CHAPTER - 9

ON THE CONVERGENCE OF LACUNARY ORTHOGONAL SERIES

Let $\{\phi_n(x)\}$ (n = o, 1, 2, ...) be an orthonormal system of L^2 - integrable functions defined in the closed interval [a, b]. Let $\lambda(x) \leq x$ denote a positive function, concave from below, defined for $x \geq 1$ and increasing monotonely to infinity. We shall call the orthogonal series

$$(9.1.1) \qquad \sum_{n=o}^{\infty} c_n \phi_n(x)$$

$\lambda(n)$ lacunary [1], if the number of non-vanishing coefficients c_k with $n < k \leq 2n$ does not exceed $\lambda(n)$. Furthermore we shall say that the coefficients have the positive number sequence $\{q_n\}$ as a majorant if the relation

$$c_n = O(q_n)$$

holds.

The notations $s_n(x)$, $T_n(x)$ and $\sigma_n(\lambda, x)$

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1) Alexits ([5], p. 130)

have the same meaning as considered in Chapter 2.

The Cesàro summability of order α of $\lambda(n)$ - lacunary orthogonal series (9.1.1) was discussed by Alexits[1]. He has proved the following theorem :

<u>Theorem A</u> : If the coefficients of $\lambda(n)$ lacunary orthogonal series (9.1.1) have as a majorant a positive monotone decreasing number sequence $\{q_n\}$, satisfying the condition

$$(9.1.2) \qquad \sum_{n=1}^{\infty} \frac{\sqrt{\lambda(n)}\, q_n}{n} < \infty ,$$

then the condition

$$(9.1.3) \qquad \sum_{n=0}^{\infty} c_n^2 < \infty ,$$

implies the (C, α) summability almost everywhere of the orthogonal series (9.1.1).

The (E, q) summability for $q > 0$ of the $\lambda(n)$- lacunary orthogonal series (9.1.1) was discussed by Sapre and Bhatnagar[2]. Similarly Nörlund summability was discussed by Kantawala P.S.[3]

In this chapter we extend the above results

1) Alexits G. ([5], p. 130)
2) Sapre and Bhatnagar [107]
3) Kantawala P.S. [50]

to $(\bar{N}, p_n)$ summability and Riesz summability of order 1. Our results are as follows.

Theorem 1 : Let $p_n(- \quad M^{\alpha}$ and the coefficients of $\lambda(n)$-lacunary orthogonal series (9.1.1) have as a majorant a positive monotone decreasing sequence of numbers $\{q_n\}$ satisfying the condition (9.1.3) and (9.1.2). Then the orthogonal series (9.1.1) is $(\bar{N}, p_n)$ summable almost everywhere.

Theorem 2 : Let $\{v_n\}$ denote a strictly increasing sequence satisfying the condition

$$(9.1.4) \qquad 1 < \ell \leqslant \frac{\lambda_{v_n + 1}}{\lambda_{v_n}} \leqslant k$$

where ℓ and k are constants, independent of n and the coefficients of $\lambda(n)$ lacunary orthogonal series (9.1.1) have as a majorant a positive monotone decreasing number sequence $\{q_n\}$ satisfying the condition (9.1.3) and (9.1.2). Then the series (9.1.1) is $(R, \lambda_n, 1)$ summable almost everywhere.

If we take in to consideration that between the indices n and 2n there are excatly n free places, then in the above theorem we may exclude the condition of lacunarity

This remark enables us to point out a special case from Theorem 1 and Theorem 2, in which the condition of lacunarity does not more appear.

<u>Theorem 3</u>: <u>If the coefficients of the orthogonal series (9.1.1) have as a majorant a positive, monotone decreasing sequence</u> $\{q_n\}$ <u>satisfying the condition</u>

$$(9.1.5) \qquad \sum_{n=1}^{\infty} \frac{q_n}{\sqrt{n}} < \infty \quad \text{and}$$

<u>(9.1.3) holds, then the orthogonal series (9.1.1) is</u> $(\bar{N}, p_n)$ <u>summable almost everywhere.</u>

<u>Theorem 4</u> : <u>If the coefficients of the orthogonal series (9.1.1) have as a majorant a positive, monotone decreasing sequence</u> $\{q_n\}$ <u>satisfying the condition (9.1.5) and (9.1.3), then the orthogonal series (9.1.1) is</u> $(R, \lambda_n, 1)$ <u>summable almost everywhere.</u>

While, discussing the convergence of $\lambda(n)$ lacunary orthogonal series Alexits has proved the following theorem.

<u>Theorem B</u> : If the coefficients of a $\lambda(n)$ - lacunary series (9.1.1) have as a majorant the positive monotone decreasing sequence $\{q_n\}$ for which

$$\sum_{n=1}^{\infty} q_n^2 \log^{\beta} n < \infty \qquad (o \leqslant \beta \leqslant 2)$$

holds, then the condition

$$\lambda(n) = O\left(\frac{n}{\log^{2-\beta} n}\right)$$

implies the convergence of the orthogonal series (9.1.1) almost everywhere.

In this chapter we generalize the above result for higher order.

Theorem 5 : If the coefficient of a $\lambda(n)$ - lacunary orthogonal series (9.1.1) have as a majorant the positive monotone decreasing sequence $\{q_n\}$ for which,

$$(9.1.6) \qquad \sum_{n=1}^{\infty} q_n^2 (\log n)^{p\beta} < \infty \qquad (o \leqslant \beta \leqslant 2)$$

holds, then the condition

$$(9.1.7) \qquad \lambda(n) = O\left(\frac{n}{\log^{2-\beta} n}\right)^{p} \qquad o < p \leqslant 1$$

implies the convergence of (9.1.1) orthogonal series almost everywhere.

For proving this theorem we need the following Lemmas.

Lemma 1[1] : If the coefficients of a $\lambda(n)$ lacunary orthogonal series have as a majorant a positive monotone decreasing sequence $\{q_n\}$, satisfying the condition (9.1.2) then the condition (9.1.3) implies the (C, α) summability almost everywhere of orthogonal series(9.1.1).

Lemma 2 : The series

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{2p} n^{2-p}} \quad \text{is convergent.}$$

Proof : Here

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{2p} n^{2-p}}$$

$$= \sum_{k=2}^{\infty} 2^k \frac{1}{(k\log 2)^{2p}\, 2^{(2-p)k}}$$

$$= O(1) \sum_{k=2}^{\infty} \frac{1}{k^{2p}}\, 2^{kp-k} ,$$

By ratio test,

$$= \lim_{k \to \infty} \frac{2^{(k+1)(p-1)}}{(k+1)^{2p}} \frac{k^{2p}}{2^{k(p-1)}}$$

$$= \lim_{k \to \infty} 2^{p-1} \left(\frac{k}{k+1}\right)^{2p}$$

$$= 2^{p-1} .$$

1) Alexits G. ([5], p. 130)

Therefore, the given series is convergent if $p < 1$.

Lemma 3 :[1] Under the condition (9.1.3) the relation $s_{v_n}(x) - \sigma_{v_n}(x) = o_x(1)$ is valid almost everywhere for every index sequence $\{v_n\}$ with $\frac{v_{n+1}}{v_n} \geq q > 1$.

Lemma 4 :[2] If the real numbers $a_o, a_1, a_2, \ldots a_N$ and the orthonormal system $\{\psi_n(x)\}$ are arbitrarily given, there exist an L_μ^2 - integrable function $\delta_N(x) \geq o$ with the condition;

$$\max_{v \leq N} \left| \sum_{k=o}^{v} a_k \psi_k(x) \right| \leq \delta_N(x)$$

$$\int_a^b \delta_N^2(x)\, d\mu(x) = O(\log^2 N) \sum_{k=o}^{N} a_k^2 .$$

In order to prove the above theorems we need the following Lemmas.

Lemma 5 :[3] Let $\{p_n\}$ be a nonnegative monotonic increasing or decreasing sequence of real numbers such that $P_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and $np_n = O(P_n)$.

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1) Alexits G. ([5], p. 118)
2) Alexits G. ([5], p. 79)
3) Sharma J.P. [110]

Let $\{n_k\}$ be an arbitary increasing sequence of numbers satisfying the following condition of lacunarity.

$$1 < q \leqslant \frac{n_{k+1}}{n_k} \leqslant r \qquad \text{for } k = o, 1, 2, \ldots\ldots$$

where q and r are constants. Then the orthogonal series (9.1.1) with coefficients satisfying the condition (9.1.3) is $(\bar{N}, p_n)$ summable almost everywhere, if the sequence of partial sums $\{s_{n_k}(x)\}$ is convergent almost everywhere.

Lemma 6[1]: Let $\{v_n\}$ denotes a strictly increasing sequence satisfying the condition (9.1.4). In order that orthogonal series (9.1.1) should be summable $(R, \lambda_n, 1)$ in a set E almost everywhere, it is necessary and sufficient that the sequence $\{s_{v_n}(x)\}$ of partial sums should converge in E almost everywhere.

Proof of Theorem 1 :

Under the condition (9.1.2) and (9.1.3), Alexit G.[2] has proved the convergence almost everywhere of the sequence $\{s_{2^n}(x)\}$ of the partial sums of the orthogonal

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1) Meder J. [76]

2) Alexits G. ([5], p.130)

series (9.1.1). Hence by Lemma 5, the series is $(\bar{N}, p_n)$ summable almost everywhere.

<u>Proof of Theorem 2</u> :

By Lemma 6, the result directly follows in the direction of theorem 1.

<u>Proof of Theorem 3</u> :

Alexits[1] has proved that the condition (9.1.5) implies the condition (9.1.3).

But the condition (9.1.5) is a special case of (9.1.2) corresponding to $\lambda(n) = n$ and therefore this case is as mentioned above, satisfied for every series. Hence our theorem follows from theorem 1.

<u>Proof of Theorem 4</u> :

Result follows from Theorem 2, in the direction of theorem 3.

<u>Proof of Theorem 5</u> :

Here,

$$\sum_{n=2}^{\infty} \frac{\sqrt{\lambda(n)}\, q_n}{n}$$

$$= O(1) \cdot \sum_{n=2}^{\infty} \frac{q_n}{n} \left(\frac{n}{\log^{2-\beta} n} \right)^{\frac{p}{2}}$$

1) Alexits G. ([5], p. 132)

$$= O(1) \sum_{n=2}^{\infty} q_n \frac{n^{\frac{p}{2}-1}}{(\log n)^{p-\frac{p\beta}{2}}}$$

Applying Schwarz inequality, we have

$$= O(1)\left\{ \sum_{n=2}^{\infty} q_n^2 (\log n)^{p\beta} \sum_{n=2}^{\infty} \frac{1}{(\log n)^{2p} n^{2-p}} \right\}^{\frac{1}{2}}$$

by Lemma 2 we have,

$$= O(1)\left\{ \sum_{n=2}^{\infty} q_n^{\ 2} (\log n)^{p\beta} \right\}^{\frac{1}{2}}$$

$$< \infty$$

Obviously we have (9.1.3) from (9.1.6). By Lemma 1 the series (9.1.1) is (C, $\alpha > o$) summable almost everywhere, so by Lemma 3 the convergence almost everywhere of the sequence $s_{2^m}(x)$ follows. It remains to show that apart from a set of measure zero the relation,

(9.1.8) $\qquad s_n(x) - s_{2^m}(x) = o_x(1) \qquad (n \to \infty,\ 2^m < n \leq 2^{m+1}),$

holds.

Let $C_{v_i(m)}$ $(i = 1, \ldots\ldots, M_n)$ denote the positive, nonzero coefficients with indices between 2^m and 2^{m+1}, by Lemma 4 there exist such a function $\delta_m(x)$ that

$$(9.1.9) \quad |s_n(x) - s_{2^m}(x)| \leqslant \delta_m(x) \quad (2^m < n < 2^{m+1})$$

and $\int_a^b \delta_m^{\,2}(x)\, d\mu(x) = O(\log^2 M_m) \sum_{i=1}^{M_m} C_{v_i}^{\,2}(m)$

are satisfied since $M_m = O(\lambda(2^m))$ and

$C_{v_i}^{\,2}(m) \leqslant q_{v_i}^{2}(m) \leqslant q_{2^m}^{2} \quad (i = 1, \ldots, M_m)$ hold, an

account of (9.1.7) we have,

$$\int_a^b \delta_m^{\,2}(x)\, d\mu(x) = O(\log^2 M_m) \sum_{i=1}^{M_m} C_{v_i}^{2}(m)$$

$$= O\left(\log^2 \left(\frac{2^m}{\log^{2-\beta} 2^m}\right)^p\right) q_{2^m}^{2}\, M_m$$

$$= O(1) \log^2 \left(\frac{2^{mp}}{(m(\log 2))^{(2-\beta)p}}\right) \cdot$$

$$q_{2^m}^{2} \frac{2^{mp}}{(m\log 2)^{(2-\beta)p}}$$

$$= O(1) \left[\log 2^{mp} - \log \left((m\log 2)^{(2-\beta)p}\right)\right]^2$$

$$q_{2^m}^{2} \frac{2^{mp}}{(m \log 2)^{(2-\beta)p}}$$

$$= O(1)\ (mp - \log m^{(2-\beta)p})^2\ q_{2^m}^2\ \frac{2^{mp}}{m^{(2-\beta)p}}$$

$$= O(1)\ [\ m^2p^2 + (\log m^{(2-\beta)p})^2\ ]\ q_{2^m}^2\ \frac{2^{mp}}{m^{(2-\beta)p}}$$

$$= O(1)\ [m^2p^2 + m^2(2-\beta)^2p^2]\ q_{2^m}^2\ \frac{2^{mp}}{m^{(2-\beta)p}}$$

$$= O(1)\ \frac{m^2p^2}{m^{2p}}\ q_{2^m}^2\ m^{p\beta}\ 2^{mp}$$

$$= O(1)\ m^{p\beta}\ q_{2^m}^2\ 2^{mp}\ .$$

Hence,

$$\sum_{m=2}^{\infty} \int_a^b \sigma_m^2(x)\ dx = O(1) \sum_{m=2}^{\infty} m^{p\beta}\ q_{2^m}^2\ 2^{mp}$$

$$= O(1) \sum_{m=2}^{\infty} q_n^2\ (\log n)^{\beta}$$

$$< \infty\ .$$

Hence by B. Levy's theorem, the series

$\sum_{m=2}^{\infty} \sigma_m^2(x)$ converges almost everywhere.

i.e. $\sigma_m(x) \longrightarrow 0$ almost everywhere.

Hence from (9.1.9) the relation (9.1.8) is satisfied almost everywhere.

Hence the proof.