CHAPTER - II

ON NÖRLUND SUMMABILITY OF GENERAL ORTHOGONAL SERIES

2.1 Let $\{\phi_n(x)\}$ $(n=0,1,2\ldots\ldots)$ be an orthonormal system (ONS) of L^2-integrable functions defined in the closed interval $[a,b]$. We consider the orthogonal series

$$\sum_{n=0}^{\infty} C_n\phi_n(x) \tag{2.1.1}$$

with real coefficients C_n's.

We denote as usual the n^{th}-partial sums, (C,1)-means, (E,1)-means, $(R, \lambda_n, 1)$- means and (N,p_n)- means of the orthogonal series (2.1.1) by $s_n(x)$, $\sigma_n(x)$, $\tau_n(x)$, $\sigma_n(\lambda,x)$ and $t_n(x)$ respectively.

The n^{th} Nörlund mean or (N,p_n)- mean of the sequence of partial sums $\{s_n(x)\}$ of the orthogonal series (2.1.1) is defined as

$$t_n(x) = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k(x), \quad n=0,1,2\ldots\ldots,$$

where $\{p_n\}$ is a sequence of non-negative real numbers, $p_0>0$, $P_n = p_0+p_1+\ldots\ldots+p_n$ and

$$s_n(x) = \sum_{k=0}^{n} C_k\phi_k(x).$$

The series (2.1.1) is said to be (N,p_n)-summable to $s(x)$, if

$$\lim_{n\to\infty} t_n(x) = s(x).$$

The sequence $\{p_n\}$ will be said to belong to the class M^{α}, for a certain real $\alpha \geq 0$, if

i) $0 < p_n < p_{n+1}$ for $n=0,1,2,\ldots..$

or $0 < p_{n+1} < p_n$ for $n=0,1,2,\ldots..$

ii) $p_0+p_1+ \ldots\ldots + p_n = P_n \uparrow \infty$

iii) $\lim_{n\to\infty} \frac{np_n}{P_n} = \alpha$.

It is well-known that the method (N,p_n) is regular, if and only if

$$\lim_{n\to\infty} \frac{p_n}{P_n} = 0 .$$

Obviously, if $\{p_n\} \in M^{\alpha}$, then the method (N,p_n) is regular.[1]

Let

$$S_n = \frac{1}{P_n} \sum_{k=0}^{n} \frac{P_k}{k+1}$$

The sequence $\{p_n\}$ will be said to belong to the class BVM^{α}, if $\{p_n\} \in M^{\alpha}$ and if $\{S_n\}$ is a sequence of bounded variation i.e.

$$\sum_{n=1}^{\infty} |S_n - S_{n-1}| < \infty .$$

1) Meder [48]

Let $\mu(x) \leqslant x$ denote a positive function concave from below, defined for $x \geqslant 1$ and increasing monotonely to infinity. We shall call the orthogonal series (2.1.1) $\mu(n)$-lacunary, if the number of non-vanishing coefficients C_k with $n < k \leqslant 2n$ does not exceed $\mu(n)$. Furthermore, we shall say that the coefficients have the positive number sequence $\{q_n\}$ as a majorant, if the relation

$$c_n = O(q_n)$$

holds.

Let $p = \{p_n\}$ and $q = \{q_n\}$ be non-negative sequences of real numbers. We write

$$r_n = \sum_{\nu=0}^{n} p_{n-\nu} q_\nu$$

and assume that r_n is non-zero for all values of n.

The n^{th} generalized Nörlund mean of the sequence of partial sums $\{s_n(x)\}$ of the orthogonal series (2.1.1) is given by

$$T_n^{(p,q)}(x) = \frac{1}{r_n} \sum_{k=0}^{n} p_{n-k} q_k s_k(x), \quad n=0,1,2\ldots\ldots$$

The method (N,p,q) reduces to the Nörlund method when $q_n=1$ and to the method $(\overline{N},q)$ when $p_n=1$.

An increasing sequence of natural numbers

$$n_1 < n_2 < \cdots\cdots < n_k < \cdots\cdots$$

is said to satisfy the condition (L), if the series

$$\sum \frac{1}{n_k}$$

satisfies the condition (L), i.e.

$$\sum_{k=m}^{\infty} \frac{1}{n_k} = O\left(\frac{1}{n_m}\right)^{1)} .$$

Sunouchi[2] has discussed the convergence of the series

$$\sum_{n=1}^{\infty} \frac{|s_n(x)-\sigma_n(x)|^k}{n} , \quad k>1$$

under the restriction of boundedness of the functions $\phi_n(x)$ by proving the following theorem.

<u>THEOREM</u> : If

(2.1.2) $$|\phi_n(x)| \leq K \qquad (n=0,1,2.....)$$

then

$$\int_a^b \sum_{n=1}^{\infty} \frac{|s_n(x)-\sigma_n(x)|^q}{n} dx \leq A \sum_{n=1}^{\infty} n^{q-2} |C_n|^q , \quad q>1 .$$

Patel[3] investigated the convergence of the series

(2.1.3) $$\sum_{n=1}^{\infty} \frac{|s_n(x)-\tau_n(x)|^k}{n} , \qquad k\geq 2$$

and

(2.1.4) $$\sum_{n=1}^{\infty} \frac{|s_n(x)-\sigma_n(\lambda,x)|^k}{n} , \qquad k\geq 2$$

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1) Bary [11]
2) Sunouchi [76]
3) Patel [62].

under the restriction of the boundedness of the functions $\phi_n(x)$.

The convergence of the series (2.1.3) and (2.1.4) for $k=2$ has been studied by Meder[1] and Patel[2].

In this chapter, we first prove the analogous result for Nörlund summability for $k=2$ and then extend it for $k \geq 2$ by asserting the following theorems :

THEOREM 1[3]: If the coefficients of the orthogonal series (2.1.1) satisfy the condition

(2.1.5) $$\sum_{n=o}^{\infty} c_n^2 < \infty$$

and

(2.1.6) $$\{p_n\} \in M^{\alpha}, \quad \alpha \geq o,$$

then the series

$$\sum_{n=1}^{\infty} \frac{(s_n(x)-t_n(x))^2}{n} < \infty$$

almost everywhere.

THEOREM 2 : If $p_o > o$, $p_n \geq o$, $np_n = O(P_n)$ and the condition (2.1.2) is satisfied, then

$$\int_a^b \sum_{n=1}^{\infty} \frac{|s_n(x)-t_n(x)|^q}{n} dx = O(1) \sum_{n=1}^{\infty} |c_n|^q n^{q-2}$$

where $q \geq 2$.

1) Meder [46]
2) Patel [60]
3) Agrawal and Kantawala [1]

Further, we also discuss in this chapter the (N,p_n)-summability of the $\mu(n)$-lacunary orthogonal series (2.1.1).

Dealing with the $(C, \alpha>o)$- summability of $\mu(n)$- lacunary orthogonal series (2.1.1) Alexits[1) has proved the following theorem.

<u>THEOREM B</u> : If the coefficients of $\mu(n)$-lacunary series (2.1.1) have as a majorant a positive monotone decreasing number sequence $\{q_n\}$ satisfying the condition

$$(2.1.7) \qquad \sum_{n=1}^{\infty} \frac{\sqrt{\mu(n)}\, q_n}{n} < \infty ,$$

then the condition (2.1.5) implies the $(C,\alpha>o)$- summability almost everywhere of the orthogonal series (2.1.1).

The (E,q)- summability for $q>o$ of the $\mu(n)$- lacunary orthogonal series (2.1.1) has been discussed by Sapre and Bhatnagar[2)].

We extend in this chapter the above results to (N,p_n)-summability as follows :

<u>THEOREM 3</u> : Let

$$(2.1.8) \qquad \{p_n\} \in BVM^{\alpha} , \ \alpha>\tfrac{1}{2}$$

<u>and the coefficients of</u> $\mu(n)$ - <u>lacunary orthogonal series</u> (2.1.1) <u>have as a majorant a positive, monotone decreasing number</u>

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1) Alexits ([4] , p.130)

2) Sapre and Bhatnagar [74]

sequence $\{q_n\}$ satisfying the the conditions (2.1.5) and (2.1.7). Then the series (2.1.1) is (N,p_n)- summable almost everywhere.

In the above theorem we may exclude the condition of the lacunary property, if we take into consideration that between the indices n and 2n there are exactly n free places; therefore if we put $\mu(x)=x$ i.e. $\mu(n)=n$, then every row is $\mu(n)$-lacunary.

This remark enables us to point out a special case from Theorem 3 in which the condition of the lacunarity does appear no more.

THEOREM 4 : If the coefficients of the orthogonal series (2.1.1) have as a majorant a positive, monotone decreasing sequence $\{q_n\}$ satisfying the condition

$$(2.1.9) \qquad \sum_{n=1}^{\infty} \frac{q_n}{\sqrt{n}} < \infty$$

and (2.1.8) holds, then the orthogonal series (2.1.1) is (N,p_n)- summable almost everywhere.

Moreover, this chapter also contains a result on generalized Nörlund summability of orthogonal series. The summability of the orthogonal series (2.1.1) by Cesàro, Euler, Riesz and Nörlund methods has been investigated by Kolmogoroff[1], Sapre[2], Zygmund[3] and Meder[4]. Dealing with the generalized Nörlund summability of orthogonal series (2.1.1) Patel and Patel[5] has

1) Kolmogoroff [34]
2) Sapre [72]
3) Zygmund [96]
4) Meder [48]
5) Patel R.K. and Patel C.M. [63]

proved the following theorem :

THEOREM C : Let $p=\{p_n\}$ and $q=\{q_n\}$ be non-negative, non--decreasing sequences of real numbers such that $r_n \to \infty$ as $n \to \infty$ and

(2.1.10) $$\frac{np_n q_n}{r_n} = O(1).$$

Let $\{n_k\}$ be an arbitrary increasing sequence of indices satisfying

$$1 < q \leq \frac{n_{k+1}}{n_k} \leq r \quad \text{for } k=o,1,2\ldots\ldots$$

where q and r are positive constants. Then the series

(2.1.11) $$\sum_{k=1}^{\infty} (s_{n_k}(x) - T_{n_k}^{(p,q)}(x))^2$$

is convergent almost everywhere in (a,b).

In this chapter we generalize the above result by proving the following theorem :

THEOREM 5 : Let $p=\{p_n\}$ and $q=\{q_n\}$ be non-negative, non-decreasing sequences of real numbers such that $r_n \to \infty$ as $n \to \infty$ and satisfy the condition (2.1.10). If an increasing sequence of natural numbers $\{n_k\}$ satisfy the condition(L), then the series (2.1.11) converges almost everywhere in (a,b).

In order to prove the above theorems we need the following lemmas :

LEMMA 1[1]: (PALEY's THEOREM) : Let $\{\phi_n(x)\}$ be an ONS over an interval (a,b) and $|\phi_n(x)| \leq M$ for $a<x<b$.

(i) If $f \in L^p$, $1<p\leq 2$ and $C_1, C_2 \ldots\ldots C_n \ldots\ldots$ are the Fourier coefficients of f with respect to $\phi_1, \phi_2, \ldots\ldots$ then

$$\left\{ \sum_{n=1}^{\infty} |C_n|^p n^{p-2} \right\}^{\frac{1}{p}}$$

is finite and

$$\left\{ \sum_{n=1}^{\infty} |C_n|^p n^{p-2} \right\}^{\frac{1}{p}} \leq A_p \left\{ \int_a^b |f|^p \, dx \right\}^{\frac{1}{p}}$$

where A_p depends only on p and M.

(ii) If $q \geq 2$ and $C_1, C_2, \ldots\ldots C_n, \ldots\ldots$ is a sequence of numbers for which

$$\sum_{n=1}^{\infty} |C_n|^q n^{q-2} < +\infty \quad ,$$

then a function $f(x) \in L^q(a,b)$ exists, for which the numbers C_n are Fourier coefficients with respect to the system $\{\phi_n(x)\}$ and

$$\left\{ \int_a^b |f|^q \, dx \right\}^{\frac{1}{q}} \leq B_q \left\{ \sum_{n=1}^{\infty} |C_n|^q n^{q-2} \right\}^{\frac{1}{q}}$$

where B_q depends only on q and M.

LEMMA 2[2]: Let $\{n_k\}$ be an increasing sequence of indices satisfying the condition $1 < q < \frac{n_{k+1}}{n_k} \leq r$ for $k = 0, 1, 2 \ldots\ldots$

1) Bary ([11], p.224), Zygmund ([98], p.121)

2) Meder [48]

where r and q are constants. If the conditions (2.1.5) and (2.1.8) hold, then the orthogonal series (2.1.1) is $(\bar{N},p_n)$-summable almost everywhere if and only if the sequence $\{s_{n_k}(x)\}$ is convergent almost everywhere.

2.3 PROOF OF THEOREM 1 : We have

$$s_n(x)-t_n(x) =$$

$$= \sum_{k=o}^{n} c_k\phi_k(x) - \frac{1}{P_n} \sum_{r=o}^{n} p_{n-r}s_r(x)$$

$$= \frac{1}{P_n} \sum_{k=o}^{n} c_k\phi_k(x) \sum_{r=o}^{n} p_{n-r} - \frac{1}{P_n} \sum_{r=o}^{n} p_{n-r} \sum_{k=o}^{r} c_k\phi_k(x)$$

$$= \frac{1}{P_n} \sum_{k=o}^{n} c_k\phi_k(x) \sum_{r=o}^{n} p_{n-r} - \frac{1}{P_n} \sum_{k=o}^{n} c_k\phi_k(x) \sum_{r=k}^{n} p_{n-r}$$

$$= \frac{1}{P_n} \sum_{k=o}^{n} c_k\phi_k(x) \sum_{r=o}^{k-1} p_{n-r}$$

Consequently

$$(2.3.1) \quad \sum_{n=1}^{\infty} \frac{1}{n} \int_a^b (s_n(x)-t_n(x))^2 \, dx = \sum_{n=1}^{\infty} \frac{1}{nP_n^2} \sum_{k=o}^{n} c_k^2 \Big(\sum_{r=o}^{k-1} p_{n-r} \Big)^2$$

If $o < p_n \uparrow$, then the conditions (2.1.5) and (2.1.6) gives

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_a^b (s_n(x)-t_n(x))^2 \, dx \leqslant \sum_{n=1}^{\infty} \frac{1}{nP_n^2} \sum_{k=o}^{n} k^2 c_k^2 p_n^2$$

$$= \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{\infty} \frac{p_n^2}{nP_n^2}$$

$$= O(1) \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{\infty} \frac{1}{n^3} = O(1) \sum_{k=1}^{\infty} c_k^2 < \infty .$$

Therefore, by B. Levy's theorem, we obtain

$$\sum_{n=1}^{\infty} \frac{(s_n(x)-t_n(x))^2}{n} < \infty$$

almost everywhere.

If $o < p_n \downarrow$, then (2.3.1) becomes

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_a^b (s_n(x)-t_n(x))^2 dx \leq \sum_{n=1}^{\infty} \frac{1}{nP_n^2} \sum_{k=o}^{n} k^2 c_k^2 p_{n-k+1}^2$$

$$= \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{\infty} \frac{p_{n-k+1}^2}{nP_n^2}$$

$$= O(1) \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{\infty} \frac{1}{n^3}$$

$$= O(1) \sum_{k=1}^{\infty} c_k^2 < \infty$$

Hence, by B. Levy's theorem, it follows that

$$\sum_{n=1}^{\infty} \frac{(s_n(x)-t_n(x))^2}{n} < \infty .$$

With that the theorem is completely proved.

2.4 PROOF OF THEOREM 2 : We have

$$s_n(x)-t_n(x) = \sum_{k=o}^{n} c_k\phi_k(x) - \frac{1}{P_n}\sum_{r=o}^{n} p_{n-r}\, s_r(x)$$

$$=\frac{1}{P_n}\sum_{k=o}^{n} c_k\phi_k(x)\sum_{r=o}^{n} p_{n-r} - \frac{1}{P_n}\sum_{r=o}^{n} p_{n-r}\sum_{k=o}^{r}\cdot$$

$$\cdot c_k\phi_k(x)$$

$$=\frac{1}{P_n}\sum_{k=o}^{n} c_k\phi_k(x)\sum_{r=o}^{n} p_{n-r} - \frac{1}{P_n}\sum_{k=o}^{n} c_k\phi_k(x)\sum_{r=k}^{n}\cdot$$

$$\cdot p_{n-r}$$

$$=\frac{1}{P_n}\sum_{k=o}^{n} c_k\phi_k(x)\sum_{r=o}^{k-1} p_{n-r}$$

$$=\frac{1}{P_n}\sum_{k=o}^{n} c_k\phi_k(x)(P_n-P_{n-k})$$

$$=\sum_{k=o}^{n} c_k\phi_k(x)R_k, \quad \text{where } R_k = \frac{P_n-P_{n-k}}{P_n}$$

Using Lemma 1, we have

$$(2.4.1)\quad \int_a^b |s_n(x)-t_n(x)|^q\,dx = \int_a^b \left|\sum_{k=o}^{n} c_k R_k\phi_k(x)\right|^q dx$$

$$\leqslant A_1 \sum_{k=1}^{n} |c_k|^q\, |R_k|^q\, k^{q-2}$$

Hence

$$\int_a^b \sum_{n=1}^{\infty} \frac{|s_n(x)-t_n(x)|^q}{n}\,dx \leqslant A_1 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} |C_k|^q |R_k|^q k^{q-2}$$

$$= A_1 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} |C_k|^q k^{q-2} \frac{(P_n-P_{n-k})^q}{P_n^q}$$

i.e.

$$(2.4.2) \quad \int_a^b \sum_{n=1}^{\infty} \frac{|s_n(x)-t_n(x)|^q}{n}\,dx \leqslant A_1 \sum_{k=1}^{\infty} |C_k|^q k^{q-2} \sum_{n=k}^{\infty} \frac{(P_n-P_{n-k})^q}{nP_n^q}$$

Now

$$\sum_{n=k}^{\infty} \frac{(P_n-P_{n-k})^q}{nP_n^q} = \sum_{n=k}^{2k-1} \frac{(P_n-P_{n-k})^q}{nP_n^q} + \sum_{n=2k}^{\infty} \frac{(P_n-P_{n-k})^q}{nP_n^q}$$

Since $P_n > o$

$$\sum_{n=k}^{2k-1} \frac{(P_n-P_{n-k})^q}{nP_n^q} \leqslant \sum_{n=k}^{2k-1} \frac{P_n^q}{nP_n^q} < \frac{k}{k} = 1$$

and for $n \geqslant 2k$

$$P_n - P_{n-k} = \sum_{r=n-k+1}^{n} p_r$$

$$= O(1) \sum_{r=n-k+1}^{n} \frac{P_r}{r}$$

$$= O\left(\frac{kP_n}{n} \right).$$

Therefore

$$\sum_{n=2k}^{\infty} \frac{(P_n-P_{n-k})^q}{nP_n^q} = O(1) \sum_{n=2k}^{\infty} \frac{k^q p_n^q}{n^{q+1} P_n^q},$$

$$= O(k^q) \sum_{n=2k}^{\infty} \frac{1}{n^{q+1}}$$

$$= O(1).$$

Consequently, from (2.4.2)

$$\int_a^b \sum_{n=1}^{\infty} \frac{|s_n(x)-t_n(x)|^q}{n} dx = O(1) \sum_{k=1}^{\infty} |c_k|^q k^{q-2}$$

This completes the proof of our theorem.

2.5 PROOF OF THEOREM 3 : Under the conditions (2.1.5) and (2.1.7) Alexits[1] has proved in Theorem B, the convergence almost everywhere of the sequence $\{s_{2^n}(x)\}$ of the partial sums of the orthogonal series (2.1.1).

Consequently, it follows by condition (2.1.8) and Lemma 2 that the series (2.1.1) is (N, p_n)- summable almost everywhere.

PROOF OF THEOREM 4 : Alexits[2] has proved that the condition (2.1.9) implies the condition (2.1.5).

Moreover, the condition (2.1.9) is a special case of (2.1.7) corresponding to $\mu(n)=n$ and this case is, as mentioned above, satisfied for every series. Hence our theorem follows from Theorem 3.

1) Alexits ([4] , p.130)

2) Alexits ([4], p.132)

2.6 <u>PROOF OF THEOREM 5</u> : We have

$$s_n(x) - T_n^{(p,q)}(x) =$$

$$= \sum_{i=o}^{n} c_i\phi_i(x) - \frac{1}{r_n}\sum_{\nu=o}^{n} p_{n-\nu}\, q_\nu\, s_\nu(x)$$

$$= \sum_{i=o}^{n} c_i\phi_i(x) - \frac{1}{r_n}\sum_{\nu=o}^{n} p_{n-\nu}\, q_\nu \sum_{i=o}^{\nu} c_i\phi_i(x)$$

$$= \frac{1}{r_n}\sum_{i=o}^{n} c_i\phi_i(x)\sum_{\nu=o}^{n} p_{n-\nu}\, q_\nu - \frac{1}{r_n}\sum_{i=o}^{n} c_i\phi_i(x)\sum_{\nu=i}^{n} p_{n-\nu}\, q_\nu$$

$$= \frac{1}{r_n}\sum_{i=o}^{n} c_i\phi_i(x)\sum_{\nu=o}^{i-1} p_{n-\nu}\, q_\nu .$$

Since $\{p_n\}$ and $\{q_n\}$ are non-decreasing,

$$p_i \leqslant p_n \quad \text{and} \quad q_i \leqslant q_n \quad \text{for} \quad i \leqslant n.$$

Consequently

$$\int_a^b (s_n(x) - T_n^{(p,q)}(x))^2\, dx \leqslant \frac{p_n^2 q_n^2}{r_n^2}\sum_{i=o}^{n} i^2 c_i^2$$

Now, replacing n by n_k in the above inequality, we have

$$\sum_{k=1}^{\infty}\int_a^b (s_{n_k}(x) - T_{n_k}^{(p,q)}(x))^2 dx \leqslant \sum_{k=1}^{\infty}\frac{p_{n_k}^2 q_{n_k}^2}{r_{n_k}^2}\sum_{i=1}^{n_k} i^2 c_i^2$$

We shall show the convergence of above series by taking the sum upto m terms.

$$\sum_{k=1}^{m} \frac{p_{n_k}^2 q_{n_k}^2}{r_{n_k}^2} \sum_{i=1}^{n_k} i^2 c_i^2$$

$$= O(1) \sum_{k=1}^{m} \frac{1}{n_k^2} \sum_{i=1}^{n_k} i^2 c_i^2$$

$$= O(1) \left[\frac{1}{n_1^2} \sum_{i=1}^{n_1} i^2 c_i^2 + \frac{1}{n_2^2} \sum_{i=1}^{n_2} i^2 c_i^2 + \cdots\cdots + \frac{1}{n_m^2} \sum_{i=1}^{n_m} i^2 c_i^2 \right]$$

$$= O(1) \left[\sum_{i=1}^{n_1} i^2 c_i^2 \sum_{k=1}^{m} \frac{1}{n_k^2} + \sum_{i=n_1+1}^{n_2} i^2 c_i^2 \sum_{k=2}^{m} \frac{1}{n_k^2} + \cdots\cdots + \sum_{i=n_{m-1}+1}^{n_m} \cdot i^2 c_i^2 \frac{1}{n_m^2} \right]$$

Since, the sequence $\{n_k\}$ satisfies the condition (L), the sequence $\{n_k^2\}$ also satisfies the condition (L) and hence

$$\sum_{k=1}^{m} \frac{1}{n_k^2} < \frac{A}{n_1^2}, \quad \sum_{k=2}^{m} \frac{1}{n_k^2} < \frac{A}{n_2^2}, \quad \cdots\cdots$$

Consequently

$$\sum_{k=1}^{m} \frac{p_{n_k}^2 q_{n_k}^2}{r_{n_k}^2} \sum_{i=1}^{n_k} i^2 c_i^2 =$$

$$= O(1) \left[\sum_{i=1}^{n_1} i^2 c_i^2 \frac{1}{n_1^2} + \sum_{i=n_1+1}^{n_2} i^2 c_i^2 \frac{1}{n_2^2} + \cdots\cdots + \sum_{i=n_{m-1}+1}^{n_m} i^2 c_i^2 \cdot \frac{1}{n_m^2} \right]$$

$$= O(1) \sum_{i=1}^{n_m} c_i^2 < \infty.$$

Hence, the result follows by B. Levy's theorem.