CHAPTER - 2

CERTAIN SUMMABILITY MEANS OF GENERAL ORTHOGONAL SERIES

Let $\{\phi_n(x)\}$ (n=o, 1,2,.....) be an orthonormal system (ONS) of L^2 - integrable functions defined in the closed interval [a, b]. We consider the orthogonal series

$$(2.1.1) \qquad \sum_{n=o}^{\infty} c_n \phi_n(x)$$

with real coefficients c_n's.

Let us denote the partial sums, (C,α) means, (R, 1) means, $(\bar{N}, p_n)$ means, Riesz means and de-la Valle'e - poussin's means, of the series (2.1.1) by

$$S_n(x) = \sum_{k=o}^{n} c_k \phi_k(x),$$

$$\sigma_n^{\alpha}(x) = \frac{1}{A_n^{\alpha}} \sum_{k=o}^{n} A_{n-k}^{\alpha-1} S_k(x)$$

$$L_n(x) = \frac{1}{\log n} \sum_{k=o}^{n} \frac{S_k(x)}{k},$$

$$\bar{T}_n(x) = \frac{1}{P_n} \sum_{k=o}^{n} p_k S_k(x),$$

$$\sigma_n(\lambda, x) = \sum_{k=o}^{n} (1 - \frac{\lambda_k}{\lambda_{n+1}}) c_k \phi_k(x),$$

$$V_n(x) = \sum_{k=n}^{2n-1} \frac{S_k(x)}{n},$$

respectively.

We denote as usual the (C,1) means, (E, 1) means, $(R, \lambda_n, 1)$ means and (N, p_n) means of the orthogonal series(2.1.1) by

$\sigma_n(x)$, $T_n(x)$, $\sigma_n(\lambda, x)$ and $t_n(x)$.

An increasing sequence of natural numbers

$n_1 < n_2 < \ldots\ldots\ldots < n_k < \ldots\ldots\ldots$

is said to satisfy the condition (L), if the series $\Sigma \frac{1}{n_k}$ satisfies condition (L) i.e.,

$$\sum_{k=m}^{\infty} \frac{1}{n_k} = O(\frac{1}{n_m})$$ [1)]

Sunouchi[2)] has proved the following theorem concerning the (C, 1) means of(2.1.1).

Theorem :- If

(2.1.2) $|\phi_n(x)| \leqslant k$ $(n=o,1,2,\ldots\ldots)$

then

---

1) Bary N.K. [12]
2) Sunouchi G. [118]

$$\int_a^b \sum_{n=1}^{\infty} \frac{|S_n(x) - \sigma_n(x)|^q}{n} \, dx \leq A \sum_{n=1}^{\infty} n^{q-2} |c_n|^q, \quad q>1.$$

Similarly, Patel[1] and Kantawala[2] have found similar orders of approximation of,

$$(2.1.3) \qquad \sum_{n=1}^{\infty} \frac{|S_n(x) - T_n(x)|^k}{n}, \qquad k \geq 2$$

$$(2.1.4) \qquad \sum_{n=1}^{\infty} \frac{|S_n(x) - \sigma_n(\lambda, x)|^k}{n}, \qquad k \geq 2.$$

$$(2.1.5) \qquad \sum_{n=1}^{\infty} \frac{|S_n(x) - t_n(x)|^k}{n}, \qquad k \geq 2$$

The convergence of the series (2.1.3), (2.1.4) and (2.1.5) for $k = 2$ have been studied by Meder[3], Patel[4] and Kantawala[5].

In this chapter we first prove the analogus results for (C, α) summability, $(\bar{N}, p_n)$ summability, $(R, 1)$ summability, Riesz summability of order 1, de-la Valle'e, poussin's summability for $k = 2$ and then we extend the above result for (C, α) summability, $(\bar{N}, p_n)$ summability by considering $k \geq 2$. We prove the following theorems:

---

1) Patel R. K. [94]
2) Kantawala [1]
3) Meder [76]
4) Patel [92]
5) Kantawala [50]

Theorem 1 :- If the coefficients of the orthogonal series(2.1.1) satisfy the condition

$$(2.1.6) \qquad \sum_{n=o}^{\infty} c_n^2 < \infty ,$$

then

$$\sum_{n=1}^{\infty} \frac{[S_n(x) - \sigma_n^{\alpha}(x)]^2}{n} < \infty$$

almost everywhere.

Theorem 2 :- If $p_o > o$, $p_n \geqslant o$, $np_n = O(p_n)$ and the condition(2.1.6) is satisfied, then the series

$$\sum_{n=1}^{\infty} \frac{(S_n(x) - \overline{T}_n(x))^2}{n} < \infty$$

almost everywhere.

Theorem 3 :- If the coefficients of the orthogonal series (2.1.1) satisfy the condition

$$\sum_{k=1}^{\infty} c_k^2 \log k < \infty ,$$

then,

$$\sum_{n=1}^{\infty} \frac{(S_n(x) - L_n(x))^2}{n} < \infty$$

almost everywhere.

Theorem 4 :- If the coefficients of the orthogonal series (2.1.1) satisfy the condition (2.1.6), and $1 < q \leq \frac{\lambda_{n+1}}{\lambda_n}$ then

$$\sum_{n=1}^{\infty} \frac{(S_n(x) - \sigma_n(\lambda, x))^2}{n^p} < \infty, \qquad p > 1$$

almost everywhere.

Theorem 5 :- If the coefficients of the orthogonal series (2.1.1) satisfy the condition (2.1.6) then

$$\sum_{n=1}^{\infty} \frac{(S_n(x) - V_n(x))^2}{n^p} < \infty, \qquad p > 1$$

almost everywhere.

Theorem 6 :- If $|\phi_n(x)| \leq k, \quad n = 0, 1, 2, \ldots\ldots$ then

$$\int_a^b \sum_{n=1}^{\infty} \frac{|S_n(x) - \sigma_n^{\alpha}(x)|^q}{n} dx \leq A \sum_{n=1}^{\infty} n^{q-2} |c_n|^q, \quad q \geq 2.$$

Theorem 7 :- If $p_0 > 0$, $p_n \geq 0$, $np_n = O(P_n)$ and the condition (2.1.2) are satisfied then

$$\int_a^b \sum_{n=1}^{\infty} \frac{|S_n(x) - \bar{T}_n(x)|^q}{n} dx = O(1) \sum_{n=1}^{\infty} |c_n|^q n^{q-2}, \quad q \geq 2.$$

Dealing with the (C, 1) summability Kolmogoroff[1] has proved the following theorem.

1) Kolmogoroff [58]

Theorem A :- Under the condition

$$\sum_{n=o}^{\infty} c_n^2 < \infty$$

the relation $S_{V_n}(x) - \sigma_{V_n}(x) = O_x(1)$ is valid almost everywhere for every index sequence $\{V_n\}$ with

$$\frac{V_{n+1}}{V_n} \geqslant q \geqslant 1.$$

Similar result was proved by Sharma[1)] for $(\bar{N}, p_n)$ summability Using lacunarity. The same result was proved by Sapre[2)] with (L) condition, weaker then lacunarity. In this chapter we extend the above results for Nörlund summability and $(\bar{N}, p_n)$ summability.

Theorem 8 :- Let $\{p_n\}$ be a nonnegative monotone sequence of real numbers such that $P_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and $np_n = O(P_n)$. If an increasing sequence of natural numbers $\{V_n\}$ satisfies the condition (L), then we have under the condition (2.1.6), the relation.

$$S_{V_n}(x) - t_{V_n}(x) = O_x(1)$$

almost everywhere.

Theorem 9 :- Under the same condition as of theorem 8. We have,

$$S_{V_n}(x) - \bar{T}_{V_n}(x) = O_x(1)$$

almost everywhere.

1) Sharma [110] 2) Sapre A. R. [106]

We need the following lemmas for proving the above theorems :

<u>Lemma 1</u>[1] :- (Paley's theorem) : Let $\{\phi_n(x)\}$ be an ONS over the interval (a, b) and

$$|\phi_n(x)| \leq M \quad \text{for} \quad a < x < b.$$

(i) If $f \in L^p$, $1 < p \leq 2$ and $c_1, c_2, \ldots\ldots c_n \ldots$ are the Fourier coefficients of f with respect to $\phi_1, \phi_2, \ldots\ldots, \phi_n \ldots$ then

$$\left\{ \sum_{n=1}^{\infty} |c_n|^p \, n^{p-2} \right\}^{\frac{1}{p}} \leq A_p \left\{ \int_a^b |f|^p \, dx \right\}^{\frac{1}{p}}$$

where A_p depends only on p and M

(ii) If $q \geq 2$ and $c_1, c_2, \ldots\ldots c_n, \ldots$ is a sequence of numbers for which

$$\sum_{n=1}^{\infty} |c_n|^q \, n^{q-2} < \infty$$

then a function $f(x) \in L^q$ (a, b) exists, for which the numbers c_n are Fourier coefficients with respect to the system $\{\phi_n(x)\}$ and

$$\left\{ \int_a^b |f|^q \, dx \right\}^{\frac{1}{q}} \leq B_q \left\{ \sum_{n=1}^{\infty} |c_n|^q \, n^{q-2} \right\}^{\frac{1}{q}},$$

1) Bary N.K. [12]

where B_q depends only on q and M.

<u>Lemma 2</u>[1] :- Suppose that p_n is non-increasing, and that $p_n \geq \sigma > o$, $n = o, 1, 2, \ldots$

Then $(\bar{N}, p_n)$ summability reduces to (N, p_n) summability.

<u>Lemma 3</u>[2] :- If the coefficients of the orthogonal series (2.1.1) satisfies the condition (2.1.6) and

$\{p_n\} \in M^{\alpha}, \quad \alpha \geq o,$

then the series,

$$\sum_{n=1}^{\infty} \frac{(S_n(x) - t_n(x))^2}{n} < \infty ,$$

almost everywhere.

<u>Lemma 4</u>[2] :- If $p_o > o$, $p_n \geq o$ and the condition, $np_n = O(P_n)$ $|\phi_n(x)| \leq k$ is satisfied then

$$\int_a^b \sum_{n=1}^{\infty} \frac{|S_n(x) - t_n(x)|^q}{n} dx = O(1) \sum_{n=1}^{\infty} |c_n|^q n^{q-2}$$

where $q \geq 2$.

---

1) Ishiguro Kazuo [45]
2) Kantawala P.S. [50]

Proof of Theorem 1.

We have,

$$S_n(x) - \sigma_n^\alpha(x) = \sum_{k=o}^{n} c_k\phi_k(x) - \frac{1}{A_n^\alpha}\sum_{V=o}^{n} A_{n-V}^{\alpha-1} S_V(x)$$

$$= \frac{1}{A_n^\alpha}\sum_{k=o}^{n} c_k\phi_k(x)\sum_{V=o}^{n} A_{n-V}^{\alpha-1} - \frac{1}{A_n^\alpha}\sum_{V=o}^{n} A_{n-V}^{\alpha-1} S_V(x)$$

$$= \frac{1}{A_n^\alpha}\sum_{k=o}^{n} c_k\phi_k(x)\sum_{V=o}^{n} A_{n-V}^{\alpha-1} - \frac{1}{A_n^\alpha}\sum_{V=o}^{n} A_{n-v}^{\alpha-1} \sum_{k=o}^{V} c_k\phi_k(x)$$

$$= \frac{1}{A_n^\alpha}\sum_{k=o}^{n} c_k\phi_k(x)\sum_{V=o}^{n} A_{n-v}^{\alpha-1} - \frac{1}{A_n^\alpha}\sum_{k=o}^{n} c_k\phi_k(x)\sum_{V=k}^{n} A_{n-V}^{\alpha-1}$$

$$= \frac{1}{A_n^\alpha}\sum_{k=o}^{n} c_k\phi_k(x)\sum_{V=o}^{k-1} A_{n-V}^{\alpha-1}$$

consequently,

$$(2.1.7)\qquad \sum_{n=1}^{\infty} \frac{1}{n}\int_a^b [S_n(x) - \sigma_n^\alpha(x)]^2\, dx$$

$$= \sum_{n=1}^{\infty} \frac{1}{n(A_n^\alpha)^2}\sum_{k=o}^{n} c_k^2\Big(\sum_{V=o}^{k-1} A_{n-V}^{\alpha-1}\Big)^2$$

Zygmund[1] has proved that A_n^{α} is increasing (as a function of n) for $\alpha > o$ and A_n^{α} is decreasing for $-1 < \alpha < o$.

For $\alpha > 1$, the condition (2.1.7) gives

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_a^b (s_n(x) - \sigma_n^{\alpha}(x))^2 \, dx$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n(A_n^{\alpha})^2} \sum_{k=o}^{n} k^2 c_k^2 (A_n^{\alpha-1})^2$$

$$= O(1) \sum_{n=1}^{\infty} \frac{1}{n} \frac{n^{2\alpha-2}}{n^{2\alpha}} \sum_{k=o}^{n} k^2 c_k^2$$

$$= O(1) \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 c_k^2$$

$$= O(1) \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{\infty} \frac{1}{n^3}$$

$$= O(1) \sum_{k=1}^{\infty} k^2 c_k^2 \frac{1}{k^2}$$

$$= O(1) \sum_{k=1}^{\infty} c_k^2 < \infty .$$

Therefore, by B.Levy's theorem[2],

$$\sum_{n=1}^{\infty} \frac{[s_n(x) - \sigma_n^{\alpha}(x)]^2}{n}$$

converges almost everywhere in (a, b).

---

1) Zygmund [149]

2) Alexits G. [5]

Again for $o < \alpha < 1$, the condition (2.1.7) gives

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_a^b [s_n(x) - \sigma_n^{\alpha}(x)]^2 \, dx$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n(A_n^{\alpha})^2} \sum_{k=o}^{n} k^2 c_k^2 (A_{n-k+1}^{\alpha-1})^2$$

$$= O(1) \sum_{n=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{\infty} \frac{(n-k+1)^{2\alpha-2}}{n^{2\alpha+1}}$$

$$= O(1) \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{\infty} \frac{1}{n^3}$$

$$= O(1) \sum_{k=1}^{\infty} c_k^2 < \infty$$

Therefore, by B. Levy's theorem,

$$\sum_{n=1}^{\infty} \frac{(s_n(x) - \sigma_n^{\alpha}(x))^2}{n} < \infty$$

almost everywhere in (a, b).

Thereby the theorem is completely proved.

<u>Proof of Theorem 2</u> :- We have

$$s_n(x) - \overline{T}_n(x) = \sum_{k=o}^{n} c_k \phi_k(x) - \frac{1}{P_n} \sum_{r=o}^{n} p_r s_r(x)$$

$$= \frac{1}{P_n} \sum_{k=o}^{n} c_k \phi_k(x) \sum_{r=o}^{n} p_r - \frac{1}{P_n} \sum_{r=o}^{n} p_r \sum_{k=o}^{r} c_k \phi_k(x)$$

$$= \frac{1}{P_n} \sum_{k=o}^{n} c_k\phi_k(x) \sum_{r=o}^{n} p_r - \frac{1}{P_n} \sum_{k=o}^{n} c_k\phi_k(x) \sum_{r=k}^{n} p_r$$

$$= \frac{1}{P_n} \sum_{k=o}^{n} c_k\phi_k(x) \sum_{r=o}^{k-1} p_r .$$

Consequently,

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_a^b (S_n(x) - \overline{T}_n(x))^2 \, dx$$

$$= \sum_{n=1}^{\infty} \frac{1}{nP_n^2} \sum_{k=o}^{n} c_k^2 (\sum_{r=o}^{k-1} p_r)^2 .$$

If $\{p_n\}$ is increasing, then the condition (2.1.6) and $np_n = O(P_n)$ gives,

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_a^b (S_n(x) - \overline{T}_n(x))^2 \, dx \leqslant \sum_{n=1}^{\infty} \frac{1}{nP_n^2} \sum_{k=o}^{n} k^2 c_k^2 p_n^2$$

$$= \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{\infty} \frac{p_n^2}{nP_n^2}$$

$$= O(1) \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{\infty} \frac{1}{n^3}$$

$$= O(1) \sum_{k=1}^{\infty} c_k^2 < \infty .$$

Hence by B. Levy's theorem,

$$\sum_{n=1}^{\infty} \frac{(S_n(x) - \overline{T}_n(x))^2}{n} < \infty$$

almost everywhere in [a, b].

If $\{p_n\}$ is non-increasing then by Lemma 2 $(\bar{N}, p_n)$ summability reduces to (N, p_n) summability and for (N, p_n) summability the same result was discussed by Agrawal and Kantawala[1)] (see Lemma 3).

Proof of Theorem 3 :

We have,

$$S_n(x) - L_n(x) = \sum_{k=o}^{n} c_k\phi_k(x) - \frac{1}{\log n} \sum_{V=1}^{n} \frac{S_V}{V}$$

$$= \sum_{k=o}^{n} c_k\phi_k(x) - \frac{1}{\log n} \sum_{V=1}^{n} \frac{1}{V} \sum_{k=1}^{V} c_k\phi_k(x)$$

$$= \sum_{k=o}^{n} c_k\phi_k(x) - \frac{1}{\log n} \sum_{k=1}^{n} c_k\phi_k(x) \sum_{V=k}^{n} \frac{1}{V}$$

$$= \frac{1}{\log n}\left[\log n - \sum_{V=1}^{n} \frac{1}{V}\right] \sum_{k=1}^{n} c_k\phi_k(x) + \frac{1}{\log n} \sum_{k=1}^{n} c_k\phi_k(x) \sum_{V=1}^{k-1} \frac{1}{V} .$$

Since

$$\lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n} - \log n\right) \text{ is finite,}$$

and

$$o < 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n} - \log n, \quad \text{for } n = 1, 2, \ldots$$

...., there exist such a constant $M > 1$ such that

1) Agrawal S. R. and Kantawala P. S. [1]

(2.1.8) $\quad o < 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n} - \log n < M$ for

$n = 1, 2, \ldots\ldots\ldots$, so by inequality.

$$(a + b)^2 \leqslant 2 (a^2 + b^2)$$

We have,

(2.1.9) $$\left(\sum_{V=1}^{n} \frac{1}{V} \right)^2 \leqslant 2 (M^2 + \log^2 n) \quad \text{for } n = 1, 2, \ldots.$$

Hence by (2.1.8) and (2.1.9) we have

$$\int_a^b (S_n(x) - L_n(x))^2 \, dx < \frac{4M^2}{\log^2 n} \sum_{k=1}^{n} c_k^2 + \sum_{k=1}^{n} c_k^2 \log^2 k.$$

Let $\sum_{n=1}^{\infty} \frac{1}{n} \int_a^b (S_n(x) - L_n(x))^2 \, dx = I_1$.

Therefore,

$$I_1 < 4M^2 \left[\sum_{n=4}^{\infty} \frac{1}{n \log^2 n} \sum_{k=1}^{n} c_k^2 + \sum_{n=4}^{\infty} \frac{1}{n \log^2 n} \sum_{k=1}^{n} c_k^2 \log^2 k \right]$$

$$= 4M^2 [\Sigma_1 + \Sigma_2] .$$

Now

$$\Sigma_1 = \sum_{n=4}^{\infty} \frac{1}{n \log^2 n} \sum_{k=1}^{n} c_k^2$$

$$\leqslant \sum_{k=1}^{\infty} c_k^2 \sum_{n=k}^{\infty} \frac{1}{n \log^2 n}$$

$$< \infty ,$$

and

$$\Sigma_2 = \sum_{n=4}^{\infty} \frac{1}{n\log^2 n} \sum_{k=1}^{n} c_k^2 \log^2 k$$

$$= \sum_{k=1}^{\infty} c_k^2 \log^2 k \sum_{n=k}^{\infty} \frac{1}{n\log^2 n}$$

$$= \sum_{k=1}^{\infty} c_k^2 \log^2 k \, \frac{1}{\log k}$$

$$= \sum_{k=1}^{\infty} c_k^2 \log k < \infty.$$

Therefore,

$$\sum_{n=1}^{\infty} \int_a^b \frac{(S_n(x) - L_n(x))^2}{n} \, dx < \infty .$$

Hence by B. Levy's theorem,

$$\sum_{n=1}^{\infty} \frac{(S_n(x) - L_n(x))^2}{n}$$

is convergent almost everywhere.

Proof of Theorem 4 :-

$$S_n(x) - \sigma_n(\lambda, x) = \sum_{k=o}^{n} c_k \phi_k(x) - \sum_{k=o}^{n} \left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right) c_k \phi_k(x)$$

$$= \frac{1}{\lambda_{n+1}} \sum_{k=o}^{n} c_k \phi_k(x) \lambda_k$$

i.e.

$$\int_a^b [S_n(x) - \sigma_n(\lambda, x)]^2 \, dx = \frac{1}{\lambda_{n+1}^2} \sum_{k=o}^{n} \lambda_k^2 c_k^2$$

i.e.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \int_a^b [S_n(x) - \sigma_n(\lambda, x)]^2 dx = \sum_{n=1}^{\infty} \frac{1}{\lambda_{n+1}^2 n^p} \sum_{k=o}^{n} \lambda_k^2 c_k^2$$

$$= \sum_{k=o}^{\infty} c_k^2 \sum_{n=k}^{\infty} \frac{\lambda_k^2}{\lambda_{n+1}^2} \frac{1}{n^p}$$

$$= \sum_{k=o}^{\infty} c_k^2 \Big[\frac{\lambda_k^2}{\lambda_{k+1}^2} \frac{1}{k^p} + \frac{\lambda_k^2}{\lambda_{k+2}^2} \frac{1}{(k+1)^p} + \ldots \Big]$$

$$= \sum_{k=o}^{\infty} c_k^2 \frac{1}{q^2} \sum_{\ell=k}^{\infty} \frac{1}{q^{2\ell}(k+\ell)^p}$$

$$= O(1) \sum_{k=o}^{\infty} c_k^2$$

$$< \infty .$$

Therefore, by B.Levy's theorem,

$$\sum_{n=1}^{\infty} \frac{[S_n(x) - \sigma_n(\lambda, x)]^2}{n^p} < \infty, \qquad p > 1 .$$

Proof of Theorem 5 :-

$$[S_n(x) - V_n(x)]^2 = [\sigma_n(\lambda, x) - V_n(x) + S_n(x) - \sigma_n(\lambda, x)]^2$$

$$\leqslant 2\left\{[\sigma_n(\lambda,x) - V_n(x)]^2 + [S_n(x) - \sigma_n(\lambda, x)]^2\right\}$$

$$\therefore \sum_{n=1}^{\infty} \frac{[S_n(x) - V_n(x)]^2}{n^p} \leqslant 2\left\{\sum_{n=1}^{\infty} \frac{(\sigma_n(\lambda, x) - V_n(x))^2}{n^p} + \sum_{n=1}^{\infty} \frac{(S_n(x) - V_n(x))^2}{n^p}\right\}$$

$$= 2[M_1 + M_2].$$

But M_1 is convergent by Patel[1)]

M_2 is convergent by Theorem 2.

Hence Theorem is proved.

Proof of Theorem 6 :-

$$S_n(x) - \sigma_n^\alpha(x) = \sum_{k=o}^{n} c_k\phi_k(x) - \frac{1}{A_n^\alpha} \sum_{V=o}^{n} A_{n-V}^{\alpha-1} S_V(x)$$

$$= \frac{1}{A_n^\alpha}\sum_{k=o}^{n} c_k\phi_k(x) \sum_{V=o}^{n} A_{n-V}^{\alpha-1} - \frac{1}{A_n^\alpha} \sum_{V=o}^{n} A_{n-V}^{\alpha-1} S_V(x)$$

1) Patel C. M. [92]

$$= \frac{1}{A_n^{\alpha}} \sum_{k=o}^{n} c_k \phi_k(x) \sum_{V=o}^{n} A_{n-V}^{\alpha-1} - \frac{1}{A_n^{\alpha}} \sum_{V=o}^{n} A_{n-V}^{\alpha-1} \sum_{k=o}^{V} c_k \phi_k(x)$$

$$= \frac{1}{A_n^{\alpha}} \sum_{k=o}^{n} c_k \phi_k(x) \sum_{V=o}^{n} A_{n-V}^{\alpha-1} - \frac{1}{A_n^{\alpha}} \sum_{k=o}^{n} c_k \phi_k(x) \sum_{V=k}^{n} A_{n-V}^{\alpha-1}$$

$$= \frac{1}{A_n^{\alpha}} \sum_{k=o}^{n} c_k \phi_k(x) \sum_{V=o}^{k=1} A_{n-V}^{\alpha-1}$$

$$= \sum_{k=o}^{n} c_k \phi_k(x) R_k \text{ where}$$

$$R_k = \frac{\sum_{V=o}^{k-1} A_{n-V}^{\alpha-1}}{A_n^{\alpha}}$$

Using Lemma 1 we have,

$$\int_a^b |S_n(x) - \sigma_n^{\alpha}(x)|^q dx = \int_a^b \left| \sum_{k=o}^{n} c_k \phi_k(x) R_k \right|^q dx$$

$$\leq A_1 \sum_{k=1}^{n} |c_k|^q |R_k|^q k^{q-2}$$

Therefore,

$$\int_a^b \sum_{n=1}^{\infty} \frac{|S_n(x) - \sigma_n^{\alpha}(x)|^q}{n} dx \leq A_1 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} |c_k|^q |R_k|^q k^{q-2}$$

$$(2.1.10) \qquad = A_1 \sum_{k=1}^{\infty} |c_k|^q \, k^{q-2} \sum_{n=k}^{\infty} \frac{1}{n} \left\{ \frac{\sum_{V=o}^{k-1} A_{n-V}^{\alpha-1}}{A_n^{\alpha}} \right\}^q$$

Zygmund has proved that A_n^{α} is a increasing (as a function of n) for $\alpha > o$ and A_n^{α} is decreasing for $-1 < \alpha < o$.

For $\alpha > 1$, we have

$$\sum_{n=k}^{\infty} \left\{ \frac{\sum_{V=o}^{k-1} A_{n-V}^{\alpha-1}}{A_n^{\alpha}} \right\}^q \frac{1}{n}$$

$$< \sum_{n=k}^{\infty} k^q \frac{1}{n^q} O(1) \frac{1}{n}$$

$$= O(1) \, k^q \sum_{n=k}^{\infty} \frac{1}{n^{q+1}}$$

$$= O(1)$$

Therefore condition (2.1.10) gives,

$$\int_a^b \sum_{n=1}^{\infty} \frac{|s_n(x) - \sigma_n^{\alpha}(x)|^q}{n} dx = O(1) \sum_{k=1}^{\infty} |c_k|^q \, k^{q-2}$$

For $o < \alpha < 1$, we have

$$\sum_{n=k}^{\infty} \left\{ \frac{\sum_{V=o}^{k-1} A_{n-V}^{\alpha-1}}{A_n^{\alpha}} \right\}^q \frac{1}{n} = \sum_{n=k}^{\infty} k^q \frac{O\,(n-k+1)^{\alpha-1}}{O\ (n)^{\alpha}} \frac{1}{n}$$

$$= \sum_{n=k}^{\infty} k^q \frac{1}{n^q} O(1) \left\{ \frac{1}{n} \right\}$$

$$= O(1)\, k^2 \sum_{n=k}^{\infty} \frac{1}{n^{q+1}}$$

$$= O(1) .$$

Therefore, condition (2.1.10) gives

$$\int_a^b \sum_{n=1}^{\infty} \frac{|S_n(x) - \sigma_n^{\alpha}(x)|^q}{n}\, dx = O(1) \sum_{k=1}^{\infty} |c_k|^q\, k^{q-2}$$

This complets the proof of our theorem.

Proof of Theorem 7 :-

We have,

$$S_n(x) - \bar{T}_n(x) = \frac{1}{P_n} \sum_{k=o}^{n} c_k \phi_k(x) \sum_{r=o}^{k-1} p_r$$

$$= \sum_{k=o}^{n} c_k \phi_k(x)\, R_k$$

where $R_k = \frac{1}{P_n} \sum_{r=o}^{k-1} p_r$.

Therefore by using Lemma 1, we have,

$$\int_a^b |S_n(x) - \bar{T}_n(x)|^q dx = \int_a^b \left|\sum_{k=o}^{n} c_k \phi_k(x)\right|^q |R_k|^q\, dx$$

$$< A_1 \sum_{k=1}^{n} |c_k|^q\, |R_k|^q\, k^{q-2}$$

Hence,

$$(2.1.11) \quad \int_a^b \sum_{n=1}^{\infty} \frac{|S_n(x) - \bar{T}_n(x)|^q}{n} dx < A_1 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} |c_k|^q |R_k|^q k^{q-2}$$

$$= A_1 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |c_k|^q k^{q-2} \frac{\left(\sum_{r=0}^{k-1} p_r\right)^q}{n(P_n)^q}$$

If p_n is increasing, then

$$\sum_{r=0}^{k-1} p_r \leq k p_n = \frac{k}{n} O(P_n)$$

$$= O\left(\frac{k P_n}{n}\right).$$

Hence,

$$\frac{\left(\sum_{r=0}^{k-1} p_r\right)^q}{n P_n^q} = O(1) \frac{k^q}{n^{q+1}}$$

Hence R.H.S. of (2.1.11) is

$$= A_1 \sum_{k=1}^{\infty} |c_k|^q k^{q-2} \sum_{n=k}^{\infty} \frac{1}{n^{q+1}}$$

Consequently, from (2.1.11)

$$\int_a^b \sum_{n=1}^{\infty} \frac{|S_n(x) - \bar{T}_n(x)|^q}{n} dx = O(1) \sum_{k=1}^{\infty} |c_k|^q k^{q-2}.$$

If p_n is non increasing then by Lemma 2 $(\bar{N}, p_n)$ summubility reduces to (N, p_n) summability and for (N, p_n) summability the same result was discussed by Agrawal and Kantawala[1] (see Lemma 4). This complets the proof of our theorem.

Proof of Theorem 8 :-

We have,

$$S_{V_n}(x) - t_{V_n}(x) = \sum_{k=o}^{V_n} c_k \phi_k(x) - \frac{1}{P_{V_n}} \sum_{k=o}^{V_n} p_{V_n-k} S_k(x)$$

$$= \frac{1}{P_{V_n}} \sum_{k=o}^{V_n} c_k \phi_k(x) \left(\sum_{i=o}^{k-1} p_{V_n-i} \right)$$

Therefore,

$$\int_a^b (S_{V_n}(x) - t_{V_n}(x))^2 dx = \frac{1}{P_{V_n}^2} \sum_{k=o}^{V_n} c_k^2 \left(\sum_{i=o}^{k-1} p_{V_n-i} \right)^2$$

Now if $\{p_n\}$ is increasing,

$$\sum_{i=o}^{k-1} p_{V_n-i} = p_{V_n-k+1} + \cdots\cdots + p_{V_n}$$

$$\leq k\, p_{V_n}$$

1) Agrawal and Kantawala [1]

Hence,

$$\int_a^b (s_{V_n}(x) - t_{V_n}(x))^2 \, dx < \frac{1}{p_{V_n}^2} \sum_{k=o}^{V_n} c_k^2 k^2 p_{V_n}^2$$

$$= O(1) \frac{1}{V_n^2} \sum_{k=o}^{V_n} k^2 c_k^2$$

$$(2.1.12) \quad \sum_{n=1}^{\infty} (s_{V_n}(x) - t_{V_n}(x))^2 \, dx = O(1) \sum_{n=1}^{\infty} \frac{1}{V_n^2} \sum_{k=o}^{V_n} k^2 c_k^2$$

Similarly of $\{p_n\}$ is decreasing,

$$\sum_{i=o}^{k-1} p_{V_n - i} = p_{V_n - k+1} + \dots + p_{V_n}$$

$$< k \, p_{V_n - k+1}$$

$$\int_a^b (s_{V_n}(x) - t_{V_n}(x))^2 \, dx < \frac{1}{p_{V_n}^2} \sum_{k=o}^{V_n} k^2 c_k^2 p_{V_n - k+1}^2$$

$$= O(1) \sum_{k=o}^{V_n} \frac{k^2 c_k^2 V_n^2}{(V_{n-k+1})^2}$$

Therefore,

$$(2.1.13) \quad \sum_{n=1}^{\infty} \int_a^b (s_{V_n}(x) - t_{V_n}(x))^2 \, dx = O(1) \sum_{n=1}^{\infty} \frac{1}{V_n^2} \sum_{k=o}^{V_n} k^2 c_k^2 .$$

From (2.1.12) and (2.1.13), for $\{p_n\}$ monotone,

$$(2.1.14) \qquad \sum_{n=1}^{\infty} \int_a^b (s_{V_n}(x) - t_{V_n}(x))^2 dx = O(1) \sum_{n=1}^{\infty} \frac{1}{V_n^2} \sum_{k=o}^{V_n} k^2 c_k^2$$

Now we will prove the convergence of the series on the right of (2.1.14). Taking the first p terms of the series on the right side of (2.1.14).

$$\frac{1}{V_1^2} \sum_{k=1}^{V_1} k^2 c_k^2 + \frac{1}{V_2^2} \sum_{k=1}^{V_2} k^2 c_k^2 + \ldots + \frac{1}{V_p^2} \sum_{k=V_{p-1}+1}^{V_p} k^2 c_k^2$$

$$= \sum_{k=1}^{V_1} k^2 c_k^2 \sum_{m=1}^{p} \frac{1}{V_m^2} + \sum_{k=V_1+1}^{V_2} k^2 c_k^2 \sum_{m=2}^{p} \frac{1}{V_m^2} + \ldots + \sum_{k=V_{p-1}+1}^{V_p} k^2 c_k^2 \frac{1}{V_p^2}$$

But as the sequence $\{V_n\}$ satisfy the condition (L) so also the sequence $\{V_n^2\}$ (see Bary [2], page8)

Therefore,

$$\sum_{m=k}^{\infty} \frac{1}{V_m^2} < c \frac{1}{V_k^2}, \quad \text{where } c \text{ is constant.}$$

Hence,

$$\sum_{n=1}^{p} \frac{1}{V_n^2} \sum_{k=o}^{V_m} k^2 c_k^2 <$$

$$\sum_{k=1}^{V_1} k^2 c_k^2 \frac{c}{V_1^2} + \sum_{k=V_1+1}^{V_2} k^2 c_k^2 \frac{c}{V_2^2} + \ldots + \sum_{k=V_{p-1}+1}^{V_p} k^2 c_k^2 \frac{c}{V_p^2}$$

$$= c \sum_{m=1}^{p} \frac{1}{V_m^2} \sum_{k=V_{m-1}+1}^{V_m} k^2 c_k^2$$

$$\leq c \sum_{m=1}^{p} \sum_{k=V_{m-1}+1}^{V_m} c_k^2 = c \sum_{k=1}^{V_p} c_k^2$$

$$< \infty .$$

From this, the convergence of series on the right in (2.1.14) follows. So by B.Levy's theorem result directly follows. Hence the proof.

Proof of Theorem 9 :-

We have,

$$S_n(x) - \bar{T}_n(x) = \sum_{k=o}^{n} c_k \phi_k(x) - \frac{1}{P_n} \sum_{k=o}^{n} p_k S_k(x)$$

$$= \frac{1}{P_n} \sum_{V=o}^{n} p_V \sum_{k=o}^{n} c_k \phi_k(x) - \frac{1}{P_n} \sum_{k=o}^{n} p_k S_k(x)$$

$$= \frac{1}{P_n} \sum_{k=o}^{n} c_k \phi_k(x) \sum_{V=o}^{k-1} p_V$$

Now,

$$\int_a^b (S_n(x) - \bar{T}_n(x))^2 \, dx = \frac{1}{P_n^2} \sum_{k=o}^{n} c_k^2 \left(\sum_{V=o}^{k-1} p_V \right)^2$$

$$\leq \frac{1}{P_n^2} \sum_{k=o}^{n} c_k^2 P_k^2 .$$

Therefore,

$$\sum_{n=1}^{\infty} \int_a^b (S_n(x) - \overline{T}_n(x))^2 \, dx < \sum_{n=1}^{\infty} \frac{1}{P_n^2} \sum_{k=o}^{n} c_k^2 \, P_k^2$$

$$= O(1) \sum_{n=1}^{\infty} \frac{1}{n^2 P_n^2} \sum_{k=o}^{n} c_k^2 \, P_k^2$$

since $\{p_n\}$ is increasing we have,

$$p_k \leqslant p_n \quad \text{for} \quad k \leqslant n \, .$$

So,

$$\sum_{n=1}^{\infty} (S_n(x) - \overline{T}_n(x))^2 \, dx = O(1) \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=o}^{n} c_k^2 \, k^2 \, .$$

Now replacing n by V_n in the above inequality we have,

$$\sum_{n=1}^{\infty} \int_a^b (S_{V_n}(x) - t_{V_n}(x))^2 \, dx = O(1) \sum_{n=1}^{\infty} \frac{1}{V_n^2} \sum_{k=1}^{V_n} k^2 c_k^2 \, .$$

The convergence of the right side of the above inequality follows from theorem 8. Hence by B.Levy's theorem the result directly follows. Hence the proof.

If $\{p_n\}$ is non-increasing then by Lemma 2, $(\overline{N}, p_n)$ summability reduces to (N, p_n) summability and result follows from theorem 8.

Hence the proof.