+(1-v)\delta_1'\delta_1\}/2}\,.$$

$$\frac{u^{\frac{1}{2}p+j-1}(1-u)^{\frac{1}{2}(m-q-p-1)}}{B\{\frac{1}{2}p+j,\frac{1}{2}(m-q-p+1)\}}\ dv\,.$$

Also by using the integral (A.3.6), we get

$$(3.2.2)\quad \alpha=\int_b^1 u^{\frac{1}{2}p-1}(1-u)^{\frac{1}{2}(m-q-p-1)}du\Big/B\{\tfrac{1}{2}p,\tfrac{1}{2}(m-q-p+1)\}$$

$$<\int_b^1 u^{\frac{1}{2}p+j-1}(1-u)^{\frac{1}{2}(m-q-p-1)}du\Big/B\{\tfrac{1}{2}p+j,\tfrac{1}{2}(m-q-p+1)\}$$

for $j=1,2,3,\ldots$.

Since the range of integration in (3.2.1) is over a finite domain, we can interchange the order of integration in (3.2.1) and then by using (3.2.2), we write

(3.2.3) i.e. $1-\beta>\alpha$.

Hence we have proved the result for $r=1$.

3.3 : Monotonicity for the λ_{max} test in general :

Without any loss of generality, we shall rewrite (2.2.1) as

(3.3.1) const. $\exp\{-\mathrm{tr}\, VV' - \mathrm{tr}(Z-\delta)(Z-\delta)'\}$

where $VV' = \tilde{B}'^{-1} S \tilde{B}^{-1}$, $2\Sigma = \tilde{B}'\tilde{B}$, $V = \begin{array}{cc} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} & \begin{matrix} p \\ q \end{matrix} \\ m & \end{array}$ for $m \geqslant p+q$

and VV' is positive definite, $\begin{array}{cc} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} & \begin{matrix} p \\ q \end{matrix} \\ r & \end{array} = Z = \tilde{B}'^{-1} \begin{pmatrix} Y \\ X \end{pmatrix}$,

$$\delta = \begin{array}{cc} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} & \begin{matrix} p \\ q \end{matrix} \\ r & \end{array} = \tilde{B}'^{-1} \begin{pmatrix} \mu \\ \nu \end{pmatrix}, \quad \tilde{B} = \begin{array}{cc} \begin{pmatrix} \tilde{B}_1 & 0 \\ B_2 & \tilde{B}_3 \end{pmatrix} & \begin{matrix} p \\ q \end{matrix} \\ \begin{matrix} p & \quad q \end{matrix} & \end{array}.$$

(3.3.2) It is easy to see that under the above transformations,

roots of $\left[(I_r + X'S_{22}^{-1}X)^{-1} \left\{I_r + (Y' \quad X')\, S^{-1} \begin{pmatrix} Y \\ X \end{pmatrix}\right\} - I_r\right]$

$= \text{roots of } \left[\left\{I_r + Z_2' (V_2V_2')^{-1} Z_2\right\}^{-1} \left\{I_r + Z'(VV')^{-1} Z\right\} - I_r\right]$

referring to (1.6 iii), $\delta_1 = \tilde{B}_1'^{-1}(\mu - \Sigma_{12}\Sigma_{22}^{-1}\nu)$,

$\tilde{B}_1' \tilde{B}_1 = 2(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}')$ and $\delta_2 = \tilde{B}_3'^{-1}\nu : q\mathrm{x}r$.

(3.3.3) If δ_1 is of rank $s \leq \min.(p,r)$, then by (A.1.8), we write $\delta_1 = \Gamma \begin{pmatrix} D_\gamma & 0 \\ 0 & 0 \end{pmatrix} \Delta$, where $\Gamma : p\mathrm{x}p$ and $\Delta : r\mathrm{x}r$ are orthogonal matrices and γ^2 are the nonzero roots of $\delta_1\delta_1'$ i.e.

γ^2 are the nonzero roots of

$$(\mu-\Sigma_{12}\Sigma_{22}^{-1}\nu)'(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}')^{-1}(\mu-\Sigma_{12}\Sigma_{22}^{-1}\nu)/2 .$$

Let $\begin{array}{cc} \left(\begin{array}{cc} D_\gamma & 0 \\ 0 & 0 \end{array}\right) & \begin{array}{l} s \\ p-s \end{array} \\ \begin{array}{cc} r & r-s \end{array} & \end{array} = \eta : pxr$ and so $\eta_{ii} = \gamma_i^2$ for $i=1,2,\ldots,s$ and $\eta_{ij}=0$ for $i\neq j$, $i=1,2,\ldots p$ & $j=1,2,\ldots r$, $\eta_{ii}=0$ for $i=s+1,\ldots,$ min.(p,r).

Transform Z to B and V to T by the relations

(3.3.4) $B = \begin{array}{c} \begin{array}{c} p \\ q \end{array}\left(\begin{array}{c} B_1 \\ B_2 \end{array}\right) \\ r \end{array} = \begin{array}{cc} \begin{array}{c} p \\ q \end{array}\left(\begin{array}{cc} \Gamma & 0 \\ 0 & I_q \end{array}\right)' \\ \begin{array}{cc} p & q \end{array} \end{array} Z\Delta'$ and $T = \begin{array}{c} \begin{array}{c} p \\ q \end{array}\left(\begin{array}{c} T_1 \\ T_2 \end{array}\right) \\ m \end{array} =$

$\left(\begin{array}{cc} \Gamma & 0 \\ 0 & I_q \end{array}\right)' V$. Then the jacobian of the transformation is one and by using (1.6 iii), roots of

$$\left[(I_r+X'S_{22}^{-1}X)^{-1}\left\{I_r+(Y' \ \ X')\,S^{-1}\left(\begin{array}{c} Y \\ X \end{array}\right)\right\}-I_r\right]$$

$$= \text{roots of } \left[\left\{I_r+B_2'\,(T_2T_2')^{-1}B_2\right\}^{-1}\left\{I_r+B'(TT')^{-1}B\right\}-I_r\right] .$$

Hence the equation (3.3.1) is

(3.3.5) const. $\exp\left\{-\mathrm{tr}\, TT'-\mathrm{tr}\,(B-\eta_0)(B-\eta_0)'\right\}$ where

$\eta_0 = \begin{array}{c} \left(\begin{array}{c} \eta \\ \eta_1 \end{array}\right)\begin{array}{c} p \\ q \end{array} \\ r \end{array}$, $\eta_1=\delta_2\Delta : qxr$, roots of $\eta_1'\eta_1$ are the roots of $\nu'\Sigma_{22}^{-1}\nu/2$.

Now by (2.1.1), we have the acceptance region

λ_{max}=maximum characteristic root of

$$\left[\{I_r+B_2'(T_2T_2')^{-1}B_2\}^{-1}\{I_r+B'(TT')^{-1}B\}-I_r\right] \leq c_o$$

By (A.1.24), we have for non-null $\underline{a}$:rx1,

$$\bigcap_{\underline{a}:rx1}\left[\underline{a}'B'(TT')^{-1}B\underline{a}-\underline{a}'B_2'(T_2T_2')^{-1}B_2\underline{a}\leq c_o\{\underline{a}'\underline{a}+\underline{a}'B_2'(T_2T_2')^{-1}B_2\underline{a}\}\right]$$

i.e. $\bigcap_{\underline{a}:rx1}\left[\underline{a}'B'(TT')^{-1}B\underline{a}\leq c_o\underline{a}'\underline{a}+(1-c_o)\underline{a}'B_2'(T_2T_2')^{-1}B_2\underline{a}\right]$.

Hence we can write $\beta=\bar{I}=Pr(\lambda_{max}\leq c_o / \text{not } H_o)$ i.e.

(3.3.6) $\bar{I}=\text{const.}\int_D\cdots\int \exp(-tr\,TT'-trBB')\,dTdB$

Where the domain D is given by

$$\begin{cases}\bigcap_{\underline{a}:rx1}\Big[\underline{a}'(B+\eta_o)'(TT')^{-1}(B+\eta_o)\underline{a}\leq c_o\,\underline{a}'\underline{a}+ \\ \qquad\qquad (1-c_o)\underline{a}'(B_2+\eta_1)'(T_2T_2')^{-1}(B_2+\eta_1)\underline{a}\Big] \\ \text{and } b_{ij}\text{'s and } t_{ij}\text{'s are to be integrated over } (-00,00).\end{cases}$$

By the use of (A.1.26), we have

(3.3.7) $\underline{a}'(B+\eta_o)'(TT')^{-1}(B+\eta_o)\underline{a}$ = max.root of

$$(B+\eta_o)\underline{a}\underline{a}'(B+\eta_o)'(TT')^{-1}$$

$$\geq\{\text{max.root of } (B+\eta_o)\underline{a}\,\underline{a}'(B+\eta_o)'\}.\{\text{min.root of}(TT')^{-1}\}$$

i.e. $\underline{a}'(B+\eta_o)'(TT')^{-1}(B+\eta_o)\underline{a}\geq\underline{a}'(B+\eta_o)'(B+\eta_o)\underline{a}/\lambda_{max}(TT')$

where λ_{max} (TT') means the maximum root of TT'.

Therefore substituting the value of (3.3.7) in (3.3.6), we have $\bar{I}<I_1$ where

$$(3.3.8) \quad I_1 = \text{const.} \int_{D_1}\!\!\cdots\!\int \exp(-\operatorname{tr} BB' - \operatorname{tr} TT')\, dB\, dT,$$

where

$$D_1 = \left\{ \begin{array}{l} \bigcap_{\underline{a}:\, r\times 1} \Big[\underline{a}'(B+\eta_o)'(B+\eta_o)\underline{a} \leq (\lambda_{\max} TT')\underline{a}'\underline{a}\, c_o + \\ (\lambda_{\max} TT')(1-c_o)\underline{a}'(B_2+\eta_1)'(T_2T_2')^{-1}(B_2+\eta_1)\underline{a} \\ \qquad \text{and } B \;\&\; T : (-\infty, \infty) \,. \end{array} \right.$$

We note that the constant value of (3.3.8) does not contain any γ's. The problem is now of integrating (3.3.8) and showing that the integral stays greater than a monotonically decreasing function of each γ_i or $|\gamma_i|$ separately. It will suffice to show the monotonic character of this integral with respect to variation of (say) $|\gamma_1|$. To this end, remembering that $\underline{a}'$ is a non-null vector $(a_1, a_2, \ldots, a_r)$, we might without any loss of generality put $a_1=1$ and write the domain D_1 of (3.3.8) as

$$(3.3.9) \quad \bigcap_{\underline{a}:\, r\times 1} \Big[\Big\{ (b_{1.11}+\gamma_1) + \sum_{j'=2}^{r} a_{j'}\; b_{1.1j'} \Big\}^2 + \sum_{i=2}^{p} \Big\{ \sum_{j=1}^{r} a_j\, (b_{1.ij}+\eta_{ij}) \Big\}^2 \leq c_o \underline{a}'\underline{a}(\lambda_{\max} TT') + (1-c_o)(\lambda_{\max} TT')\; \underline{a}'(B_2+\eta_1)'(T_2T_2')^{-1}(B_2+\eta_1)\; \underline{a} - \underline{a}'(B_2+\eta_1)'(B_2+\eta_1)\; \underline{a} \Big]$$

where η_{ij}'s are defined in (3.3.3) and $B_1=(b_{1.tt'})$.

To carry out the integration (3.3.8), we first integrate out over $b_{1.11}$ and then check the total integral, which we call I_2 is proportional to

$$(3.3.10) \quad \int \cdots \int \left[\int_{\sup \underline{a}}^{\operatorname{Inf} \underline{a}} \int_{e_{2a}}^{e_{1a}} \exp(-b_{1.11}^2) db_{1.11} \right] \exp(-\operatorname{tr} TT').$$

$$\prod_{i,j} db_{1.ij} \cdot \exp(-\operatorname{tr} B_2 B_2' - \sum_{j'=2}^{r} v_{j'}^2 - \sum_{i,j} b_{1.ij}^2) dB_2 \, dT \prod_{j'} dv_{j'};$$

the symbols being defined in the following way :-

$v_{j'} = b_{1.1j'}$ $j'=2,3,\ldots, r$ and $b_{1.ij}$ $i=2,3,\ldots p;$ $j=1,2,\ldots,r,$

and $e_{1a} = -\gamma_1 - f_{1a} + f_{2a}$ and $e_{2a} = -\gamma_1 - f_{1a} - f_{2a}$ where

$$(3.3.11) \qquad f_{1a} = \sum_{j'=2}^{r} a_{j'} v_{j'} \quad \text{and}$$

$$\text{and } f_{2a} = \Big[(\lambda_{\max} TT') \, c_o \, \underline{a}'\underline{a} + (1-c_o).$$

$$\underline{a}'(B_2+\eta_1)'(T_2T_2')^{-1}(B_2+\eta_1) \, \underline{a} \; -$$

$$\underline{a}'(B_2+\eta_1)'(B_2+\eta_1) \, \underline{a} - \sum_{i=2}^{p} \Big\{ \sum_{j=1}^{r} a_j (b_{1.ij} + \eta_{ij}) \Big\}^2 \Big]^{1/2}.$$

Further-more, we have the properties of (3.3.10) as (1) the constant of proportionality in (3.3.10) is free from γ_i's, (2) $T:(p+q)xm = (t_{kk'})$, $-\infty \leq t_{kk'} \leq \infty$, (3) $B_2:qxr = (b_{2.tt'})$, $-\infty \leq b_{2.tt'} \leq \infty$, while $v_{j'}$ and $b_{1.ij}$ from $-\infty$ to ∞ subject to f_{2a} always staying real, (4) for f_{2a} only the positive square root is to be taken,

(5) $f_{1\underline{a}}$ and $f_{2\underline{a}}$ are free from γ_1. Now with $a_1=1$, let $\underline{a}^*$ denote the value of $\underline{a}$ for which $f_{2\underline{a}}$ is minimum. Then it is clear that this $\underline{a}^*$ is free from γ_1 and $v_{j'}$ and is a function of $t_{kk'}$'s, $b_{2.tt'}$'s, c_0 and possibly also η_{ij}'s and η_2. Notice that $a_1^* = 1$. Also let $e_{1\underline{a}^*}$ and $e_{2\underline{a}^*}$ stand for the values of $e_{1\underline{a}}$ and $e_{2\underline{a}}$ on the substitution of $\underline{a}^*$ for $\underline{a}$. It is now clear that $\operatorname{Inf}_{\underline{a}} e_{1\underline{a}} < e_{1\underline{a}^*}$, $\operatorname{Sup}_{\underline{a}} e_{2\underline{a}} > e_{2\underline{a}^*}$, so that Interval$\{\operatorname{Sup}_{\underline{a}} e_{2\underline{a}}, \operatorname{Inf}_{\underline{a}} e_{1\underline{a}}\}$ $<$ Interval $(e_{2\underline{a}^*}, e_{1\underline{a}^*})$.

Let us now introduce an I_2^* such that but for the constant and positive factor of proportionality (the same as I_2), it is defined by

$$(3.3.12)\quad \int\cdots\int\left[\int_{e_{2\underline{a}^*}}^{e_{1\underline{a}^*}} \exp(-b_{1.11}^2)\,db_{1.11}\right] \exp(-\operatorname{tr} TT'-\operatorname{tr}B_2B_2').$$

$$\exp(-\sum_{j'} v_{j'}^2 - \sum_{i,j} b_{1.ij}^2)\,dT\,dB_2 \prod_{j'} dv_{j'} \prod_{i,j} db_{1.ij}\ .$$

It will be seen that, while I_1 is the integral of the a.e. positive function of (3.3.8) over the domain D_1, which is the intersection of a class of domains, I_2^* is the integral of the same a.e. positive function of (3.3.8) over the intersection of a sub-class of the previous class. In fact, the sub-class is formed by excluding from D_1 all $\underline{a}$'s for which Inf $_{\underline{a}}$ $e_{1\underline{a}} \leq e_{1\underline{a}} < e_{1\underline{a}*}$ and/or $e_{2\underline{a}*} < e_{2\underline{a}} \leq$ Sup $_{\underline{a}}$ $e_{2\underline{a}}$. This shows that $I_2 < I_2^*$.

It is now easy to check that, aside from a constant and positive factor of proportionality, we have

$$\frac{\partial I_2^*}{\partial \gamma_1} = \int \cdots \int \left[\exp(-e^2_{2\underline{a}*}) - \exp(-e^2_{1\underline{a}*}) \right] \exp(-\mathrm{tr}TT' - \mathrm{tr}B_2B_2') \cdot$$

$$\exp(-\sum_{j'} v^2_{j'} - \sum_{i,j} b^2_{1.ij})\, dT\, dB_2 \prod_{j'} dv_{j'} \prod_{i,j} db_{1.ij} \quad .$$

$$\text{(3.3.13) i.e.} \quad \frac{\partial I_2^*}{\partial \gamma_1} = \int \cdots \int \left[\exp\{-(f_{1\underline{a}*} + \gamma_1 + f_{2\underline{a}*})^2\} - \right.$$

$$\left. \exp\{-(f_{1\underline{a}*} + \gamma_1 - f_{2\underline{a}*})^2\} \right] \exp(-\mathrm{tr}TT' - \sum_{j'} v^2_{j'}) \cdot$$

$$\exp(-\mathrm{tr}B_2B_2' - \sum_{i,j} b^2_{1.ij})\, dT\, dB_2 \prod_{j'} dv_{j'} \prod_{i,j} db_{1.ij}$$

by the use of (3.3.11).

Now we recall that $f_{2\underline{a}*}$ is a function of $t_{kk'}$'s, $b_{2.tt'}$'s, $b_{1.ij}$'s, γ_1, c_o and possibly also of the other η_{ij}'s while $f_{1\underline{a}*}$ which is a linear function of $v_{j'}$'s with a coefficient vector $\underline{a}^*$ which is a function of the same quantities that occur in $f_{2\underline{a}*}$. Thus, since $v_{j'}$ are each a $N(o,1)$, therefore the conditional distribution of $f_{1\underline{a}*}$ given $\underline{a}^*$, i.e. given $t_{kk'}$'s, $b_{2.tt'}$'s, $b_{1.ij}$'s is normal with mean zero and variance $\sigma^2_{a*} = \sum_{j'=2}^{r} a^{*2}_{j'}$. Therefore aside from a constant and positive factor of proportionality, we can rewrite (3.3.13) as

$$(3.3.14)\quad \frac{\partial I_2^*}{\partial \gamma_1} = \int\cdots\int \Big[\exp\{-(f_{1a*}+\gamma_1+f_{2a*})^2\} - \exp\{-(f_{1a*}+{}_1-f_{2a*})^2\}\Big] \exp(-f^2_{1a*}/\sigma^2_{a*}) \cdot \exp(-\mathrm{tr}\ TT'-\mathrm{tr}\ B_2B_2'-\sum_{i,j} b^2_{1.ij})df_{1a*}\ dT \cdot dB_2 \prod_{i,j} db_{1.ij} \cdot$$

Integrating out over f_{1a*}, it is easy to check that the right hand side of (3.3.14) reduces to

$$(3.3.15)\quad \int\cdots\int\Big[\exp\{-(\gamma_1+f_{2a*})^2/(1+\sigma^2_{a*})\} - \exp\{-(\gamma_1-f_{2a*})^2/(1+\sigma^2_{a*})\}\Big] \exp(-\mathrm{tr}\ TT'). \exp(-\mathrm{tr}\ B_2B_2'-\sum_{i,j} b^2_{1.ij})dT\ dB_2 \prod_{i,j} db_{1.ij} \cdot$$

Remembering that f_{2a*} is a.e. positive, it is now easy to check that according as γ_1 is positive or negative, we have a.e., $(\gamma_1+f_{2a*})^2 >$ or $<(\gamma_1-f_{2a*})^2$ i.e. a.e.,

$$(3.3.16)\quad \exp\{-(\gamma_1+f_{2a*})^2/(1+\sigma^2_{a*})\} < \text{or} > \exp\{-(\gamma_1-f_{2a*})^2/(1+\sigma^2_{a*})\} \cdot$$

Thus, the integral (3.3.15) is negative or positive according as γ_1 is positive or negative, which proves that I_2^* is a monotonically decreasing function of each $|\gamma_i|$ separately, so that the power of the test stays greater than a monotonically increasing function of each $|\gamma_i|$ separately, and is unbiased at least against all alternative γ_i's for which $I_2^* \leq (1-\alpha)$. Hence $\beta = I \leq (1-\alpha)$. i.e. $1-\beta \geq \alpha$,

Which proves the proposition, that the test is unbiased.

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