CHAPTER - VI

ON STRONG APPROXIMATION OF ORTHOGONAL SERIES

6.1 Let $\{\phi_n(x)\}$, $n=0,1,2\ldots\ldots$ be an orthonormal system (ONS) of L^2-integrable functions defined in the closed interval $[a,b]$. We consider the orthogonal series

$$(6.1.1)\qquad \sum_{n=0}^{\infty} C_n\phi_n(x)$$

with real coefficients C_n's.

The series (6.1.1) is said to be strongly summable (C,α) to the sum $s(x)$, if

$$\frac{1}{A_n^{\alpha}}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}\,(s_k(x)-s(x))^2=o_x(1) \quad \text{as } n\longrightarrow\infty\,,$$

where

$$s_k(x)=\sum_{i=0}^{k} C_i\phi_i(x)\,.$$

The series (6.1.1) is said to be strongly summable (E,q), $q>0$ to the sum $s(x)$, if

$$\frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}(s_k(x)-s(x))^2=o_x(1) \quad \text{as } n\longrightarrow\infty\,.$$

The notations $t_n(x)$, $\{p_n\}\in M^{\alpha}$, $\{p_n\}\in \overline{M}^{\alpha}$ and $\{p_n\}\in M^*$ mean the same as referred in Chapters II and III.

The series (6.1.1) is said to be strongly (N,p_n)-summable to s(x) with $\{p_n\} \in M^{\alpha}, \alpha > 0$, $(p_n\uparrow)$, if

$$\frac{1}{n+1}\sum_{k=0}^{n}(T_k(x)-s(x))^2 \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty$$

where

$$T_k(x)=\frac{1}{P_k}\sum_{\nu=0}^{k}(p_{k-\nu}-p_{k-\nu-1})s_{\nu}(x), \quad p_{-1}=0^{1)}.$$

Let $\{\lambda_n\}$ be an increasing unbounded sequence of non-negative numbers and for any series

$$\sum a_n,$$

we write

$$C^k(\omega)=\sum_{\lambda_m\leqslant\omega}\left(1-\frac{\lambda_m}{\omega}\right)^k a_m \quad (k>0).$$

Let $\mu=\{\mu_m\}$ be a strictly positive sequence of numbers. A sequence $\{X_m\}$ is said to be $[R,\lambda,1,\mu]$ summable to s, if and only if

$$\frac{1}{\lambda_{n+1}}\sum_{m=0}^{n}(\lambda_{m+1}-\lambda_m)\,|X_m-s|^{\mu_m} \longrightarrow 0 .$$

This is called a generalized form of strong Riesz summability.

Let $\omega(f,\delta,c,d)$ denote the continuity modulus of the

---

1) Meder [49]

function $f(x)$ in the interval $[c,d]$ i.e.

$$\omega(f,\delta,c,d)= \sup_{\substack{|t-x|\leq\delta \\ t,x\in[c,d]}} |f(t)-f(x)| .$$

We denote by $\omega(\delta)$ a majorant function of $\omega(f,\delta,c,d)$ i.e. a function satisfying the condition

$$\omega(\delta)\geq\omega(f,\delta,c,d).$$

The ONS $\{\phi_n(x)\}$ is called polynomial-like , if its n^{th} kernel

$$K_n(t,x)=\sum_{k=0}^{n} \phi_k(t)\phi_k(x)$$

has the following structure :

$$(6.1.2)\quad K_n(t,x)=\sum_{k=1}^{r} F_k(t,x) \sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(n)} \phi_{n+i}(t)\phi_{n+j}(x),$$

where p and r are natural numbers independent of n and the constants $|\gamma_{i,j,k}^{(n)}|$ have a common bound independent of n, while the measurable functions $F_k(t,x)$ satisfy the condition

$$F_k(t,x)=O\left(\frac{1}{|t-x|}\right)$$

for every $x\in[a,b]$. We assume that ϕ_{n+i} with negative index is identically equal to zero.

The ONS $\{\phi_n(x)\}$ is said to be constant-preserving, if $\phi_o(x)$ = constant.

The n^{th}(C,1)- means of Fourier series and the Walsh expansion of a function f(x) satisfying the Lipschitz condition were approximated by Bernstein[1] and Fine[2] respectively. The strong (C,1)- summability of Fourier series, conjugate Fourier series and orthogonal series was investigated by Alexits[3], Alexits and Kralik,[4] Alexits and Leindler[5], Sun Yong Sheng[6] and Turan[7]. Alexits and Kralik[8] have also discussed the strong de la Vallèe Poussin summability for the orthogonal series.

Dealing with the strong (C,1)-summability of the orthogonal series (6.1.1) Alexits[9] has proved the following theorem.

<u>THEOREM A</u> : Let $\{\phi_n(x)\}$ be a constant-preserving polynomial-like ONS with respect to the weight function $\varrho(x)$ satisfying the conditions

(6.1.3) $$\sum_{k=o}^{n} \phi_k^2(x)=O(n)$$

and

1) Bernstein [13]
2) Fine [19]
3) Alexits([3], [4])
4) Alexits and Kralik([6],[8])
5) Alexits and Leindler [9]
6) Sun Yong Sheng [80]
7) Turan [90]
8) Alexits and Kralik [7]
9) Alexits([4], p.295)

and

(6.1.4) $\quad 0 \leq g(x) \leq$ const.
uniformly in the subinterval $[c,d]$ of $[a,b]$.
Let $s_n(x)$ denote the n^{th} partial sum of the expansion of an L^2-integrable and on $[c,d]$ continuous function $f(x)$ with the continuity modulus $\omega(f,\delta,c,d)$. If $\omega(f,\delta,c,d)$ possesses a majorant function $\omega(\delta)$ such that $\omega(\delta)/\delta^{\frac{1}{2}-\gamma}$ with some fixed $\gamma > 0$ increases monotonely to infinity as $\delta \longrightarrow 0$, then the relation

$$\frac{1}{n+1}\sum_{\nu=0}^{n} |f(x)-s_\nu(x)| = O[\omega(\tfrac{1}{n})]$$

holds uniformly on every interval $[c+\epsilon, d-\epsilon] \subset (c,d)$.

In this chapter we generalize the above result to strong $(C, \alpha>0)$-summability and also prove the analogous result for the strong Euler means as follows :

THEOREM 1 : Let $\{\phi_n(x)\}$ be a constant-preserving polynomial-like ONS satisfying the condition(6.1.3) uniformly in the subinterval $[c,d]$ of $[a,b]$. Let $s_n(x)$ denote the n^{th} partial sum of the expansion of an L^2-integrable and on $[c,d]$ continuous function $f(x)$ with the continuity modulus $\omega(f,\delta, c,d)$. If $\omega(\delta)/\delta^{\frac{1}{2}-\gamma}$ with some fixed $\gamma > 0$ increases monotonely to infinity as $\delta \longrightarrow 0$, then the relation

$$\frac{1}{A_n^{\alpha}}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |s_\nu(x)-f(x)| = O[\omega(\tfrac{1}{n})]$$

holds uniformly on every interval $[c+\epsilon, d-\epsilon] \subset (c,d)$.

THEOREM 2 : Let $\{\phi_n(x)\}$ be a constant-preserving polynomial-like ONS satisfying the condition (6.1.3) uniformly in the sub-interval $[c,d]$ of $[a,b]$. Let $s_n(x)$ denote the n^{th} partial sum of the expansion of L^2-integrable and on $[c,d]$ continuous function $f(x)$ with the continuity modulus $\omega(f,\delta,c,d)$. If $\omega(\delta)\delta^{\gamma}$ with some fixed $\gamma \geq 0$ increases monotonely to infinity as $\delta \to 0$, then the relation

$$\frac{1}{(1+q)^n}\sum_{\nu=0}^{n}\binom{n}{\nu}q^{n-\nu}\left|f(x)-s_\nu(x)\right| = O[\omega(\tfrac{1}{n})]$$

holds uniformly on every interval $[c+\epsilon, d-\epsilon] \subset (c,d)$.

Concerning the strong Cesàro-summability of orthogonal series (6.1.1) Sunouchi[1] has proved the following theorem.

THEOREM B : If the orthogonal series (6.1.1) with

$$\sum_{n=0}^{\infty} c_n^2 < \infty \qquad (6.1.5)$$

is (C,1)-summable to f(x) almost everywhere in $[a,b]$, then

$$\lim_{n\to\infty}\frac{1}{A_n^{\beta}}\sum_{\nu=0}^{n}A_{n-\nu}^{\beta-1}\left|s_\nu(x)-f(x)\right|^k = 0$$

almost everywhere in $[a,b]$ for any $\beta>0$ and $k>0$.

Maddox[2] has generalized Sunouchi's result which concerns with the weakening of the hypothesis rather than with strengthening of the conclusion by proving the following theorem.

1) Sunouchi [78]

2) Maddox [44]

THEOREM C : Let

$$\sum \frac{\lambda_n}{\lambda_{n+1}} c_n^2 < \infty$$

and suppose that for $k > o$, the sequence $(C^k(\lambda_{n+1}))$ corresponding to the orthogonal series (6.1.1) is summable $[R,\lambda,1,2]$ to $f(x)$ almost everywhere on $[a,b]$. Then for any sequence $\{\mu_m\}$ with $o < \inf \mu_m \leq \mu_m \leq 2$, we have that the series (6.1.1) is $[R,\lambda, 1,\mu]$ summable to $f(x)$ almost everywhere on $[a,b]$.

We prove in this chapter the theorem analogous to the Theorems B and C, where we extend these results to Nörlund summability as follows :

THEOREM 3 : If the orthogonal series (6.1.1) is (N,p_n)-summable to $f(x)$ almost everywhere and the conditions $\{p_n\} \in \overline{M}^{\alpha}$, $\alpha > \frac{1}{2}$ and (6.1.5) are satisfied then

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=o}^{n} |T_k(x) - f(x)|^{\mu_k} = o$$

holds almost everywhere for any sequence $\{\mu_k\}$ with $o < \inf \mu_k \leq \mu_k \leq 2$.

6.2 For the proof of the theorems we need some preliminary lemmas.

LEMMA[1] 1 : If $\{p_n\} \in M^{\alpha}$, $\alpha > \frac{1}{2}$, then

$$\lim_{n \to \infty} \frac{n}{P_n^2} \sum_{k=o}^{n} \frac{P_k^2}{(k+1)^2} = \frac{1}{2\alpha - 1} .$$

---

1) Meder [48]

LEMMA 2[1]: If $\{p_n\} \in M^*$, then

$$\sup_{k, n \geq k} \frac{|p_{n-k} P_n - p_n P_{n-k}|}{k p_n p_{n-k}} < +\infty .$$

Remark : Lemma 2 holds, if we replace the class M^* by the class $\overline{M}^{\alpha}$ with $\alpha > o$.

LEMMA 3[2]: For any value of $q > o$, the following evaluation is valid :

$$\max_{o \leq k \leq n} \binom{n}{k} q^k \leq A_q \frac{(1+q)^n}{\sqrt{n}} , \quad n=1,2 \ldots .$$

where the constant A_q does not depend on n.

6.3 PROOF OF THEOREM 1 : Since the ONS $\{\phi_n(x)\}$ is constant-preserving

$$\frac{1}{A_n^{\alpha}} \sum_{\nu=o}^{n} A_{n-\nu}^{\alpha-1} \left| f(x) - s_{\nu}(x) \right|$$

$$= \frac{1}{A_n^{\alpha}} \sum_{\nu=o}^{n} A_{n-\nu}^{\alpha-1} \left| \int_a^b \{f(t)-f(x)\} \; K_{\nu}(t,x) dt \right| .$$

For $x \in [c+\epsilon, d-\epsilon]$, we divide the integral on the R.H.S. of above equality into three parts :

$$I_{\nu 1} = \int_a^c + \int_d^b , \quad I_{\nu 2} = \int_c^{x-1/n} + \int_{x+1/n}^{d} , \quad I_{\nu 3} = \int_{x-1/n}^{x+1/n} .$$

Now, for $n \geq n_{\epsilon} > \frac{1}{\epsilon}$

1) Meder [49]

2) Ziza [95]

$$\sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} |I_{\nu 1}| =$$

$$= \sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} \left| \left(\int_a^c + \int_d^b \right) \{f(t)-f(x)\} \; K_\nu(t,x)dt \right|$$

$$= \sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} \left| \left(\int_a^c + \int_d^b \right) \{f(t)-f(x)\} \sum_{k=1}^{r} F_k(t,x) \sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(\nu)} \cdot \right.$$

$$\left. \cdot \phi_{\nu+i}(t)\phi_{\nu+j}(x) \; dt \right|$$

$$= \sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} \left| \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(\nu)} \phi_{\nu+j}(x) \left(\int_a^c + \int_d^b \right) \{f(t)-f(x)\} \cdot \right.$$

$$\left. \cdot F_k(t,x)\phi_{\nu+i}(t)dt \right|.$$

Let us put

$$g_k(t,x) = \begin{cases} (f(t)-f(x))F_k(t,x), & t \in [a,c] \cup [d,b] \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} |I_{\nu 1}| =$$

$$= \sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} \left| \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(\nu)} \phi_{\nu+j}(x) \int_a^b g_k(t,x)\phi_{\nu+i}(t)dt \right|$$

$$\leq \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} \left|\gamma_{i,j,k}^{(\nu)}\right| \; \left|\phi_{\nu+j}(x)\right| \; \left| \int_a^b g_k(t,x)\phi_{\nu+i}(t)dt \right| .$$

Applying Cauchy's inequality, we get

$$\sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} |I_{\nu 1}| \leq$$

$$\leq \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \left[\sum_{\nu=n_\epsilon}^{n} \left(\frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} \right)^2 (\gamma_{i,j,k}^{(\nu)})^2 \phi_{\nu+j}^2(x) \sum_{\nu=n_\epsilon}^{n} \left\{ \int_a^b g_k(t,x) \cdot \phi_{\nu+i}(t)dt \right\}^2 \right]^{\frac{1}{2}}.$$

Now

$$(6.3.1) \quad \sum_{\nu=n_\epsilon}^{n} \phi_{\nu+j}^2(x) \leq \sum_{\nu=0}^{n} \phi_{\nu+j}^2(x) = \sum_{\nu=0}^{n+j} \phi_\nu^2(x)$$

as the function $\phi_\nu(x)$ with negative index is considered to be identically equal to zero. Also due to Bessel's inequality

$$(6.3.2) \quad \sum_{\nu=n_\epsilon}^{n} \left\{ \int_a^b g_k(t,x)\phi_{\nu+i}(t)dt \right\}^2 \leq \int_a^b g_k^2(t,x)dt.$$

Hence, using (6.3.1), (6.3.2) and the condition (6.1.3), we get

$$\sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} |I_{\nu 1}| = O\left(\frac{1}{n} \right) \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \left[\sum_{\nu=0}^{n+j} \phi_\nu^2(x) \int_a^b g_k^2(t,x)dt \right]^{\frac{1}{2}}$$

$$= O\left(\frac{1}{\sqrt{n}}\right) \sum_{k=1}^{r} \left\{ \int_a^b g_k^2(t,x)dt \right\}^{\frac{1}{2}}.$$

Moreover, taking into consideration that $F_k(t,x)=O(|t-x|^{-1})$, it follows that $F_k(t,x)$ remains bounded for $t\in[a,c] \cup [d,b]$ as $|t-x|\geq\epsilon$ and therefore,

$$(6.3.3)\quad \sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} |I_{\nu 1}| = O\left(\frac{1}{\sqrt{n}}\right) \left[\left(\int_a^c + \int_d^b \right) \{f(t)-f(x)\}^2\, dt \right]^{\frac{1}{2}}$$

$$= O\left(\frac{1}{\sqrt{n}}\right).$$

Since $\omega(\delta)/\delta^{\frac{1}{2}-\gamma}$ increases monotonely to infinity as $\delta \to 0$

$$\omega\left(\frac{1}{n}\right) n^{\frac{1}{2}-\gamma} \geq 1$$

for sufficiently large n and therefore

$$\sqrt{n}\;\omega\left(\frac{1}{n}\right) > n^{\frac{1}{2}-\gamma}\,\omega\left(\frac{1}{n}\right) > 1 .$$

Consequently, it follows from (6.3.3) that

$$\sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} |I_{\nu 1}| = O\left[\frac{1}{\sqrt{n}}\sqrt{n}\;\omega\left(\frac{1}{n}\right)\right].$$

Thus

$$(6.3.4)\quad \sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} |I_{\nu 1}| = O\left[\omega\left(\frac{1}{n}\right)\right]$$

holds for all $x \in [c+\epsilon, d-\epsilon]$.

Now, we proceed to estimate the sum

$$\sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} |I_{\nu 2}| .$$

We have

$$\sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} |I_{\nu 2}| =$$

$$= \sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} \left| \left(\int_c^{x-1/n} + \int_{x+1/n}^{d} \right) \{f(t)-f(x)\} \; K_\nu(t,x)dt \right|$$

$$= \sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} \left| \left(\int_c^{x-1/n} + \int_{x+1/n}^{d} \right) \{f(t)-f(x)\} \sum_{k=1}^{r} F_k(t,x) \sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(\nu)} \cdot \right.$$

$$\left. \cdot \phi_{\nu+i}(t)\phi_{\nu+j}(x)dt \right|$$

$$= \sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} \left| \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(\nu)} \phi_{\nu+j}(x) \left(\int_c^{x-1/n} + \int_{x+1/n}^{d} \right) \{f(t)-f(x)\} \cdot \right.$$

$$\left. \cdot F_k(t,x)\phi_{\nu+i}(t)dt \right| .$$

Let us put

$$h_k(t,x) = \begin{cases} (f(t)-f(x))F_k(t,x), & t\in [c,x-1/n] \cup [x+1/n,d] \\ 0 & \text{otherwise.} \end{cases}$$

Then, we get

$$\sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} |I_{\nu 2}| \leq$$

$$\leq \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} \left|\gamma_{i,j,k}^{(\nu)}\right| \left|\phi_{\nu+j}(x)\right| \left| \int_a^b h_k(t,x)\phi_{\nu+i}(t)dt \right| .$$

Applying Cauchy's inequality, we get

$$\sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} |I_{\nu 2}| \leq$$

$$\leq \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \left[\sum_{\nu=n_\epsilon}^{n} \left(\frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} \right)^2 \left(\gamma_{i,j,k}^{(\nu)} \right)^2 \phi_{\nu+j}^2(x) \sum_{\nu=n_\epsilon}^{n} \left\{ \int_a^b h_k(t,x) \cdot \right. \right.$$

$$\left. \left. \cdot \phi_{\nu+i}(t)dt \right\}^2 \right]^{\frac{1}{2}}$$

$$= O\left(\frac{1}{n}\right) \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \left[\sum_{\nu=n_\epsilon}^{n} \phi^2_{\nu+j}(x) \sum_{\nu=n_\epsilon}^{n} \left\{ \int_a^b h_k(t,x)\phi_{\nu+i}(t)dt \right\}^2 \right]^{\frac{1}{2}}.$$

On account of (6.3.1), the condition (6.1.3) and Bessel's inequality, we have

$$\sum_{\nu=n_\epsilon}^{n} \frac{A^{\alpha-1}_{n-\nu}}{A^{\alpha}_{n}} |I_{\nu 2}| =$$

$$= O\left(\frac{1}{\sqrt{n}}\right) \sum_{k=1}^{r} \left\{ \int_a^b h^2_k(t,x)dt \right\}^{\frac{1}{2}}$$

$$= O\left(\frac{1}{\sqrt{n}}\right) \sum_{k=1}^{r} \left[\left(\int_c^{x-1/n} + \int_{x+1/n}^{d} \right) \left\{ f(t)-f(x) \right\}^2 F^2_k(t,x)dt \right]^{\frac{1}{2}}$$

$$= O\left(\frac{1}{\sqrt{n}}\right) \left[\left(\int_c^{x-1/n} + \int_{x+1/n}^{d} \right) \left\{ f(t)-f(x) \right\}^2 \frac{1}{(t-x)^2} dt \right]^{\frac{1}{2}}$$

$$= O\left(\frac{1}{\sqrt{n}}\right) \left[\left(\int_c^{x-1/n} + \int_{x+1/n}^{d} \right) \frac{\omega^2(f, |t-x|)}{(t-x)^2} dt \right]^{\frac{1}{2}}.$$

i.e.

$$(6.3.5) \quad \sum_{\nu=n_\epsilon}^{n} \frac{A^{\alpha-1}_{n-\nu}}{A^{\alpha}_{n}} |I_{\nu 2}| = O\left(\frac{1}{\sqrt{n}}\right) \left[\left(\int_{c-x}^{-1/n} + \int_{1/n}^{d-x} \right) \frac{\omega^2(f, |u|)}{u^2} du \right]^{\frac{1}{2}}$$

Since $\omega(\delta)/\delta^{\frac{1}{2}-\gamma}$ increases monotonely as $\delta \to 0$, the first integral on the R.H.S. of (6.3.5) i.e.

$$\int_{c-x}^{-1/n} \frac{\omega^2(f, |u|)}{u^2} du \leq \int_{c-x}^{-1/n} \frac{\omega^2(|u|)}{|u|^{1-2\gamma} |u|^{1+2\gamma}} du \leq$$

$$\leq \omega^2(\tfrac{1}{n})n^{1-2\gamma} \int_{c-x}^{-1/n} \frac{du}{(-u)^{1+2\gamma}}$$

$$= \omega^2(\tfrac{1}{n}) \frac{n^{1-2\gamma}}{2\gamma} (n^{2\gamma} - (x-c)^{-2\gamma})$$

$$\leq \frac{n}{2\gamma}\omega^2(\tfrac{1}{n}).$$

i.e. $$\int_{c-x}^{-1/n} \frac{\omega^2(f,|u|)}{u^2}\,du = O\left[n\omega^2(\tfrac{1}{n})\right].$$

Now, we proceed to estimate the second integral on the R.H.S. of (6.3.5)

Since $\omega(f,\delta)$ increases monotonely and bounded in the interval $[1/n, d-x]$, effecting second mean value theorem, we obtain

$$\int_{1/n}^{d-x} \frac{\omega^2(f,|u|)}{u^2}\,du = \omega^2(f,1/n)\int_{1/n}^{\xi} \frac{du}{u^2} + \omega^2(f,d-x)\int_{\xi}^{d-x} \frac{du}{u^2} \leq$$

$$\leq \omega^2(\tfrac{1}{n}) \int_{1/n}^{\xi} \frac{du}{u^2} + \omega^2(f,d-c)\int_a^b \frac{du}{u^2}$$

$$\leq \omega^2(\tfrac{1}{n})(n-\xi^{-1}) + \omega^2(d-c)(a^{-1}-b^{-1})$$

$$\leq n\,\omega^2(\tfrac{1}{n}) + \omega^2(d-c)(a^{-1}-b^{-1}).$$

i.e.

$$\int_{1/n}^{d-x} \frac{\omega^2(f,|u|)}{u^2}\,du = O\left[n\omega^2(\tfrac{1}{n})\right].$$

Hence, it follows from (6.3.5) and above estimates that

$$(6.3.6)\quad \sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} |I_{\nu 2}| = O\left(\frac{1}{\sqrt{n}}\right) O\left[\sqrt{n}\,\omega(\tfrac{1}{n})\right] = O\left[\omega(\tfrac{1}{n})\right].$$

is true for all $x \in [c+\epsilon, d-\epsilon]$.

Finally to estimate the integral sum

$$\sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} |I_{\nu 3}|$$

We first of all obtain from Schwarz's inequality

$$I_{\nu 3}^2 \leq \int_{x-1/n}^{x+1/n} \{f(t)-f(x)\}^2 \, dt \int_{x-1/n}^{x+1/n} K_\nu^2 (t,x)dt$$

$$= \int_{x-1/n}^{x+1/n} \{ f(t)-f(x) \}^2 \, dt \sum_{k=o}^{\nu} \phi_k^2(x)$$

$$= O(\nu) \int_{x-1/n}^{x+1/n} \{ f(t)-f(x)\}^2 \, dt$$

$$= O(\nu)\, \omega^2(f,\tfrac{1}{n}) \tfrac{2}{n}$$

$$= O(\tfrac{\nu}{n})\, \omega^2(\tfrac{1}{n}).$$

Hence

$$\sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} |I_{\nu 3}| \leq \left\{ \sum_{\nu=n_\epsilon}^{n} \left(\frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} \right)^2 \sum_{\nu=n_\epsilon}^{n} I_{\nu 3}^2 \right\}^{\frac{1}{2}}$$

$$\leq \left\{ \sum_{\nu=o}^{n} \left(\frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} \right)^2 \sum_{\nu=n_\epsilon}^{n} I_{\nu 3}^2 \right\}^{\frac{1}{2}}$$

$$= O\left(\left(\frac{n^{\alpha-1}}{n^{\alpha}} \right)^{\frac{1}{2}} \right) \left\{ \sum_{\nu=o}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} \sum_{\nu=n_\epsilon}^{n} O(\tfrac{\nu}{n})\, \omega^2(\tfrac{1}{n}) \right\}^{\frac{1}{2}}$$

$$= O(\tfrac{1}{n}) O[\omega(\tfrac{1}{n})] \left\{ \sum_{\nu=n_\epsilon}^{n} \nu \right\}^{\frac{1}{2}}$$

Thus

$$(6.3.7)\qquad \sum_{\nu=n_\epsilon}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^{\alpha}} |I_{\nu 3}| = O\left[\omega\left(\tfrac{1}{n}\right)\right]$$

is valid for all $x \in [c+\epsilon, d-\epsilon]$.

Consequently, it follows from (6.3.4), (6.3.6) and (6.3.7) that the relation

$$\frac{1}{A_n^{\alpha}} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |s_\nu(x)-f(x)| = O\left[\omega\left(\tfrac{1}{n}\right)\right]$$

holds uniformly on $[c+\epsilon, d-\epsilon]$.

With this the theorem is proved.

6.4 PROOF OF THEOREM 2 : Since the ONS $\{\phi_n(x)\}$ is constant-preserving

$$\frac{1}{(1+q)^n} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} |f(x)-s_\nu(x)|$$

$$= \frac{1}{(1+q)^n} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} \left| \int_a^b \{f(t)-f(x)\} K_\nu(t,x)dt \right|.$$

For $x \in [c+\epsilon, d-\epsilon]$, we divide the integral on the R.H.S. of the above equality into three parts :

$$I_{\nu 1} = \int_a^c + \int_d^b, \quad I_{\nu 2} = \int_c^{x-1/n^{3/2}} + \int_{x+1/n^{3/2}}^{d}, \quad I_{\nu 3} = \int_{x-1/n^{3/2}}^{x+1/n^{3/2}}.$$

Now, for $n \geq n_\epsilon > \frac{1}{\epsilon}$

$$\sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} |I_{\nu 1}| =$$

$$= \sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} \left| \left(\int_a^c + \int_d^b \right) \{f(t)-f(x)\} \; K_\nu(t,x)dt \right|$$

$$= \sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} \left| \left(\int_a^c + \int_d^b \right) \{f(t)-f(x)\} \sum_{k=1}^{r} F_k(t,x) \sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(\nu)} \cdot \right.$$

$$\left. \cdot \varnothing_{\nu+i}(t) \varnothing_{\nu+j}(x)dt \right|$$

$$= \sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} \left| \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(\nu)} \varnothing_{\nu+j}(x) \left(\int_a^c + \int_d^b \right) \{f(t)-f(x)\} \; \cdot \right.$$

$$\left. \cdot F_k(t,x) \varnothing_{\nu+i}(t)dt \right|$$

$$\leq \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} \left| \gamma_{i,j,k}^{(\nu)} \right| \left| \varnothing_{\nu+j}(x) \right| \left| \int_a^b g_k(t,x) \varnothing_{\nu+i}(t)dt \right| ,$$

where

$$g_k(t,x) = \begin{cases} (f(t)-f(x)) \; F_k(t,x), & t \in [a,c] \cup [d,b] \\ 0 & \text{otherwise} . \end{cases}$$

By Cauchy's inequality and Lemma 3, we have

$$\sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} \; |I_{\nu 1}| \leq$$

$$\leq \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \left[\sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu}^2 \; q^{2(n-\nu)} \; \left(\gamma_{i,j,k}^{(\nu)} \right)^2 \varnothing_{\nu+j}^2(x) \sum_{\nu=n_\epsilon}^{n} \; \cdot \right.$$

$$\left. \cdot \left\{ \int_a^b g_k(t,x) \varnothing_{\nu+i}(t)dt \right\}^2 \right]^{\frac{1}{2}}$$

Using (6.3.1), (6.3.2) and the condition (6.1.3), we get

$$\sum_{\nu=n_\epsilon}^{n}\binom{n}{\nu}q^{n-\nu}|I_{\nu 1}| = O\left(\frac{(1+q)^n}{\sqrt{n}}\right)\sum_{k=1}^{r}\sum_{i,j=-p}^{p}\left[\sum_{\nu=0}^{n+j}\phi_\nu^2(x)\int_a^b g_k^2(t,x)dt\right]^{\frac{1}{2}}$$

$$= O((1+q)^n)\sum_{k=1}^{r}\left[\int_a^b g_k^2(t,x)dt\right]^{\frac{1}{2}}.$$

Moreover, taking into consideration that $F_k(t,x)=O(|t-x|^{-1})$, it follows that $F_k(t,x)$ remains bounded for $t\in[a,c]\cup[d,b]$ as $|t-x|\geq\epsilon$ and therefore

$$\sum_{\nu=n_\epsilon}^{n}\binom{n}{\nu}q^{n-\nu}\ |I_{\nu 1}| = O((1+q)^n)\left[\left(\int_a^c+\int_d^b\right)\left\{f(t)-f(x)\right\}^2dt\right]^{\frac{1}{2}}.$$

i.e.

$$(6.4.1)\qquad \sum_{\nu=n_\epsilon}^{n}\binom{n}{\nu}q^{n-\nu}\ |I_{\nu 1}| = O((1+q)^n).$$

Since $\omega(\delta)\delta^\gamma$ increases monotonely as $\delta\longrightarrow 0$,

$$\omega\left(\frac{1}{n}\right)\ \frac{1}{n^\gamma} > 1$$

for sufficiently large n and also $\gamma\geq 0$ gives

$$\omega\left(\frac{1}{n}\right)\geq\omega\left(\frac{1}{n}\right)\ \frac{1}{n^\gamma} > 1.$$

Consequently, it follows from (6.4.1) that

$$(6.4.2)\qquad \sum_{\nu=n_\epsilon}^{n}\binom{n}{\nu}q^{n-\nu}\ |I_{\nu 1}| = (1+q)^n O\left[\omega\left(\frac{1}{n}\right)\right]$$

holds for all $x\in[c+\epsilon,\ d-\epsilon]$.

Now, we proceed to estimate the sum

$$\sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} \left| I_{\nu 2} \right| .$$

$$\sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} \left| I_{\nu 2} \right| =$$

$$= \sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} \left| \left(\int_{c}^{x-1/n^{3/2}} + \int_{x+1/n^{3/2}}^{d} \right) \left\{ f(t)-f(x) \right\} K_\nu(t,x)dt \right|$$

$$= \sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} \left| \left(\int_{c}^{x-1/n^{3/2}} + \int_{x+1/n^{3/2}}^{d} \right) \left\{ f(t)-f(x) \right\} \sum_{k=1}^{r} F_k(t,x) \sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(\nu)} \cdot \right.$$

$$\left. \cdot \phi_{\nu+i}(t) \phi_{\nu+j}(x)dt \right|$$

$$\leqslant \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} \left| \gamma_{i,j,k}^{(\nu)} \right| \left| \phi_{\nu+j}(x) \right| \left| \int_{a}^{b} h_k(t,x) \phi_{\nu+i}(t)dt \right| ,$$

where

$$h_k(t,x) = \begin{cases} (f(t)-f(x))F_k(t,x), & t \in [c, x-1/n^{3/2}] \cup [x+1/n^{3/2}, d] \\ 0 & \text{otherwise.} \end{cases}$$

Effecting Cauchy's inequality, we obtain

$$\sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} \left| I_{\nu 2} \right| \leqslant$$

$$\leqslant \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \left[\sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu}^2 q^{2(n-\nu)} \left(\gamma_{i,j,k}^{(\nu)} \right)^2 \phi_{\nu+j}^2(x) \sum_{\nu=n_\epsilon}^{n} \cdot \right.$$

$$\left. \cdot \left\{ \int_{a}^{b} h_k(t,x) \phi_{\nu+i}(t)dt \right\}^2 \right]^{\frac{1}{2}}$$

Using Lemma 3, (6.3.1), the condition (6.1.3) and Bessel's inequality, we have

$$\sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} |I_{\nu 2}| =$$

$$= O((1+q)^n) \sum_{k=1}^{r} \left\{ \int_a^b h_k^2(t,x)\,dt \right\}^{\frac{1}{2}}$$

$$= O((1+q)^n) \sum_{k=1}^{r} \left[\left(\int_c^{x-1/n^{3/2}} + \int_{x+1/n^{3/2}}^{d} \right) \{f(t)-f(x)\}^2 F_k^2(t,x)\,dt \right]^{\frac{1}{2}}$$

$$= O((1+q)^n) \left[\left(\int_c^{x-1/n^{3/2}} + \int_{x+1/n^{3/2}}^{d} \right) \{f(t)-f(x)\}^2 \frac{1}{(t-x)^2}\,dt \right]^{\frac{1}{2}}$$

$$= O((1+q)^n) \left[\left(\int_c^{x-1/n^{3/2}} + \int_{x+1/n^{3/2}}^{d} \right) \frac{\omega^2(f, |t-x|)}{(t-x)^2}\,dt \right]^{\frac{1}{2}}$$

$$= O((1+q)^n) \left[\left(\int_{c-x}^{-1/n^{3/2}} + \int_{1/n^{3/2}}^{d-x} \right) \frac{\omega^2(f, |u|)}{u^2}\,du \right]^{\frac{1}{2}}.$$

i.e.

$$(6.4.3) \quad \sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} |I_{\nu 2}| = O((1+q)^n) \left[\left(\int_{c-x}^{-1/n^{3/2}} + \int_{1/n^{3/2}}^{d-x} \right) . \frac{\omega^2(f, |u|)}{u^2}\,du \right]^{\frac{1}{2}}.$$

To estimate the first integral on the R.H.S. of (6.4.3), we use the monotone increasing nature of $\omega(f,\delta)$ and therefore we have

$$\int_{c-x}^{-1/n^{3/2}} \frac{\omega^2(f,|u|)}{u^2}\,du \leq \omega^2(f, \frac{1}{n^{3/2}}) \int_{c-x}^{-1/n^{3/2}} \frac{du}{u^2}$$

$$\leq \omega^2(f, \frac{1}{n}) \int_a^b \frac{du}{u^2}$$

$$\leq \omega^2(\frac{1}{n})(a^{-1}-b^{-1}).$$

i.e.

$$\int_{c-x}^{-1/n^{3/2}} \frac{\omega^2(f,|u|)}{u^2}\,du = O[\omega^2(\frac{1}{n})] .$$

For the second integral on the R.H.S. of (6.4.3) taking into consideration the monotone increasing nature of $\omega(f,\delta)$ and boundedness in the interval $[1/n^{3/2}, d-x]$, applying second mean value theorem, we obtain

$$\int_{1/n^{3/2}}^{d-x} \frac{\omega^2(f,|u|)}{u^2}\,du = \omega^2(f, \frac{1}{n^{3/2}}) \int_{1/n^{3/2}}^{\xi'} \frac{du}{u^2} + \omega^2(f, d-x) \int_{\xi'}^{d-x} \frac{du}{u^2}$$

$$\leq \omega^2(f,\frac{1}{n}) \int_a^b \frac{du}{u^2} + \omega^2(f, d-c) \int_a^b \frac{du}{u^2}$$

$$\leq \omega^2(\frac{1}{n})(a^{-1}-b^{-1}) + \omega^2(d-c)(a^{-1}-b^{-1}).$$

i.e.

$$\int_{1/n^{3/2}}^{d-x} \frac{\omega^2(f,|u|)}{u^2}\,du = O[\omega^2(\frac{1}{n})] ,$$

Consequently, we have from (6.4.3)

$$(6.4.4)\quad \sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} \left|I_{\nu 2}\right| = (1+q)^n O\left[\omega\left(\tfrac{1}{n}\right)\right]$$

for all $x \in [c+\epsilon, d-\epsilon]$.

In order to estimate the integral sum

$$\sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} \left|I_{\nu 3}\right|$$

We first of all obtain from Schwarz's inequality

$$I_{\nu 3}^2 \leq \int_{x-1/n^{3/2}}^{x+1/n^{3/2}} \left\{f(t)-f(x)\right\}^2 dt \int_{x-1/n^{3/2}}^{x+1/n^{3/2}} K_\nu^2(t,x)\,dt$$

$$\leq \int_{x-1/n^{3/2}}^{x+1/n^{3/2}} \left\{f(t)-f(x)\right\}^2 dt \sum_{k=0}^{\nu} \phi_k^2(x)$$

$$= O(\nu) \int_{x-1/n^{3/2}}^{x+1/n^{3/2}} \left\{f(t)-f(x)\right\}^2 dt$$

$$= O(\nu)\, \omega^2\left(f, \frac{1}{n^{3/2}}\right) \frac{2}{n^{3/2}}$$

$$= O\left(\frac{\nu}{n^{3/2}}\right) \omega^2\left(f, \frac{1}{n}\right)$$

$$= O\left(\frac{\nu}{n^{3/2}}\right) \omega^2\left(\frac{1}{n}\right).$$

Hence due to Lemma 3

$$\sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} \left|I_{\nu 3}\right| \leq \left[\sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu}^2 q^{2(n-\nu)} \sum_{\nu=n_\epsilon}^{n} I_{\nu 3}^2\right]^{\frac{1}{2}}$$

$$= \left[O\left(\frac{(1+q)^n}{\sqrt{n}}\right) \quad (1+q)^n \, \omega^2\left(\frac{1}{n}\right) \frac{1}{n^{3/2}} \sum_{\nu=n_\epsilon}^{n} \nu \right]^{\frac{1}{2}}$$

$$= (1+q)^n \, O\left[\omega\left(\frac{1}{n}\right)\right].$$

i.e.

$$(6.4.5) \quad \sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} \left| I_{\nu 3} \right| = (1+q)^n \, O\left[\omega\left(\frac{1}{n}\right)\right]$$

is true for all $x \in [c+\epsilon, d-\epsilon]$.

Consequently, it follows from (6.4.2), (6.4.4) and (6.4.5) that the relation

$$\frac{1}{(1+q)^n} \sum_{\nu=o}^{n} \binom{n}{\nu} q^{n-\nu} \left| f(x) - s_\nu(x) \right| = O\left[\omega\left(\frac{1}{n}\right)\right]$$

holds for all $x \in [c+\epsilon, d-\epsilon]$.

This completes the proof of the theorem.

6.5 PROOF OF THEOREM 3 : We have

$$\frac{1}{n+1} \sum_{k=o}^{n} (T_k(x) - f(x))^2 \leqslant$$

$$\leqslant \frac{2}{n+1} \sum_{k=o}^{n} (T_k(x) - t_k(x))^2 + \frac{2}{n+1} \sum_{k=o}^{n} (t_k(x) - f(x))^2 .$$

Since, the series (6.1.1) is (N, p_n)-summable, it follows that the 2nd term on the R.H.S. of above relation tends to zero almost everywhere. Hence, in order to prove the theorem for the case $\mu_k = 2$, it remains to prove that the relation

$$\sum_{k=o}^{n} (T_k(x)-t_k(x))^2 = o(n+1)$$

holds almost everywhere.

We have

$T_k(x)-t_k(x) =$

$$= \frac{1}{p_k}\sum_{r=o}^{k} (p_{k-r}-p_{k-r-1})s_r(x) - \frac{1}{P_k}\sum_{r=o}^{k} p_{k-r}s_r(x)$$

$$= \frac{1}{p_k}\sum_{r=o}^{k} (p_{k-r}-p_{k-r-1}) \sum_{m=o}^{r} c_m\emptyset_m(x) - \frac{1}{P_k}\sum_{r=o}^{k} p_{k-r} \sum_{m=o}^{r} c_m\emptyset_m(x)$$

$$= \frac{1}{p_k}\sum_{m=o}^{k} c_m\emptyset_m(x) \sum_{r=m}^{k} (p_{k-r}-p_{k-r-1}) - \frac{1}{P_k}\sum_{m=o}^{k} c_m\emptyset_m(x) \sum_{r=m}^{k} p_{k-r}$$

$$= \frac{1}{p_k}\sum_{m=o}^{k} p_{k-m}c_m\emptyset_m(x) - \frac{1}{P_k}\sum_{m=o}^{k} P_{k-m}c_m\emptyset_m(x)$$

$$= \frac{1}{p_kP_k} \sum_{m=o}^{k} (p_{k-m}P_k-p_k P_{k-m})c_m\emptyset_m(x) .$$

Hence, we find that

$$\sum_{k=o}^{\infty} \int_a^b \frac{(T_k(x)-t_k(x))^2}{k+1} dx$$

$$= \sum_{k=o}^{\infty} \frac{1}{k+1}\sum_{m=o}^{k} \left(\frac{p_{k-m}P_k-p_kP_{k-m}}{p_kP_k} \right)^2 c_m^2$$

$$= \sum_{m=o}^{\infty} c_m^2 \sum_{k=m}^{\infty} \frac{1}{(k+1)} \left(\frac{p_{k-m}P_k-p_kP_{k-m}}{p_kP_k} \right)^2$$

$$= \sum_{m=o}^{\infty} c_m^2 \sum_{k=m}^{2m} \frac{1}{(k+1)} \left(\frac{p_{k-m}P_k - p_k P_{k-m}}{p_k P_k}\right)^2 +$$

$$+ \sum_{m=o}^{\infty} c_m^2 \sum_{k=2m+1}^{\infty} \frac{1}{(k+1)} \left(\frac{p_{k-m}P_k - p_k P_{k-m}}{p_k P_k} \right)^2$$

$$= \sum_{m=o}^{\infty} c_m^2 \sum_{k=m}^{2m} \frac{1}{(k+1)} \left(\frac{p_{k-m}P_k - p_k P_{k-m}}{m p_k p_{k-m}} \right)^2 \frac{m^2 p_{k-m}^2}{P_k^2} +$$

$$+ \sum_{m=o}^{\infty} c_m^2 \sum_{k=2m+1}^{\infty} \frac{1}{(k+1)} \left(\frac{p_{k-m}P_K - p_k P_{k-m}}{m p_k p_{k-m}} \right)^2 \frac{m^2 p_{k-m}^2}{P_k^2} .$$

Using Lemma 2, we have

$$\sum_{k=o}^{\infty} \int_a^b \frac{(T_k(x) - t_k(x))^2}{k+1} dx$$

$$= O(1) \sum_{m=o}^{\infty} m^2 c_m^2 \sum_{k=m}^{2m} \frac{p_{k-m}^2}{(k+1)P_k^2} + O(1) \sum_{m=o}^{\infty} m^2 c_m^2 \sum_{k=2m+1}^{\infty} \frac{p_{k-m}^2}{(k+1)P_k^2}$$

$$= O(1) \sum_{m=o}^{\infty} m^2 c_m^2 \frac{1}{(m+1)P_m^2} \sum_{k=m}^{2m} p_{k-m}^2 + O(1) \sum_{m=o}^{\infty} m^2 c_m^2 \sum_{k=2m+1}^{\infty} \frac{1}{k^3}$$

$$= O(1) \sum_{m=o}^{\infty} c_m^2 \frac{m}{P_m^2} \sum_{k=o}^{m} p_k^2 + O(1) \sum_{m=o}^{\infty} m^2 c_m^2 \sum_{k=2m+1}^{\infty} \frac{1}{k^3}$$

$$= O(1) \sum_{m=o}^{\infty} c_m^2 \frac{m}{P_m^2} \sum_{k=o}^{m} \frac{P_k^2}{k^2} + O(1) \sum_{m=o}^{\infty} c_m^2 .$$

Hence, by Lemma 1, we obtain

$$\sum_{k=o}^{\infty} \int_a^b \frac{(T_k(x) - t_k(x))^2}{k+1} dx = O(1) \sum_{m=o}^{\infty} c_m^2 < \infty .$$

Consequently, by B.Levy's theorem it follows that

$$\sum_{k=0}^{\infty} \frac{(T_k(x)-t_k(x))^2}{k+1} < \infty$$

almost everywhere in (a,b) and therefore, by Kronecker's lemma, we have

$$(6.5.1) \qquad \sum_{k=0}^{n} (T_k(x)-t_k(x))^2 = o(n+1).$$

Now, write

$$R_k = |T_k - f|.$$

Then, we have $o < C < \frac{\mu_k}{2} \leq 1$ for some C.

Hence $$R_k^{\mu_k} \leq R_k^2 \quad \text{if} \quad R_k \geq 1$$

and $$R_k^{\mu_k} \leq R_k^{2C} \quad \text{if} \quad R_k < 1$$

i.e.

$$(6.5.2) \qquad R_k^{\mu_k} \leq R_k^2 + J_k^C ,$$

where $J_k = o$ if $R_k \geq 1$ and $J_k = R_k^2$ if $R_k < 1$.

Then, by Holder's inequality

$$\frac{1}{n+1} \sum_{k=0}^{n} J_k^C \leq$$

$$\leq \frac{1}{n+1} \sum_{k=0}^{n} R_k^{2C}$$

$$\leqslant \frac{1}{n+1}\left[\sum_{k=o}^{n}(R_k^{2C})^{\frac{1}{C}}\right]^C\left[\sum_{k=o}^{n}1^{\frac{1}{1-C}}\right]^{1-C}$$

$$= \frac{1}{n+1}\left[\sum_{k=o}^{n}R_k^2\right]^C (n+1)^{1-C}$$

$$= \left[\frac{1}{n+1}\sum_{k=o}^{n}R_k^2\right]^C \longrightarrow o \quad \text{as} \quad n \rightarrow \infty$$

due to (6.5.1). Whence from (6.5.2)

$$\frac{1}{n+1}\sum_{k=o}^{n}R_k^{\mu_k} \longrightarrow o \quad \text{as} \quad n \rightarrow \infty,$$

almost everywhere in $[a, b]$.

i.e. $$\frac{1}{n+1}\sum_{k=o}^{n}\left|T_k(x)-f(x)\right|^{\mu_k} \longrightarrow o \quad \text{as} \quad n \rightarrow \infty$$

almost everywhere in $[a, b]$.

With this the theorem is proved.