Chapter 4

SIMULTANEOUS CONFIDENCE BOUNDS ON THE REGRESSION LIKE PARAMETERS AND DEPARTURES FROM MULTICOLLINEARITY OF MEANS

4.1: Introduction:-

Confidence bounds given here for regression-like parameters and departures from Multicollinearity of means are improvements over those given by S.N.Roy (23,24,79) with some additional results.

Let $\begin{array}{cc}\begin{pmatrix}\underline{y}_{ij}\\ \underline{x}_{ij}\end{pmatrix} & \begin{array}{c}p\\ q\end{array}\end{array}$ be independent multivariate normals with mean vectors $\begin{array}{cc}\begin{pmatrix}\underline{\mu}_{i}\\ \underline{\nu}_{i}\end{pmatrix} & \begin{array}{c}p\\ q\end{array}\end{array}$ and variance-covariance matrix

$\Sigma = \begin{array}{cc}\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma'_{12} & \Sigma_{22}\end{pmatrix} & \begin{array}{c}p\\ q\end{array}\\ \begin{array}{cc}p & \quad q\end{array} & \end{array}$ for $i=1,2,\ldots k$ and $j=1,2,\ldots,n_i$. Let

$\beta = \Sigma_{12}\Sigma_{22}^{-1}$ and $\underline{\xi}_i = \underline{\mu}_i - \beta\,\underline{\nu}_i$, $i=1,2,\ldots k$. Then we are considering the following problems:-

(i) Confidence bounds on regression like parameters β;

(ii) Confidence bounds on $\underline{\xi}_i$'s;

(iii) Confidence bounds on the parameters of the departures of means from the multicollinearity of means of second kind;

(iv) Confidence bounds on $(\xi_i - \bar{\xi})$ where $\bar{\xi} = \sum_{i=1}^{k} n_i \xi_i / n$ where $\sum_{i=1}^{k} n_i = n$;

(v) Confidence bounds on $\xi_i - \xi_j$ $(i \neq j)$ which is a subset of (iv).

The solutions of the above problems are given in the following sections in the order mentioned above.

4.2: Confidence bounds on the regression like parameters β:-

Let $S = \begin{array}{cc|c} S_{11} & S_{12} & p \\ S'_{12} & S_{22} & q \\ p & q & \end{array}$ be the S.P.M. due to error and then the distribution of S is Wishart $(n-k, p+q; \Sigma; S)$ if $n-k \geqslant (p+q)$. (See chapter 1).

By using (A.1.19a), it is easy to see that

$$\operatorname{tr} \Sigma^{-1} S = \operatorname{tr} \Sigma_{1.2}^{-1} S_{1.2} + \operatorname{tr} \Sigma_{1.2}^{-1} (S_{12} S_{22}^{-1} S'_{12} - \beta S'_{12}) - \operatorname{tr} \beta' \Sigma_{1.2}^{-1} S_{12}$$
$$+ \operatorname{tr} \Sigma_{22}^{-1} S_{22} + \operatorname{tr} \beta' \Sigma_{1.2}^{-1} \beta S_{22},$$

where $\beta = \Sigma_{12} \Sigma_{22}^{-1}$, $\Sigma_{1.2} = \Sigma_{11} - \beta \Sigma'_{12}$, $S_{1.2} = S_{11} - S_{12} S_{22}^{-1} S'_{12}$.

Since $\operatorname{tr} PQ = \operatorname{tr} QP$, the above result is equal to

$$\text{(4.2.1)} \quad \operatorname{tr} \Sigma^{-1} S = \operatorname{tr} \Sigma_{1.2}^{-1} S_{1.2} + \operatorname{tr} \Sigma_{1.2}^{-1} (B\tilde{T} - \beta \tilde{T})(B\tilde{T} - \beta \tilde{T})' + \operatorname{tr} \Sigma_{22}^{-1} S_{22}$$

where $B = S_{12} S_{22}^{-1}$, $S_{22} = \tilde{T}\tilde{T}'$.

Also $|S| = |S_{22}| \cdot |S_{1.2}|$.

(4.2.2) Hence, it follows immediately that the distribution of $S_{1.2}$ and the joint distribution of $B\tilde{T}$ and S_{22} are independently distributed and their respective distributions are given by $W(n-k-q,p;\Sigma_{1.2};S_{1.2})$ and $MN(B\tilde{T},\Sigma_{1.2}).W(n-k,q;\Sigma_{22};S_{22})$. Hence we consider all $\lambda_i \leq g^2$ or $\lambda_{max} \leq g^2$ where λ_i's are the roots of

(4.2.3) $$S_{1.2}^{-1}(B-\beta)\,S_{22}(B-\beta)'.$$

We note that λ_{max} has the distribution of the largest characteristic root of the matrix $S_{1.2}^{-1}\,BS_{22}B'$ when $\beta=0$. The joint distribution of these central roots (when $p \geq q$ or $p \leq q$) and also of the largest root being known all that we have to do to make (4.2.3) a simultaneous confidence statement with a joint coefficient $(1-\alpha)$ is to choose $\theta_\alpha/(1-\theta_\alpha)=g^2$ where θ_α depends on p,q & $n-k-q$ and the quantity on the right hand side is defined by

(4.2.4) $$Pr\{\text{Central } \lambda_{max} \geq \theta_\alpha/(1-\theta_\alpha)\} = \alpha .$$

substituting this in (4.2.3), we can derive the simultaneous confidence bound on β from

(4.2.5) $$\lambda_{max}\{S_{1.2}^{-1}\,(B-\beta)S_{22}(B-\beta)'\} \leq \theta_\alpha/(1-\theta_\alpha).$$

Applying (A.1.24), we can rewrite (4.2.5) as

(4.2.6) $$\underline{a}'(B-\beta)S_{22}(B-\beta)'\underline{a}/\underline{a}'S_{1.2}\underline{a} \leq \theta_\alpha/(1-\theta_\alpha) = \lambda_\alpha \text{ (say)}$$

for all non-null $\underline{a}$: $p\times 1$.

Again applying (A.1.25) and with certain modifications we obtain the simultaneous confidence bound on β as

$$(4.2.7)\quad \underline{a}'B\underline{c}-\left\{\lambda_\alpha(\underline{a}'S_{1.2}\underline{a})(\underline{c}'S_{22}^{-1}\underline{c})\right\}^{\frac{1}{2}} \leq \underline{a}'\beta\underline{c} \leq \underline{a}'B\ \underline{c} + \left\{\lambda_\alpha(\underline{a}'S_{1.2}\underline{a})\ (\underline{c}'S_{22}^{-1}\underline{c})\right\}^{\frac{1}{2}}$$

for all non-null $\underline{a}$: pxl and $\underline{c}$: qxl .

Since (4.2.7) is true for all non-null $\underline{c}$: qxl we can choose $\underline{c}$ so as to maximise $\underline{a}'\beta\underline{c}$. Then it is easy to see that (4.2.7) implies $(\underline{a}'\beta\beta'\underline{a})^{\frac{1}{2}} \leq (\underline{a}'BB'\underline{a})^{\frac{1}{2}} + \left\{\lambda_\alpha\ (\underline{a}'S_{1.2}\underline{a})\ (\lambda_{\max}\ S_{22}^{-1})\right\}^{\frac{1}{2}}$. A similar result follows for the other side of the inequality and thus (4.2.7) should imply

$$(4.2.8)\quad (\underline{a}'BB'\underline{a})^{\frac{1}{2}} - \left\{\lambda_\alpha(\underline{a}'S_{1.2}\underline{a})/\lambda_{\min}S_{22}\right\}^{\frac{1}{2}} \leq (\underline{a}'\beta\beta'\underline{a})^{\frac{1}{2}} \leq (\underline{a}'BB'\underline{a})^{\frac{1}{2}} + \left\{\lambda_\alpha(\underline{a}'S_{1.2}\underline{a})/\lambda_{\min}S_{22}\right\}^{\frac{1}{2}}$$

for all non-null $\underline{a}$: pxl .

Similarly by maximising (4.2.7) with respect to $\underline{a}$: pxl, we have

$$(4.2.9)\quad (\underline{c}'B'B\underline{c})^{\frac{1}{2}} - \left\{\lambda_\alpha(\lambda_{\max}S_{1.2})(\underline{c}'S_{22}^{-1}\underline{c})\right\}^{\frac{1}{2}} \leq (\underline{c}'\beta'\beta\underline{c})^{\frac{1}{2}} \leq (\underline{c}'B'B\underline{c})^{\frac{1}{2}} + \left\{\lambda_\alpha(\lambda_{\max}S_{1.2})(\underline{c}'S_{22}^{-1}\underline{c})\right\}^{\frac{1}{2}}$$

for all non-null $\underline{c}$: qxl .

Then by similarly maximising (4.2.8) with respect to $\underline{a}$: pxl or (4.2.9) with respect to $\underline{c}$: qxl, we have the result

$$(4.2.10)\quad (\lambda_{\max}BB')^{\frac{1}{2}} - \left\{\lambda_\alpha(\lambda_{\max}S_{1.2})/\lambda_{\min}S_{22}\right\}^{\frac{1}{2}} \leq (\lambda_{\max}\beta\beta')^{\frac{1}{2}} \leq (\lambda_{\max}BB')^{\frac{1}{2}} + \left\{\lambda_\alpha(\lambda_{\max}S_{1.2})/\lambda_{\min}S_{22}\right\}^{\frac{1}{2}} \quad .$$

Truncation:- Since (4.2.7) is true for all non-null vectors $\underline{c} : q\times 1$ and $\underline{a} : p\times 1$, we can specialise $\underline{c}$ and $\underline{a}$ by putting one, two or more components equal to zero, and then in each case, take arbitrary values of the other components and reason in the same manner as above. Thus proceeding, we shall have in all $(2^p-1)\ (2^q-1)$ statements in number, all with a simultaneous confidence coefficient $\geq (1-\alpha)$.

4.3 :- Confidence bounds on $\underline{\xi}_i$'s (Multicollinearity of means of first kind :-

It is easy to see from (1.5.1) that S(S.P.M. due to error) and $n_i^{\frac{1}{2}} \begin{pmatrix} \underline{\bar{y}}_i \\ \underline{\bar{x}}_i \end{pmatrix}_1 \begin{matrix} p \\ q \end{matrix}$ $i=1,2,\ldots,k$ are independently distributed and their distributions are $W(n-k, p+q; \Sigma; S)$ if $n-k \geq p+q$ and $N\left[n_i^{\frac{1}{2}} \begin{pmatrix} \underline{\mu}_i \\ \underline{\nu}_i \end{pmatrix} \begin{matrix} p \\ q \end{matrix} ,\ \Sigma = \begin{array}{cc|c} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{pmatrix} & & \begin{matrix} p \\ q \end{matrix} \\ \begin{matrix} p \qquad & q \end{matrix} & & \end{array} \right]$ $i=1,2,\ldots,k$

respectively and putting $Y: p\times k = (\sqrt{n_1}\,\underline{\bar{y}}_1, \ldots, \sqrt{n_k}\,\underline{\bar{y}}_k)$, $X: q\times k = (\sqrt{n_1}\,\underline{\bar{x}}_1, \ldots, \sqrt{n_k}\,\underline{\bar{x}}_k)$, $\mu = (\sqrt{n_1}\,\underline{\mu}_1, \ldots, \sqrt{n_k}\,\underline{\mu}_k) ; p\times k$, $\nu : q\times k = (\sqrt{n_1}\,\underline{\nu}_1, \ldots, \sqrt{n_k}\,\underline{\nu}_k)$ and $\xi = \mu - \Sigma_{12}\Sigma_{22}^{-1}\nu$, it can be shown to be similar to lemma 2 of chapter 2 that

$$F: p\times k = \tilde{T}_1'^{-1}(I \ - T_2'\tilde{T}_3'^{-1}) \begin{pmatrix} Y-\xi \\ X \end{pmatrix} \tilde{D}^{-1} \quad \text{where } \tilde{D}'\tilde{D} =$$

$I_k-X'(S_{22}+XX')^{-1}X$: kxk and $S+\begin{pmatrix} Y-\xi \\ X \end{pmatrix}\begin{pmatrix} Y-\xi \\ X \end{pmatrix}' = \tilde{T}'\tilde{T}$,

& $\tilde{T}=\begin{array}{c} \left(\begin{array}{cc} \tilde{T}_1 & 0 \\ T_2 & \tilde{T}_3 \end{array}\right)\begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} p & q \end{array} \end{array}$, is distributed as

(4.3.1) const. $|I_p-FF'|^{(n-k-p-q-1)/2}$ and if λ is any nonzero root of $\{(I_k-F'F)^{-1}-I_k\}$, then by lemma 3 of chapter 2 and (A.1.22), λ is also the nonzero root of
$(I_k+X'S_{22}^{-1}X)^{-1}(Y-\xi-S_{12}S_{22}^{-1}X)'S_{1.2}^{-1}(Y-\xi-S_{12}S_{22}^{-1}X)$.

The distribution of λ_{max} if $p\geqslant k$ or $p\leqslant k$, can be obtained from (4.3.1) on following S.N.Roy (73,74,79) and Pillai (61,63).

(4.3.2) Hence we consider all $\lambda_i\leqslant g^2$ or $\lambda_{max}\leqslant g^2$ where λ_i's are the roots of $L^{-1}(D-\xi)'S_{1.2}^{-1}(D-\xi)$, where $L=I_k+X'S_{22}^{-1}X$, $D=Y-S_{12}S_{22}^{-1}X$ & $S_{1.2}=S_{11}-S_{12}S_{22}^{-1}S_{12}'$.

Since the distribution of λ_{max} in the null hypothesis is known, what we have to do to make (4.3.2) a simultaneous confidence statement with a joint coefficient $(1-\alpha)$ is to choose $g^2=\theta_\alpha/(1-\theta_\alpha)=\lambda_\alpha$(say) where θ_α depends on p,k and n-k-q, and the quantity on the right hand side is defined by

(4.3.3) Pr(Central $\lambda_{max}\geqslant\lambda_\alpha$) $=\alpha$.

Substituting this in (4.3.2), we have for the simultaneous confidence bound, the statement

(4.3.4) $$\lambda_{\max}\left[L^{-1}(D-\xi)' S_{1.2}^{-1} (D-\xi)\right] \leq \lambda_{\alpha} .$$

Applying (A.1.25), we can write (4.3.4) as

(4.3.5) $$\underline{a}'(D-\xi) L^{-1}(D-\xi)' \underline{a} \leq \lambda_{\alpha} \cdot \underline{a}' S_{1.2} \underline{a}$$

for all non-null $\underline{a}$: px1.

Again applying (A.1.25) and after certain modifications, we have the simultaneous confidence statement on ξ :pxk as

(4.3.6) $$\underline{a}' D \underline{c} - \{\lambda_{\alpha} (\underline{a}' S_{1.2} \underline{a})(\underline{c}' L \underline{c})\}^{\frac{1}{2}} \leq \underline{a}' \xi \underline{c} \leq \underline{a}' D \underline{c} + \{\lambda_{\alpha} (\underline{a}' S_{1.2} \underline{a})(\underline{c}' L \underline{c})\}^{\frac{1}{2}}$$

for all non-null $\underline{a}$: px1 and all non-null $\underline{c}$: kx1.

Similar to (4.2), we arrive at the following other statements which are derived from (4.3.6):

(4.3.7) $$(\underline{a}' DD' \underline{a})^{\frac{1}{2}} - \{\lambda_{\alpha} (\underline{a}' S_{1.2} \underline{a})(\lambda_{\max} L)\}^{\frac{1}{2}} \leq (\underline{a}' \xi \xi' \underline{a})^{\frac{1}{2}} \leq (\underline{a}' DD' \underline{a})^{\frac{1}{2}} + \{\lambda_{\alpha} (\underline{a}' S_{1.2} \underline{a})(\lambda_{\max} L)\}^{\frac{1}{2}}$$

for all non-null $\underline{a}$: px1 ;

(4.3.8) $$(\underline{c}' D' D \underline{c})^{\frac{1}{2}} - \{\lambda_{\alpha} (\lambda_{\max} S_{1.2})(\underline{c}' L \underline{c})\}^{\frac{1}{2}} \leq (\underline{c}' \xi' \xi \underline{c})^{\frac{1}{2}} \leq (\underline{c}' D' D \underline{c})^{\frac{1}{2}} + \{\lambda_{\alpha} (\lambda_{\max} S_{1.2})(\underline{c}' L \underline{c})\}^{\frac{1}{2}}$$

for all non-null $\underline{c}$: kx1 and

(4.3.9) $$(\lambda_{\max} DD')^{\frac{1}{2}} - \{\lambda_{\alpha} (\lambda_{\max} S_{1.2})(\lambda_{\max} L)\}^{\frac{1}{2}} \leq (\lambda_{\max} \xi \xi')^{\frac{1}{2}} \leq (\lambda_{\max} DD')^{\frac{1}{2}} + \{\lambda_{\alpha} (\lambda_{\max} S_{1.2})(\lambda_{\max} L)\}^{\frac{1}{2}} .$$

To discuss the shortness of (4.3.6), we consider the non-central distribution of λ_{max} defined in (4.3.2) i.e. λ_{max} is the largest root of the equation in λ :

(4.3.10) $$|(D-\xi)L^{-1}(D-\xi)'-\lambda S_{1.2}|=0.$$

It is easy to see that the distribution of the non-central λ_{max} is really the distribution f_{max} where f_{max} is the largest root of the equation in f obtained by (i) replacing in (4.3.10) ξ_i by ξ_i^* ($\neq \xi_i$) and so ξ by ξ^* ($\neq \xi$) and (ii) assuming the true population parameters as ξ_i and so ξ. The distribution is extremely difficult, but it can be shown easily from (2.2.2) or (3.3.5) that it involves as the parameters, the positive roots of $\gamma_1, \gamma_2, \dots, \gamma_s$ ($s \leq$ min.(p,k)) of the determinantal equation in γ:

$$|(\xi^*-\xi)(\xi^*-\xi)'-\gamma\Sigma_{1.2}|=0 \text{ where } \Sigma_{1.2}=\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}'$$

and the roots of $(\nu'\Sigma_{22}^{-1}\nu)$ which disappear in the null hypothesis. The $(\xi^*-\xi)(\xi^*-\xi)'$ is necessarily at least positive semi-definite of rank s= min.(p,k), say so that out of the p roots of the equation in γ, (p-s) are zero and s positive. Referring to chapter 3 (3.3), we observe that there is a good upper bound to the shortness of (4.3.6) and the shortness is the monotonic decreasing function of the deviation parameters and tends to zero as these tend to infinity. With one population we have s=1 and

$\lambda = n_1(\underline{\bar{y}}-\underline{\xi}_1-S_{12}S_{22}^{-1}\underline{\bar{x}})'S_{1.2}^{-1}(\underline{\bar{y}}-\underline{\xi}_1-S_{12}S_{22}^{-1}\underline{\bar{x}})/(1+n_1\underline{\bar{x}}'S_{22}^{-1}\underline{\bar{x}})$

and $\gamma = n_1(\underline{\xi}_1^*-\underline{\xi}_1)'\Sigma_{1.2}^{-1}(\underline{\xi}_1^*-\underline{\xi}_1)$, $\omega = \underline{\nu}'\Sigma_{22}^{-1}\underline{\nu}\eta$ and it is well-known that on the null hypothesis $(n_1-p-q)\lambda/p$ is distributed as central F_{p,n_1-p-q} with p and n_1-p-q d.f. and on the alternative as the distribution given in (2.3.2) with the same d.f. and with a deviation parameter γ and ω . It is easy to check that in this case the confidence statement (4.3.6) reduces to

$$(4.3.11)\quad \underline{a}'\underline{d}-\{pF_\alpha(\underline{a}'S_{1.2}\underline{a})\,\ell/n_1(n_1-p-q)\}^{\frac{1}{2}} \leq \underline{a}'\,\underline{\xi}_1 \leq \underline{a}'\underline{d}+\{pF_\alpha(\underline{a}'S_{1.2}\underline{a})\,\ell/n_1(n_1-p-q)\}^{\frac{1}{2}}$$

for $\underline{d} = \underline{\bar{y}} - S_{12}S_{22}^{-1}\underline{\bar{x}}$, $\ell = 1+n_1\,\underline{\bar{x}}'S_{22}^{-1}\underline{\bar{x}}$, $S_{1.2}=S_{11}-S_{12}S_{22}^{-1}S_{12}'$,

for any non-null $\underline{a}$:px1 and $Pr(F_{p,n_1-p-q} \geq F_\alpha/H_0)=\alpha$.

The shortness of (4.3.11) can be shown from chapter 3 (3.2) and tends to zero as $\gamma \to \infty$, whatever ω may be.

Truncation:- Since (4.3.6) is true for all non-null vectors $\underline{a}$: px1 and $\underline{c}$:kx1, we can specialise $\underline{c}$ and $\underline{a}$ by putting one, two or more components equal to zero, and then in each case, taking arbitrary values of the other components and reasoning in the same manner as above, we shall have in all

$(2^p-1)(2^k-1)$ statements in number all with a simultaneous confidence coefficient $\geq (1-\alpha)$.

4.4: Simultaneous confidence bounds on the departure of means from the Multicollinearity of means of second kind :-

(i) Here the null hypothesis is $H_0(\underline{\mu}_i=G\underline{\nu}_i$ for $i=1,2,..k)$ where G:pxq is a known matrix. Then by transforming $\underline{y}_{ij}$ to $\underline{z}_{ij}$ by the relation $\underline{y}_{ij}-G\underline{x}_{ij}=\underline{z}_{ij}$, we see that $\underline{z}_{ij}$ are independent multivariate normals with mean vector $\underline{\mu}_i-G\underline{\nu}_i$ and variance covariance matrix $(I-G)\Sigma(I-G)'$ for $i=1,2,.....k$ and $j=1,2,...,n_i$.

The confidence bound for this case can be obtained from (4.3.6) by making the necessary changes, namely $S_{1.2}$=S.P.M. due to error for $\underline{z}_{ij}$'s, $L=I_k$, $D=(\sqrt{n_1}\bar{\underline{z}}_1,...,\sqrt{n_k}\bar{z}_k)$: pxk and $\xi=(\sqrt{n_1}\underline{\xi}_1,..,\sqrt{n_k}\,\underline{\xi}_k)$:pxp where $\underline{\xi}_i=\underline{\mu}_i-G\underline{\nu}_i$ i.e. we can write the confidence statements as

$$(4.4.1)\quad \underline{a}'D\underline{c}-\{\lambda_\alpha(\underline{a}'S_{1.2}\underline{a})\ \underline{c}'\underline{c}\}^{\frac{1}{2}}\leq \underline{a}'\xi\,\underline{c}\leq \underline{a}'D\underline{c}+\{\lambda_\alpha(\underline{a}'S_{1.2}\underline{a})\ \underline{c}'\underline{c}\}^{\frac{1}{2}}$$

for all non-null vectors $\underline{a}$:pxl and $\underline{c}$: kxl, and

$$\Pr\{\text{Central } \lambda_{\max}(DD'S_{1.2}^{-1}) \geqslant \theta_\alpha/(1-\theta_\alpha)=\lambda_\alpha\} = \alpha ,$$

$D=Y-GX, S_{1.2}=(I_p \;\; -G)S\begin{pmatrix} I_p \\ -G' \end{pmatrix}$, S=S.P.M. due to error.

Other particular cases can be derived from (4.4.1) similar to (4.3).

(ii) <u>This also belongs to linear hypothesis in multivariate analysis of variance of means:-</u>

(a) <u>Univariate case</u>:- Suppose $\underline{x}_i$'s are independent $N(E(x_i), \sigma^2)$ such that putting $\underline{x}'=(x_1,\dots,x_n)$, we have $E(\underline{x})=A\underline{\mu}$, $\underline{\mu}$:mx1 (m < n), A:nxm is a matrix of rank $r \leq m < n$, given by the experimental situation and $\underline{\mu}$:mx1 is a set of unknown parameters.

Putting $A:nxm=(\underset{r}{A_1} \;\; \underset{m-r}{A_2})n$, let us assume, as we can without any loss of generality, that A_1:nxr is a set of independent column vectors which might be taken as the basis of A:mxn. Suppose now that it is required to test $H_0(C\underline{\mu}=\underline{Q})$ where C:qxm is of rank $s \leq \min.(q,r) \leq m < n$. Putting

$$C\underline{\mu} = \begin{array}{c} \begin{array}{r} s \\ q-s \end{array}\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \\ \quad\; r \quad\;\; n-r \end{array} \begin{array}{c} \begin{pmatrix} \underline{\mu}_1 \\ \underline{\mu}_2 \end{pmatrix}\begin{array}{l} r \\ n-r \end{array} \\ 1 \end{array} = \begin{array}{c} \begin{pmatrix} \underline{\eta}_1 \\ \underline{\eta}_2 \end{pmatrix}\begin{array}{l} s \\ q-s \end{array} \\ 1 \end{array} ,$$

we assume without any loss of generality, that $(C_{11} \;\; C_{12})$ can be taken as the basis of C, then the test given by S.N.Roy (79,23) is

(4.4.2) to reject $H_o(\eta_1 = \eta_1)$ if

$$(n-r)(\underline{x}-B\eta_1)'M'N^{-1}M(\underline{x}-B\eta_1)/s\, e \geq F_\alpha \text{ where}$$

$$M=C_{11}(A_1'A_1)^{-1}A_1', N=MM'=C_{11}(A_1'A_1)^{-1}C_{11}, B=M'N^{-1},$$

$$e=\underline{x}'\underline{x}-\underline{x}'A_1(A_1'A_1)^{-1}A_1\underline{x} \text{ and}$$

$$\Pr\left\{F_{s,n-r} \geq F_\alpha / H_o(\eta_1=\underline{0})\right\} = \alpha\ .$$

To obtain the confidence bound on η_1, we apply (A.1.25) and rewrite (4.4.2) as

(4.4.3) $\underline{d}'M\underline{x} - \left\{sF_\alpha e(\underline{d}'N\underline{d})/(n-r)\right\}^{\frac{1}{2}} \leq \underline{d}'\eta_1 \leq \underline{d}'M\underline{x} + \left\{sF_\alpha e(\underline{d}'N\underline{d})/(n-r)\right\}^{\frac{1}{2}}$ for any non-null vector $\underline{d}$: sx1.

If we maximise (4.4.3) with respect to all non-null $\underline{d}$: sx1, we can write it as

(4.4.4) $(\underline{x}'M'M\underline{x})^{\frac{1}{2}} - \left\{(\lambda_{\max} N)e\ sF_\alpha/(n-r)\right\}^{\frac{1}{2}} \leq (\eta_1'\eta_1)^{\frac{1}{2}} \leq (\underline{x}'M'M\underline{x})^{\frac{1}{2}} + \left\{(\lambda_{\max} N)e\ sF_\alpha/(n-r)\right\}^{\frac{1}{2}}$.

(b) Multivariate case:- We turn to the multivariate set-up, namely X:pxn whose column vectors are independently distributed, the r-th vector $\underline{x}_r$ being $N(E(\underline{x}_r), \Sigma)$ for $r=1,2,\ldots,n$. Let $E(X')=A\mu$, A:nxm, μ:mxp. Then it is easy to set the confidence bounds on the parameters $C\mu$, by considering first for all non-null $\underline{b}$:px1, the distribution of $\underline{b}'X$ which is univariate normal and so the similar confidence bound on $C\mu\underline{b}$ can be written from (4.4.2) for any non-null $\underline{b}$:px1.

Then we convert this test procedure by Union-intersection principle (75) to the multivariate set-up and arrive at the confidence bound on $\eta_1=(C_{11} \quad C_{12})\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}$ as

$$(4.4.5) \quad \underline{d}'MX'\underline{b}-\{(\underline{b}'S\underline{b})(\underline{d}'N\underline{d})\ s\lambda_\alpha/(n-r)\}^{\frac{1}{2}} \leq \underline{d}'\eta_1\ \underline{b} \leq$$
$$\underline{d}'MX'\underline{b}+\{(\underline{b}'S\underline{b})(\underline{d}'N\underline{d})\ s\lambda_\alpha/(n-r)\}^{\frac{1}{2}}$$

for all non-null $\underline{d}$: sx1 and $\underline{b}$: px1, M & N are defined in (4.4.2), S is the S.P.M. due to error and λ_α(depending on p,s,n-r) is the α% point of the distribution of the largest root of $S^* S^{-1}(n-r)/s$ under $H_o(\eta_1=0)$, S^*=S.P.M. due to hypothesis= $XM'N^{-1}MX'$.

Now for maximising over $\underline{d}$ and $\underline{b}$, we can proceed similar to (4.2) and arrive at the following:

$$(4.4.6) \quad (\underline{b}'S_1^*\underline{b})^{\frac{1}{2}}-\{(\underline{b}'S\underline{b})(\lambda_{max}\ N)s\lambda_\alpha/(n-r)\}^{\frac{1}{2}} \leq (\underline{b}'\eta_1'\eta_1\ \underline{b})^{\frac{1}{2}} \leq$$
$$(\underline{b}'S_1^*\underline{b})^{\frac{1}{2}}+\{(\underline{b}'S\underline{b})(\lambda_{max}\ N)s\lambda_\alpha/(n-r)\}^{\frac{1}{2}}$$

for any non-null $\underline{b}$: px1 and $S_1^* = XM'MX'$;

$$(4.4.7) \quad (\underline{d}'MX'XM'\underline{d})^{\frac{1}{2}}-\{(\lambda_{max}\ S)(\underline{d}'N\underline{d})s\lambda_\alpha/(n-r)\}^{\frac{1}{2}} \leq (\underline{d}'\eta_1\eta_1'\underline{d})^{\frac{1}{2}} \leq$$
$$(\underline{d}'MX'XM'\underline{d})^{\frac{1}{2}}+\{(\lambda_{max}\ S)(\underline{d}'N\underline{d})s\lambda_\alpha/(n-r)\}^{\frac{1}{2}}$$

for any non-null $\underline{d}$: sx1 and

(4.4.8) $(\lambda_{\max} S_1^*)^{\frac{1}{2}} - \{(\lambda_{\max} S)(\lambda_{\max} N)\ s\lambda_\alpha/(n-r)\}^{\frac{1}{2}} \leq (\lambda_{\max} \eta_1 \eta_1')^{\frac{1}{2}} \leq$

$(\lambda_{\max} S_1^*)^{\frac{1}{2}} + \{(\lambda_{\max} S)(\lambda_{\max} N)\ s\lambda_\alpha/(n-r)\}^{\frac{1}{2}}$,

all with confidence coefficient $\geq (1-\alpha)$.

Also we note that the hypothesis of testing $C\mu = 0 : q \times p$ is the same as $C\mu P = 0$ where $P : p \times u$ ($u \leq p$, of rank u) i.e. $C\mu^* = 0$ where $\mu^* = \mu P$, and so testing $\eta_1 = (C_{11}\ \ C_{12})\mu : s \times p$ is replaced by testing $\eta_1^* = (C_{11}\ \ C_{12})\mu P : s \times u$. Hence (4.4.5) would be replaced by a statement in which everything else is the same except that under λ_α, p would be replaced by u and written as λ_α^*, X would be replaced by $P'X : u \times n$, S would be replaced by $P'SP$ and all non-null $\underline{b} : p \times 1$ would be replaced by $\underline{b}^* : u \times 1$. Similarly in (4.4.6) and (4.4.8), in addition, S_1^* would be replaced by $P'S_1^*P$.

With a confidence coefficient $\geq (1-\alpha)$, (4.4.5) and (4.4.8) will now be replaced by the respective confidence statements:

(4.4.9) $\underline{d}'MX'P\underline{b}^* - \{(\underline{b}^{*\prime}P'SP\underline{b}^*)(\underline{d}'N\underline{d})s\ \lambda_\alpha^*/(n-r)\}^{\frac{1}{2}} \leq \underline{d}'\eta_1 P\underline{b}^* \leq$

$\underline{d}'MX'P\underline{b}^* + \{(\underline{b}^{*\prime}P'SP\underline{b}^*)(\underline{d}'N\underline{d})s\ \lambda_\alpha^*/(n-r)\}^{\frac{1}{2}}$

for all non-null $\underline{d} : s \times 1$ and $\underline{b}^* : u \times 1$, and

(4.4.10) $(\lambda_{\max} P'S_1^*P)^{\frac{1}{2}} - \{(\lambda_{\max} P'SP)(\lambda_{\max} N)s\lambda_\alpha^*/(n-r)\}^{\frac{1}{2}} \leq$

$(\lambda_{\max} P'\eta_1'\eta_1 P)^{\frac{1}{2}} \leq$

$(\lambda_{\max} P'S_1^*P)^{\frac{1}{2}} + \{(\lambda_{\max} P'SP)(\lambda_{\max} N)s\lambda_\alpha^*/(n-r)\}^{\frac{1}{2}}$.

Truncation:- Truncation problems for (4.4.5) and (4.4.9) are similar to those discussed in (4.2) and (4.3).

4.5: Confidence bounds on $(\underline{\xi}_i - \underline{\xi})$'s :-

With the same notations as (1.5), let

(4.5.1) $$\begin{array}{c}p\\q\end{array}\underset{n}{\begin{pmatrix}Y\\X\end{pmatrix}}(B-A)(Y' \;\; X') = \begin{array}{c}p\\q\end{array}\underset{k-1}{\begin{pmatrix}Z_1\\Z\end{pmatrix}}(Z_1' \;\; Z') = \begin{array}{c}p\\q\end{array}\underset{k}{\begin{pmatrix}R_1\\R\end{pmatrix}}(R_1' \;\; R')$$

where $$\begin{array}{c}p\\q\end{array}\underset{k}{\begin{pmatrix}R_1\\R\end{pmatrix}} = \begin{array}{c}p\\q\end{array}\begin{pmatrix}\sqrt{n}_1(\bar{\underline{y}}_1-\bar{\underline{y}}), & \ldots\ldots, & \sqrt{n}_k(\bar{\underline{y}}_k-\bar{\underline{y}})\\ \underset{1}{\sqrt{n}_1(\bar{\underline{x}}_1-\bar{\underline{x}})}, & \ldots\ldots, & \underset{1}{\sqrt{n}_k(\bar{\underline{x}}_k-\bar{\underline{x}})}\end{pmatrix},\ \bar{\underline{y}} = \sum_{i=1}^{k} n_i\bar{\underline{y}}_i/n,$$

$$\bar{\underline{y}}_i = \sum_{j=1}^{n_i}\underline{y}_{ij}/n_i,\ \bar{\underline{x}} = \sum_{i=1}^{k} n_i\,\bar{\underline{x}}_i/n,\ \bar{\underline{x}}_i = \sum_{j=1}^{n_i}\underline{x}_{ij}/n_i,\ n = \sum_{i=1}^{k} n_i$$

and also $$\begin{array}{c}p\\q\end{array}\underset{p-1}{\begin{pmatrix}Z_1\\Z\end{pmatrix}} = \begin{array}{c}p\\q\end{array}\underset{n}{\begin{pmatrix}Y\\X\end{pmatrix}}\Delta_1,\ \Delta_1\,\Delta_1' = B-A \text{ and } \Delta_1'\,\Delta_1 = I_{k-1}$$

for (B-A) is an idempotent matrix of rank (k-1)(A.1.12).

Hence by applying (A.1.10) to (4.5.1), there exists a semi-orthogonal matrix $\Delta:(k-1)\mathrm{x}k$ such that

(4.5.2) $$\begin{array}{c}p\\q\end{array}\underset{k}{\begin{pmatrix}R_1\\R\end{pmatrix}} = \begin{array}{c}p\\q\end{array}\underset{k-1}{\begin{pmatrix}Z_1\\Z\end{pmatrix}}\Delta\ .$$

(4.5.3) Moreover by (1.5.5), $S = \begin{array}{cc|c}S_{11} & S_{12} & p\\ S_{12}' & S_{22} & q\\ p & q & \end{array}$ and $\underset{k-1}{\begin{pmatrix}Z_1\\Z\end{pmatrix}}\begin{array}{c}p\\q\end{array}$

are independently distributed and their respective distributions are $W(n-k,p+q;\Sigma;S)$ and $MN\left[\begin{array}{c}p\\q\end{array}\underset{n}{\begin{pmatrix}\mu\\\nu\end{pmatrix}}\Delta_1\ ,\ \Sigma=\begin{array}{cc}\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{12}' & \Sigma_{22}\end{pmatrix} & \begin{array}{c}p\\q\end{array}\\ \begin{array}{cc}p & \quad q\end{array} & \end{array}\right]$

such that $\begin{pmatrix}\mu\\\nu\end{pmatrix}\Delta_1\,\Delta_1'\,(\mu'\ \ \nu')=\begin{pmatrix}\delta_1\\\delta_2\end{pmatrix}(\delta_1'\ \ \delta_2')$ where

$$\begin{array}{c}p\\q\end{array}\underset{k}{\begin{pmatrix}\delta_1\\\delta_2\end{pmatrix}}=\begin{array}{c}p\\q\end{array}\begin{pmatrix}\sqrt{n_1}(\underline{\mu}_1-\overline{\underline{\mu}}), & \dots, & \sqrt{n_k}(\underline{\mu}_k-\overline{\underline{\mu}})\\ \sqrt{n_1}(\underline{\nu}_1-\overline{\underline{\nu}}), & \dots, & \sqrt{n_k}(\underline{\nu}_k-\overline{\underline{\nu}})\\ 1 & & 1\end{pmatrix},\ \overline{\underline{\mu}}=\sum_{i=1}^{k} n_i\,\underline{\mu}_i/n\ ,$$

$$\text{also}\quad \begin{pmatrix}\mu\\\nu\end{pmatrix}\Delta_1(Z_1'\ \ Z')=\begin{pmatrix}\delta_1\\\delta_2\end{pmatrix}(R_1'\ \ R')$$

$\overline{\underline{\nu}}=\sum_{i=1}^{k} n_i\,\underline{\nu}_i/n$, and so as (4.5.2), we write

$$\begin{array}{c}p\\q\end{array}\underset{k}{\begin{pmatrix}\delta_1\\\delta_2\end{pmatrix}}=\begin{array}{c}p\\q\end{array}\underset{n}{\begin{pmatrix}\mu\\\nu\end{pmatrix}}\Delta_1\,\Delta\ ,\text{and } \eta_1=(\mu-\Sigma_{12}\Sigma_{22}^{-1}\nu)\Delta_1\,.$$

Similar to lemma 2, of chapter 2, the distribution of $F_1:p\times(k-1)=\tilde{Q}_1'^{-1}(I-Q_2'\tilde{Q}_3'^{-1})\begin{pmatrix}Z_1-\eta_1\\Z\end{pmatrix}\tilde{E}^{-1}$, where

$\eta_1=(\mu-\Sigma_{12}\Sigma_{22}^{-1}\nu)\,\Delta_1$, $S+\begin{pmatrix}Z_1-\eta_1\\Z\end{pmatrix}\begin{pmatrix}Z_1-\eta_1\\Z\end{pmatrix}'=\tilde{Q}'\tilde{Q}$, $\tilde{Q}=\begin{array}{cc}\begin{pmatrix}\tilde{Q}_1 & 0\\ Q_2 & \tilde{Q}_3\end{pmatrix} & \begin{array}{c}p\\q\end{array}\\ \begin{array}{cc}p & \quad q\end{array} & \end{array}$

and $\tilde{E}'\tilde{E}=I-Z'(S_{22}+ZZ')^{-1}Z$, is given by

$$\text{(4.5.4)}\qquad \text{const.}\ \left|I_p-F_1F_1'\right|^{(n-k-p-q-1)/2}\ .$$

(4.5.5) If λ is the nonzero root of $\left\{(I_{k-1}-F_1'F_1)^{-1}-I_{k-1}\right\}$, then by using the lemma 3 of chapter 2 and (A.1.22), λ is also the nonzero root of

$$(I_{k-1}+Z'S_{22}^{-1}Z)^{-1}(Z_1-\eta_1-S_{12}S_{22}^{-1}Z)'S_{1.2}^{-1}(Z_1-\eta_1-S_{12}S_{22}^{-1}Z), \text{ or}$$

$$(Z_1-\eta_1-S_{12}S_{22}^{-1}Z)(I_{k-1}+Z'S_{22}^{-1}Z)^{-1}(Z_1-\eta_1-S_{12}S_{22}^{-1}Z)'S_{1.2}^{-1} .$$

Now we have $(I_{k-1}+Z'S_{22}^{-1}Z)^{-1}=I_{k-1}-Z'(S_{22}+ZZ')^{-1}Z$,

and so $P=(Z_1-\eta_1-S_{12}S_{22}^{-1}Z)(I_{k-1}+Z'S_{22}^{-1}Z)^{-1}(Z_1-\eta_1-S_{12}S_{22}^{-1}Z)'$

$$=(Z_1-\eta_1-S_{12}S_{22}^{-1}Z)(Z_1-\eta_1-S_{12}S_{22}^{-1}Z)'-$$
$$(Z_1-\eta_1-S_{12}S_{22}^{-1}Z)Z'(S_{22}+ZZ')^{-1}Z(Z_1-\eta_1-S_{12}S_{22}^{-1}Z)'$$
$$=(R_1-\eta-S_{12}S_{22}^{-1}R)(R_1-\eta-S_{12}S_{22}^{-1}R)'-$$
$$(R_1-\eta-S_{12}S_{22}^{-1}R)R'(S_{22}+RR')^{-1}R(R_1-\eta-S_{12}S_{22}^{-1}R)'$$

with the help of (4.5.2) and $\eta=\delta_1-\Sigma_{12}\Sigma_{22}^{-1}\delta_2=\eta_1\Delta$. Hence

(4.5.6) $P=(R_1-\eta-S_{12}S_{22}^{-1}R)(I_k+R'S_{22}^{-1}R)^{-1}(R_1-\eta-S_{12}S_{22}^{-1}R)$.

(4.5.7) Using this expression in (4.5.5), we shall have λ as the nonzero root of $L_1^{-1}(D_1-\eta)'S_{1.2}^{-1}(D_1-\eta)$ where $L_1=(I_k+R'S_{22}^{-1}R)^{-1}$, $D_1=R_1-S_{12}S_{22}^{-1}R$ and $\eta=\delta_1-\Sigma_{12}\Sigma_{22}^{-1}\delta_2$.

(4.5.8) The distribution of λ_{max} if $p\le(k-1)$ or $p\ge(k-1)$ can be easily obtained from (4.5.4) and so we consider all $\lambda_i\le g^2$ or $\lambda_{max}\le g^2$ where λ's are defined in (4.5.7).

(4·5·9) Since the distribution of λ_{max} in the null hypothesis is known, ~~so~~ what we have to do to make (4.5.8) a simultaneous confidence statement with a joint coefficient $(1-\alpha)$, is to choose $g^2=\theta_\alpha/(1-\theta_\alpha)=\lambda_\alpha$ where θ_α depends on $p,(k-1)$ & $(n-k-q)$, and $\Pr(\text{Central } \lambda_{\max} \geqslant \lambda_\alpha) = \alpha$.

Substituting this in (4.5.8), we have for the simultaneous confidence bound the statement

(4.5.10) $\lambda_{\max}\left\{ L_1^{-1}(D_1-\eta)' S_{1.2}^{-1} (D_1-\eta) \right\} \leq \lambda_\alpha$ with a confidence coefficient $\geqslant (1-\alpha)$.

Applying (A.1.24) and then (A.1.25) and after certain modifications, we obtain the simultaneous confidence bound on η as

(4.5.11) $a'D_1b - \{\lambda_\alpha(a'S_{1.2}a)(b'L_1b)\}^{\frac{1}{2}} \leq a'\eta b \leq a'D_1b + \{\lambda_\alpha(a'S_{1.2}a)(b'L_1b)\}^{\frac{1}{2}}$ for all non-null a:px1

and all non-null b:kx1 except for the vector $(\sqrt{n_1}, \sqrt{n_2}, \ldots, \sqrt{n_k})'$ for $\eta(\sqrt{n_1}, \sqrt{n_2}, \ldots, \sqrt{n_k})' = 0$:px1, and so we can always without any loss of generality fix one element of b. Let us suppose the k-th element of b to be zero.

For maximising (4.5.11) over a:px1 or b:kx1, we can proceed similarly as in (4.2) and arrive at

(4.5.12) $(a'D_1D_1'a)^{\frac{1}{2}} - \{\lambda_\alpha(a'S_{1.2}a)(\lambda_{\max} L_1)\}^{\frac{1}{2}} \leq (a'\eta\eta'a)^{\frac{1}{2}} \leq (a'D_1D_1'a)^{\frac{1}{2}} + \{\lambda_\alpha(a'S_{1.2}a)(\lambda_{\max} L_1)\}^{\frac{1}{2}}$

for all non-null a : px1;

$$(4.5.13)\quad (\underline{b}'D_1'D_1\underline{b})^{\frac{1}{2}} - \{\lambda_\alpha(\lambda_{\max}\, S_{1.2})(\underline{b}'L_1\underline{b})\}^{\frac{1}{2}} \le (\underline{b}'\eta'\eta\underline{b})^{\frac{1}{2}} \le (\underline{b}'D_1'D_1\underline{b})^{\frac{1}{2}} + \{\lambda_\alpha(\lambda_{\max}\, S_{1.2})(\underline{b}'L_1\underline{b})\}^{\frac{1}{2}}$$

for all non-null $\underline{b}$:kx1 where the last element of $\underline{b}$ is zero;

$$(4.5.14)\quad \text{and } (\lambda_{\max}\, D_1D_1')^{\frac{1}{2}} - \{\lambda_\alpha(\lambda_{\max}\, S_{1.2})(\lambda_{\max}\, L_1)\}^{\frac{1}{2}} \le (\lambda_{\max}\eta\eta')^{\frac{1}{2}} \le (\lambda_{\max}\, D_1D_1')^{\frac{1}{2}} + \{\lambda_\alpha(\lambda_{\max}\, S_{1.2})(\lambda_{\max}\, L_1)\}^{\frac{1}{2}}$$

all with a confidence coefficient greater than or equal to $(1-\alpha)$.

To discuss the shortness of (4.5.11), we consider the non-central distribution of $\lambda_{\max}$ defined in (4.5.8) i.e. $\lambda_{\max}$ is the largest root of the determinantal equation in λ,

$$(4.5.15)\quad |(R_1-\eta-S_{12}S_{22}^{-1}R)'(I_k+R'S_{22}^{-1}R)^{-1}(R_1-\eta-S_{12}S_{22}^{-1}R)-S_{1.2}\lambda|=0.$$

It is easy to see that the distribution of the non-central $\lambda_{\max}$ is really the distribution of $f_{\max}$ where $f_{\max}$ is the largest root of the equation in f obtained by (i) replacing in (4.5.15) $\underline{\xi}_i$ by $\underline{\xi}_i^*$ ($\ne \underline{\xi}_i$) and so $\bar{\underline{\xi}}$ by $\bar{\underline{\xi}}^*$($\ne \bar{\underline{\xi}}$) i.e. η by η^* ($\ne\eta$) and (ii) assuming the true population parameters as η. The distribution is extremely difficult but after converting R_1 & R to Z_1 & Z and using (2.2.2) or (3.3.5) it can be shown that it involves the parameters, the positive roots $\gamma_1, \gamma_2, \ldots, \gamma_s$ ($s \le \min.(p,k-1)$) of the determinantal equation in γ: $|(\eta^{*}-\eta)(\eta^{*}-\eta)'-\gamma\,\Sigma_{1.2}|=0$, and the roots of $\underline{\nu}(B-A)\nu'\,\Sigma_{22}^{-1}\,\omega(\delta_2'\,\Sigma_{22}^{-1}\,\delta_2)$ which disappear in null hypothesis. The $(\eta^{*}-\eta)(\eta^{*}-\eta)'$ is necessarily at least positive semi-definite of rank min.$(p,k-1)=s$ (say), so that out of the p roots of the equation in γ, $(p-s)$ are zero and s positive. Referring to Chapter 3 (3.3), we observe that there is a good upper bound to the shortness of (4.5.11) and the shortness is the monotonic decreasing function of the deviation parameters and tends to zero as these tend to infinity. With two populations (i.e. $k=2$), we have $s=1$ and

$$\text{(4.5.16)}\quad \lambda=\frac{n_{12}\{(\underline{\bar{y}}_{1.2}-\underline{\bar{y}}_{2.2})-(\underline{\xi}_1-\underline{\xi}_2)\}'S_{1.2}^{-1}\{(\underline{\bar{y}}_{1.2}-\underline{\bar{y}}_{2.2})-(\underline{\xi}_1-\underline{\xi}_2)\}}{1+T_{12}^2}$$

where $n_{12}=n_1n_2/(n_1+n_2)$, $\underline{\bar{y}}_{1.2}=\underline{\bar{y}}_i-S_{12}S_{22}^{-1}\underline{\bar{x}}_i$ $i=1,2$,

$T_{12}^2=n_{12}(\underline{\bar{x}}_1-\underline{\bar{x}}_2)'S_{22}^{-1}(\underline{\bar{x}}_1-\underline{\bar{x}}_2)$ & $\underline{\xi}_i=\underline{\mu}_i-\Sigma_{12}\Sigma_{22}^{-1}\underline{\nu}_i, i=1,2;$

$\gamma=n_{12}(\underline{\xi}_1^{*}-\underline{\xi}_2^{*}-\underline{\xi}_1+\underline{\xi}_2)'\,\Sigma_{1.2}^{-1}\,(\underline{\xi}_1^{*}-\underline{\xi}_2^{*}-\underline{\xi}_1+\underline{\xi}_2),$

$\omega=n_{12}(\underline{\nu}_1-\underline{\nu}_2)'\,\Sigma_{22}^{-1}\,(\underline{\nu}_1-\underline{\nu}_2)$, and it is wel-known that on the null hypothesis $\lambda(n_1+n_2-p-q-1)/p$ is distributed as $F_{p,n_1+n_2-p-q-1}$ with p and $(n_1+n_2-p-q-1)$ d.f. and on the alternative as the distribution given in (2.3.2), with the same d.f. and with a deviation parameter γ and ω. It is easy to check in this case that the confidence statement (4.5.11) reduces to

$$(4.5.17)\quad \underline{a}'(\underline{\bar{y}}_{1.2}-\underline{\bar{y}}_{2.2})-\{pF_{\alpha}(\underline{a}'S_{1.2}\underline{a})(1+T_{12}^{2})/m\ n_{12}\}^{\frac{1}{2}} \leq$$

$$\underline{a}'(\underline{\xi}_1-\underline{\xi}_2)\leq \underline{a}'(\underline{\bar{y}}_{1.2}-\underline{\bar{y}}_{2.2})+\{pF_{\alpha}(\underline{a}'S_{1.2}\underline{a})(1+T_{12}^{2})/m\ n_{12}\}^{\frac{1}{2}}$$

for all non-null vector $\underline{a}$: px1, and $m=n_1+n_2-p-q-1$ &

$$Pr\{F_{p,n_1+n_2-p-q-1}\geqslant F_{\alpha}\ /H_0(\gamma=0)\}=\alpha.$$

The shortness of (4.5.17) can easily be shown from Chapter 3(3.2) and tends to zero as γ tends to infinity whatever ω may be.

Truncation:- Since (4.5.11) is true for all non-null vectors $\underline{a}$: px1 and all non-null $\underline{b}$: kx1 which has the last element as zero, we can specialise $\underline{b}$ and $\underline{a}$ by putting one, two or more components equal to zero, and then in each case, take arbitrary values of the other components and reason in the same manner as above. Thus proceeding, we shall have in all $(2^p-1)(2^{k-1}-1)$ statements in number all with a simultaneous confidence coefficient $\geqslant(1-\alpha)$.

4.6:- Confidence bounds on $(\underline{\xi}_i-\underline{\xi}_j)$ $(i\neq j)$ which is a sub-set of (4.5):-

With the same notations as (4.5) and with

$\underline{y}_{h.2} = \bar{\underline{y}}_h - S_{12}S_{22}^{-1}\bar{\underline{x}}_h$, $\underline{\xi}_h = \underline{\mu}_h - \Sigma_{12}\Sigma_{22}^{-1}\underline{\nu}_h$, $n_{ht} = n_h n_t/(n_h+n_t)$, and $T_{ht}^2 = (\bar{\underline{x}}_h - \bar{\underline{x}}_t)' S_{22}^{-1}(\bar{\underline{x}}_h - \bar{\underline{x}}_t) n_{ht}$, we may note that

$$U_{ht} = n_{ht}(\underline{y}_{h.2} - \underline{y}_{t.2} - \underline{\xi}_h + \underline{\xi}_t)' S_{1.2}^{-1}(\underline{y}_{h.2} - \underline{y}_{t.2} - \underline{\xi}_h + \underline{\xi}_t)/(1+T_{ht}^2)$$

$$= \frac{n_{ht}}{1+T_{ht}^2} \underset{\substack{\underline{a}:p\times 1 \\ \text{non-null}}}{\text{Sup.}} \frac{\underline{a}'(\underline{y}_{h.2} - \underline{y}_{t.2} - \underline{\xi}_h + \underline{\xi}_t)(\underline{y}_{h.2} - \underline{y}_{t.2} - \underline{\xi}_h + \underline{\xi}_t)'\underline{a}}{(\underline{a}'S_{1.2}\underline{a})}$$

Thus for a given pair (h,t), the statement that $U_{ht} \leq F_\alpha$ is exactly equivalent to the statement that, for all non-null $\underline{a}$'s :

$$\underline{a}'(\underline{y}_{h.2} - \underline{y}_{t.2}) - \{F_\alpha(\underline{a}'S_{1.2}\underline{a})(1+T_{ht}^2)/n_{ht}\}^{\frac{1}{2}} \leq \underline{a}'(\underline{\xi}_h - \underline{\xi}_t) \leq$$
$$\underline{a}'(\underline{y}_{h.2} - \underline{y}_{t.2}) + \{F_\alpha(\underline{a}'S_{1.2}\underline{a})(1+T_{ht}^2)/n_{ht}\}^{\frac{1}{2}} .$$

We observe that when the true population means are $\underline{\xi}_h$'s, U_{ht} $(n-k-p-q+1)/p$ is distributed as $F_{p,n-k-p-q+1}$ with p and (n-k-p-q+1) d.f. where $n = \sum_i n_i$.

Now considering all pairs (h,t) out of k-samples (and k-populations), it is easy to ~~s~~see that the statement that the largest U_{ht} out of all pairs is $\leq F_\alpha$, which again is equivalent to the statement that, for all non-null $\underline{a}$'s and all pairs (ℓ,t) out of k,

$$(4.6.1)\quad \underline{a}'(\underline{y}_{h.2}-\underline{y}_{t.2})-\{F_\alpha(\underline{a}'S_{1.2}\underline{a})(1+T^2_{ht})/n_{ht}\}^{\frac{1}{2}}\leq \underline{a}'(\underline{\xi}_h-\underline{\xi}_t)\leq$$
$$\underline{a}'(\underline{y}_{h.2}-\underline{y}_{t.2})+\{F_\alpha(\underline{a}'S_{1.2}\underline{a})(1+T^2_{ht})/n_{ht}\}^{\frac{1}{2}}\quad .$$

If the confidence coefficient of (4.6.1) is to be $(1-\alpha)$, then $F_\alpha=F_\alpha(p,q,n_1,n_2,\dots, n_k)$ will be given by (4.6.2) $\Pr\{\text{largest } U_{ht} \text{ out of} \binom{k}{2} \text{pairs} \geq F_\alpha/\text{null hypothesis}\} = \alpha$.

It is obvious that the distribution of the largest U_{ht} involves as parameters just p, q and $n_1,n_2,\dots,n_k$. It is easy to see that the distribution is manageable only when the number of parameters is small. It may be noted that when $k=2$, (the largest U_{ht}) $(n_1+n_2-p-q-1)/p$ will of course be F distributed with d.f. p and $n_1+n_2-p-q-1$. Also the shortness of the confidence bounds (4.6.1) can be formally written as

$$\Pr\{\text{largest } U_{ht} \text{ out of} \binom{k}{2} \text{pairs} \leq F_\alpha(p,q,n_1,\dots,n_k)/\text{alternative}\}.$$

It is important to observe that while each $U_{ht}\frac{n-k-p-q+1}{p}$ is individually distributed (on the null hypothesis) as F with d.f. p and $n-k-p-q+1$, the $\binom{k}{2}$ U_{ht}'s are not independent, nor do we know what the distribution of the largest central U_{ht} is, to say nothing of the non-central case, so that the confidence statement (4.6.1) has not been reduced to practical terms as was done for the other cases discussed. The distribution problem arising in this situation needs investigation.

For the associated problem of testing $H_0(\xi_1=\ldots=\xi_k)$, we set up as before the rule that if, for non-null $\underline{a}$:px1 and all pairs (h,t), the bounds (4.6.1) include zero, we accept H_0 and reject it otherwise. The properties (including power) of this test are tied up in an obvious manner with those of the multiple confidence interval statement (4.6.1).

*****