CHAPTER 5

SUMMABILITY OF ORTHOGONAL EXPANSIONS IN POLYNOMIAL LIKE ORTHONORMAL SYSTEM

Let $\{\phi_n(x)\}$ $(n = 0, 1, 2, \ldots..)$ be an orthonormal system (ONS) of $L^2_{\rho(x)}$- integrable functions defined in the closed interval $[a, b]$, with respect to a positive, bounded weight function $\rho(x)$. We consider the orthogonal series

(5.1.1) $$\sum_{n=0}^{\infty} C_n \phi_n(x)$$

with real coefficients C_n's.

The (N, p_n) means and $(\bar{N}, p_n)$ means of the sequence of partial sums $\{S_n(x)\}$ of the orthogonal series (5.1.1) is given by

$$t_n(x) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k(x)$$

$$\bar{T}_n(x) = \frac{1}{P_n} \sum_{k=0}^{n} p_k S_k(x)$$

where $$S_n(x) = \sum_{k=0}^{n} C_k \phi_k(x).$$

The series (5.1.1) is said to be (N, p_n) summable to $S(x)$ and, $(\bar{N}, p_n)$ summable to $S(x)$ respectively if

$$\lim_{n \to \infty} t_n(x) = S(x)$$

and

$$\lim_{n \to \infty} \bar{T}_n(x) = S(x).$$

An ONS $\{\phi_n(x)\}$ is called constant preserving, if $\phi_o(x) =$ constant.

An ONS $\{\phi_n(x)\}$ is called polynomial like if, if its n^{th} Kernel

$$k_n(t, x) = \sum_{k=o}^{n} \phi_k(t)\, \phi_k(x)$$

has the following structure :

$$(5.1.2) \qquad k_n(t,x) = \sum_{k=1}^{\gamma} F_k(t, x) \sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(n)}\, \phi_{n+i}(t)\, \phi_{n+j}(x)$$

where p and γ are natural numbers independent of n and the constants $|\gamma_{i,j,k}^{(n)}|$ have a common bound independent of n, while the measurable functions $F_k(t, x)$ satisfy the condition

$$F_k(t, x) = O\left(\frac{1}{|t - x|}\right)$$

for every $t, x \in [a, b]$. We assume that ϕ_{n+i} with negative index is considered to be identically equal to zero.

Define

$$\bar{N}_n(t, x) = \frac{1}{P_n} \sum_{v=0}^{n} p_v k_v(t, x)$$

$$\bar{Q}_n(x) = \int_a^b | \bar{N}_n(t, x) | \rho(t) \, dt$$

$$N_n(t, x) = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} \, k_v(t, x)$$

and

$$\bar{Z}_n(x) = \int_a^b |N_n(t, x)| \, \rho(t) dt$$

called the n^{th} $(\bar{N}, p_n)$ Kernel, Lebesgue $(\bar{N}, p_n)$ function, (N, p_n) Kernel and Lebesgue (N, p_n) function of the ONS $\{\phi_n(x)\}$, respectively.

The partial sums $S_n(x)$ of the expansions of an $L_{\rho(x)}$ - integrable function $f(x)$ in the functions of an ONS $\{\phi_n(x)\}$ can be represented by

$$I_n(f, x) = \int_a^b f(t) \, \psi_n(t, x) \, \rho(t) \, dt$$

where

$$\psi_n(t, x) = \sum_{k=0}^{n} \phi_k(t) \, \phi_k(x) .$$

The n^{th} sums, of an expansion summed by a linear summation process has also the same integral form, where $\psi_n(t, x)$ denotes the sum

$$\sum_{k=0}^{n} \alpha_{nk} \phi_k(t) \phi_k(x)$$

The integral $I_n(f, x)$ is said to be singular (with singular point x), if for an arbitrary positive number δ and for an arbitrary subinterval $[\alpha, \beta]$ of $[a, b]$, the following condition hold :

(5.1.3) $$\lim_{n \to \infty} \int_I \psi_n(t, x) \rho(t)\, dt = 1 \text{ and}$$

$$\lim_{n \to \infty} \int_J \psi_n(t, x) \rho(t)\, dt = 0$$

with $I = [a, b] \cap [x - \delta, x + \delta]$,

$J = [\alpha, \beta] - [x - \delta, x + \delta]$.

(5.1.4) $$\operatorname*{ess.\,lub}_{t \in [a, b]-[x - \delta, x + \delta]} |\psi_n(t,x)| \leq \psi(\delta)$$

where $\psi(\delta)$ is a number depending on δ but independent of n.

If $\psi_n(t, x)$ satisfies uniformly the conditions (5.1.3) and (5.1.4) in a x-set E, then the Integral

$I_n(f, x)$ is said to be uniformly singular on E.

The effect of Lebesgue functions on the convergence of orthogonal series was first investigated by Kolmogoroff-Seliverstoff[1] and Plessner[2] for the case of Fourier trigonometric series. It was extended to the convergence and Cesàro summability and summability by first logarithmic means by Kaczmarz[3], Tandori[4], Meder[5], and Patel and Sapre[6].

The behaviour of the Lebesgue functions for polynomial - like ONS is investigated by Ratajski[7] and Alexits[8]. The convergence and summability of orthogonal expansions for polynomial - like system has been studied by Zinovev[9] and Alexits[10].

Alexits[11] has proved the following theorem :

<u>Theorem A</u> : -

If the ONS $\left\{\phi_n(x)\right\}$ is polynomial - like and the condition

$$\sum_{k=0}^{n} \phi_k^2(x) = O_x(n)$$

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1) Kolmogoroff - Seliverstoff([59],[60])
2) Plessner [97]
3) Kaczmarz [51]
4) Tandori([127],[135],[137])
5) Meder [76]
6) Patel and Sapre [93]
7) Ratajski ([101],[102])
8) Alexits ([5],p.206)
9) Zinovev [147]
10) Alexits ([5], p.267)
11) Alexits ([5], p.206,267)

is fulfilled in the set E, then the relation

$$L_n^1(x) = O_x(1)$$

holds almost everywhere in E.

Theorem B :-

Let $\{\phi_n(x)\}$ be a complete, constant-preserving polynomial - like ONS with respect to the weight function $\rho(x)$. Suppose that the functions $F_k(t, x)$ are continuous in the square $a \leqslant t \leqslant b$, $a \leqslant x \leqslant b$ with eventual exception of the diagonal $t = x$ and that the two conditions

$$\sum_{k=o}^{n} \phi_k^2(x) = O(n)$$

and

(5.1.5) $$o < \rho(x) \leqslant \text{constant}$$

are also satisfied in the subinterval [C, d] of [a, b]. If the $L^2_{\rho(x)}$ - integrable function f(x) is continuous in [C, d], then its expansion

(5.1.6) $$f(x) \sim \sum_{n=o}^{\infty} C_n \phi_n(x),$$

is uniformly (C, 1) - summable in every inner subinterval of [C, d], the sum being f(x).

Similar results were proved by Kantawala[1] for Riesz means and Euler means.

In this Chapter we extend the above results to n^{th} Lebesgue (N, p_n) function and n^{th} Lebesgue $(\bar{N}, p_n)$ function for polynomial like ONS and to the (N, p_n) summability and $(\bar{N}, p_n)$ summability of orthogonal expansion for the constant-preserving polynomial-like ONS. Our results are as follows.

Theorem 1 :- *If the ONS $\{\phi_n(x)\}$ is polynomial-like and the condition*

(5.1.7) $$\phi_n(x) = O_x(1)$$

is fulfilled in the set E, then the relation

$$\bar{Q}_n(x) = O_x(1)$$

holds almost everywhere on E.

Theorem 2 :- *Let $\{\phi_n(x)\}$ be a complete constant preserving polynomial-like ONS with respect to the weight function $\rho(x)$. Suppose that the functions $F_k(t,x)$ are continuous in the square $a \leq t \leq b$, $a \leq x \leq b$ with eventual exception of the diagonal $t = x$ and that the two conditions,* (5.1.8) $$\phi_n(x) = O(1)$$

and (5.1.5) are satisfied in the subinterval [c, d] of [a, b]. If the $L^2_{\rho(x)}$ - integrable function f(x) is continuous in [c, d], then its expansion (5.1.6) is uniformly $(\bar{N}, p_n)$ summable in every inner sub-interval of [c, d], the sum being f(x).

Theorem 3 :- If the ONS $\{\phi_n(x)\}$ is polynomial like and the condition (5.1.7) is fulfilled in the set E, then the relation

$$\bar{Z}_n(x) = O_x(1)$$

holds almost everywhere in E.

Theorem 4 :- Let $\{\phi_n(x)\}$ be a complete constant-preserving polynomial - like ONS with respect to the weight function $\rho(x)$. Suppose that the function $F_k(t, x)$ are continuous in the square $a \leqslant t \leqslant b$, $a \leqslant x \leqslant b$ with eventual exception of the diagonal $t = x$ and that the conditions (5.1.5) and (5.1.8) are also satisfied in the subinterval [c, d] of [a, b]. If the $L^2_{\rho(x)}$ - integrable functions f(x) is continuous in [c,d], then its expansion (5.1.6) is uniformly (N, p_n) summable in every inner subinterval of [c, d], the sum being f(x). For proving these theorems we need following Lemmas.

Lemma 1[1] :- If $\{p_n\} \in M^\alpha$, $\alpha > \frac{1}{2}$ then,

$$\lim_{n\to\infty} \frac{n}{P_n^{\,2}} \sum_{k=0}^{n} \frac{p_k^{\,2}}{(k+1)^2} = \frac{1}{2\alpha - 1}$$

Lemma 2[2] :- In order that an ONS $\{\phi_n(x)\}$ should be complete, the validity of Parseval's equation

$$\int_a^b f^2(x)\ d\mu(x) = \sum_{n=0}^{\infty} C_n^{\,2}$$

for all $f \in L^2_\mu$ is necessary and sufficient.

Lemma 3[3] :- If the function $f(t) \in L_{\rho(t)}$ is uniformly continuous in a subset E of $[a, b]$ and the conditions (5.1.3), (5.1.4) and

$$\int_a^b |\ \psi_n(t,x)\ |\ \rho(t)\ dt = O(1)$$

are uniformly satisfied for $x \in E$, then the relation

$$I_n(f, x) \longrightarrow f(x)$$

holds uniformly in E.

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1) Meder [78]
2) Alexits ([5], P. 15)
3) Alexits ([5], P. 260)

Lemma 4 [3] :- A monotone sequence of continuous functions, whose limit function is continuous, converges uniformly.

Proof of Theorem 1 :- We have

$$\bar{Q}_n(x) = \int_a^b |\bar{N}_n(t, x)| \, \rho(t) \, dt$$

and

$$\bar{N}_n(t, x) = \frac{1}{P_n} \sum_{v=0}^{n} p_v k_v(t,x)$$

Let $P_n(t,x)$ and $N_n(t,x)$ be the characteristic functions of the sets in which

$$\sum_{v=0}^{n} p_v k_v(t, x) \geq 0 \quad \text{and} \quad < 0 \quad \text{respectively.}$$

From the defination of n^{th} Lebesgue $(\bar{N}, p_n)$ functions

$$\bar{Q}_n(x) = \int_a^b | \bar{N}_n(t, x) | \rho(t) \, dt$$

$$(5.1.9) \quad \bar{Q}_n(x) = \frac{1}{P_n} \int_a^b P_n(t,x) \sum_{v=0}^{n} p_v k_v(t,x) \, \rho(t) \, dt$$

$$- \frac{1}{P_n} \int_a^b N_n(t,x) \sum_{v=0}^{n} p_v k_v(t,x) \, \rho(t) dt$$

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3) Alexits ([5], P.266)

Now our aim is to show that each of the sum on R.H.S. of (5.1.9) is of the order of magnitude $O_x(P_n)$ for every $x \in E \cap (a+\varepsilon, b-\varepsilon)$ with arbitrary $\varepsilon > o$ and therefore, $\bar{Q}_n(x) = O_x(1)$ holds for almost every $x \in E$. We divide the integral

$\int_a^b P_n(t, x)\, k_v(t, x)\, \rho(t)\, dt$ for $n \geqslant n_\varepsilon > \frac{1}{\varepsilon}$ in to

two parts:

Now,

$$I_{v1} = \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} , \quad I_{v2} = \int_a^{x-\frac{1}{n}} + \int_{x+\frac{1}{n}}^{b}$$

We first estimate $|I_{v1}|$.

Using Schwarz's inequality ,

$$I_{v1}^2 \leqslant \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} P_n^2(t, x)\, \rho(t)\, dt \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} k_v^2(t, x)\, \rho(t)\, dt$$

Now the condition (5.1.7) and $P_n^2(t, x) \leqslant 1$ implies that

$$I_{v1}^2 \leq \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} P_n^2(t,x)\, \rho(t)\, dt \sum_{k=o}^{v} \phi_k^2(x)$$

$$= O_x(vn^{-1}).$$

Hence by Cauchy's inequality and Lemma 1

$$\sum_{v=o}^{n} p_v \,|\, I_{v1} \,| \leq \left\{ \sum_{v=o}^{n} p_v^2 \sum_{v=o}^{n} I_{v1}^2 \right\}^{\frac{1}{2}}$$

$$= O_x(1)\left\{ \sum_{v=o}^{n} p_v^2 \sum_{v=o}^{n} vn^{-1} \right\}^{\frac{1}{2}}$$

$$= O_x(1)\left\{ \sum_{v=o}^{n} \frac{P_v^2}{(v+1)^2}\, n \right\}^{\frac{1}{2}}$$

$$= O_x(1)\left\{ \frac{n}{P_n^2} \sum_{v=o}^{n} \frac{P_v^2}{(v+1)^2}\, P_n^2 \right\}^{\frac{1}{2}}$$

$$= O_x(1)\ O(P_n)$$

$$= O_x(P_n).$$

(5.1.10) $$\sum_{v=o}^{n} P_v \,|\, I_{v1} \,| = O_x(P_n).$$

Now we proceed to estimate,

$$\sum_{v=o}^{n} p_v \,|\, I_{v2} \,|$$

$$\sum_{v=o}^{n} p_v \mid I_{v2} \mid = \sum_{v=o}^{n} p_v \mid \left(\int_a^{x-\frac{1}{n}} + \int_{x+\frac{1}{n}}^{b} \right) P_n(t, x)$$

$$k_v(t, x)\, \rho(t)\, dt \mid$$

Let us put,

$$g_k(t, x) = \begin{cases} P_n(t, x)\, F_k(t, x) & \text{for } t \in [a, x-\frac{1}{n}] \cup [x+\frac{1}{n}, b] \\ o & \text{otherwise} \end{cases}$$

Since the system $\{\phi_n(x)\}$ is polynomial like and therefore using defination (5.1.2) of the Kernel $k_n(t, x)$, we have

$$\sum_{v=o}^{n} p_v \mid I_{v2} \mid = \sum_{v=o}^{n} p_v \mid \left(\int_a^{x-\frac{1}{n}} + \int_{x+\frac{1}{n}}^{b} \right) P_n(t, x)$$

$$\sum_{k=1}^{\gamma} F_k(t, x) \sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(v)}\ \phi_{v+i}(t)$$

$$\phi_{v+j}(x)\ \rho(t)\ dt \mid$$

Using the defination of the function $g_k(t, x)$, we obtain,

$$\leq \sum_{k=1}^{\gamma} \sum_{i,j=-p}^{p} \sum_{v=o}^{n} p_v \mid \gamma_{i,j,k}^{(v)} \mid \mid \phi_{v+j}(x) \mid \left(\int_a^{x-\frac{1}{n}} + \int_{x+\frac{1}{n}}^{b} \right)$$

$$P_n(t, x)\, F_k(t, x)\, \phi_{v+i}(t)\, \rho(t)\, dt \mid$$

$$= O_x(1) \sum_{k=1}^{v} \sum_{i=-p}^{p} \sum_{v=o}^{n} p_v \left| \int_a^b g_k(t, x)\, \phi_{v+i}(t)\, \rho(t)\, dt \right|$$

$$\therefore \sum_{v=o}^{n} p_v \, | I_{v2} | = O_x(1) \sum_{k=1}^{\gamma} \sum_{i=-p}^{p} \sum_{v=o}^{n} p_v \left| \int_a^b g_k(t, x)\, \phi_{v+i}(t)\, \rho(t)\, dt \right|$$

Now $F_k(t, x) = O\left(\frac{1}{|t - x|}\right)$ and $|t - x| \geqslant n^{-1}$,

imply that

$$| g_k(t, x) | \leqslant P_n(t, x)\, | F_k(t, x) |$$

$$= O(n)$$

i.e. $g_k(t, x)$ is bounded for fixed n,

i.e. $g_k(t, x)$ is integrable, which means that integrals on the R.H.S. of the above relation are expansion - coefficients of $L^2_{\rho(t)}$ - integrable function.

So,

$$\sum_{v=o}^{n} p_v \, | I_{v2} | = O_x(1) \sum_{k=1}^{\gamma} \sum_{i=-p}^{p} \left\{ \sum_{v=o}^{n} p_v^2 \left\{ \sum_{v=o}^{n} \int_a^b g_k(t, x)\, \phi_{v+i}(t)\, \rho(t)\, dt \right\}^2 \right\}^{\frac{1}{2}}$$

Now by Bessel's inequality,

$$\sum_{v=o}^{n} \left\{ \int_a^b g_k (t, x)\, \phi_{v+i} (t)\, \rho(t)\, dt \right\}^2 \leqslant \int_a^b g_k^2(t, x)\, \rho(t) dt$$

$$= O_x (n)^{1)}$$

So,

$$\sum_{v=o}^{n} p_v \mid I_{v2} \mid = O_x (1) \sum_{k=1}^{\gamma} \sum_{i=-p}^{p} \left\{ n \sum_{v=o}^{n} p_v^2 \right\}^{\frac{1}{2}}$$

$$= O_x (1) \sum_{k=1}^{\gamma} \sum_{i=-p}^{p} \left\{ \frac{n}{P_n^2} \sum_{v=o}^{n} \frac{P_v^2}{(v+1)^2} P_n^2 \right\}^{\frac{1}{2}}$$

Hence by Lemma 1 we have,

$$(5.1.11) \qquad \sum_{v=o}^{n} p_v \mid I_{v2} \mid = O_x (P_n).$$

Hence it follows from (5.1.10) and (5.1.11) that

$$\sum_{v=o}^{n} p_v \int_a^b P_n (t, x)\, k_v (t, x)\, \rho(t)\, dt = O_x (P_n),$$

is true for almost every $x \in E \cap [a + \varepsilon, b - \varepsilon]$ and in similar way we obtain that the estimate

$$\sum_{v=o}^{n} p_v \int_a^b N_n (t, x)\, k_v (t, x)\, \rho(t)\, dt = O_x (P_n)$$

holds almost every $x \in E \cap [a+ \varepsilon, b - \varepsilon]$.

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1) Alexits ([5], p. 7)

Hence due to (5.1.9) we estimate

$$\bar{\varrho}_n(x) = O_x(1)$$

holds almost everywhere in E.

This completes the proof of our theorem.

Proof of Theorem 2 :-

For $x \in [C+\delta, \; d-\delta]$

We have,

$$\left| \left(\int_a^{x-\delta} + \int_{x+\delta}^{b} \right) f(t) \; \bar{N}_n(t, x) \; g(t) \, dt \right|$$

$$= \left| \left(\int_a^{x-\delta} + \int_{x+\delta}^{b} \right) f(t) \frac{1}{P_n} \sum_{v=o}^{n} p_v k_v(t, x) \, g(t) dt \right|$$

$$= \left| \frac{1}{P_n} \left(\int_a^{x-\delta} + \int_{x+\delta}^{b} \right) f(t) \sum_{v=o}^{n} p_v \sum_{k=1}^{\gamma} F_k(t, x) \sum_{i,j=-p}^{p} \right.$$

$$\left. \gamma_{i,j,k}^{(v)} \; \phi_{v+i}(t) \, \phi_{v+j}(x) \, g(t) \, dt \right|$$

Now let us put,

$$h_k(t, x) = \begin{cases} f(t) \; F_k(t, x), & t \in [a, x-\delta] \cup [x+\delta, b] \\ o \qquad , & \text{otherwise.} \end{cases}$$

We have

$$\left| \left(\int_a^{x-\delta} + \int_{x+\delta}^b \right) f(t)\ \bar{\bar{N}}_n(t, x)\ \rho(t)\ dt \right|$$

$$\leq \frac{1}{P_n} \sum_{k=1}^{\gamma} \sum_{i,j=-p}^{p} \sum_{v=o}^{n} p_v \left| \gamma_{i,j,k}^{(\gamma)} \right| \left| \phi_{v+j}(x) \right|$$

$$\left| \int_a^b h_k(t, x)\ \phi_{v+i}(t)\ \rho(t)\ dt \right|$$

$$= O(1) \frac{1}{P_n} \sum_{k=1}^{\gamma} \sum_{i=-p}^{p} \frac{1}{P_n} \sum_{v=o}^{n} p_v \left| \int_a^b h_k(t, x) \right.$$

$$\left. \phi_{v+i}(t)\ \rho(t)\ dt \right|$$

By Cauchy's inequality we have,

$$\left| \left(\int_a^{x-\delta} + \int_{x+\delta}^b \right) f(t)\ \bar{\bar{N}}_n\ (t, x)\ \rho(t)\ dt \right|$$

$$= O(1) \sum_{k=1}^{\gamma} \sum_{i=-p}^{p} \frac{1}{P_n} \left[\sum_{v=o}^{n} p_v \sum_{v=o}^{n} p_v \left\{ \int_a^b h_k(t, x) \right.\right.$$

$$\left.\left. \phi_{v+i}(t)\ \rho(t)\ dt \right\}^2 \right]^{\frac{1}{2}}$$

i.e.

(5.1.12) $\quad \left| \left(\int_a^{x-\delta} + \int_{x+\delta}^b \right) f(t)\ \bar{N}_n\ (t, x)\ \rho(t) dt \right.$

$$= O(1) \sum_{k=1}^{\gamma} \sum_{i=-p}^{p} \left[\frac{1}{P_n} \sum_{v=o}^{n} p_v \left\{ \int_a^b h_k(t, x) \, \phi_{v+i}(t) \, \varrho(t) \, dt \right\}^2 \right]^{\frac{1}{2}}$$

The integral on the R.H.S. of the above relations are $(v + i)^{th}$ expansion coefficients of the function $h_k(t, x)$. Also the system $\{\phi_n(x)\}$ is complete and hence with fixed i and by Lemma 2 we have,

$$(5.1.13) \qquad \sum_{v=-i}^{\infty} C_{v+i}^{2}(x) = \int_a^b h_k^2(t, x) \, \varrho(t) dt \ .$$

Now we proceed to prove that the function

$G_k(x) = \int_a^b h_k^2(t, x) \, \varrho(t) \, dt$ is continuous on

$[C + \delta, d - \delta]$, it is given that $f(t)$ is continuous in $[C, d]$.

Define $f(t) = o, \quad t \notin [C, d]$.

Since $E_k(t, x)$ is continuous in the square $a \leqslant t \leqslant b$,

$a \leqslant x \leqslant b$,

except for the diagonal points $t = x$, for each t and for every $x \in [a, b]$, $x \neq t$. $F_k(t, x)$ is continuous as a function of x only. Hence given $\varepsilon > o$, $\exists\ \delta_{1x} > o$, such that

$$o < | h | < \delta_{1x} < \delta < \frac{\varepsilon}{4k} \text{ implies,}$$

$$| F_k^2 (t, x + h) - F_k^2 (t, x) | < \frac{\varepsilon}{2M_1^2 (d - C)}$$

where M_1 denotes the bound for f (since the δ chosen above is arbitrary, we may like $\delta < \frac{\varepsilon}{4k}$, where $k > o$ denote the bound for the function $f^2 (t)\ F_k^2 (t, x)$ and $f^2 (t)\ F_k^2 (t, x + h)$ in the intervals $[x - \delta , x - \delta + h]$ and $[x + \delta , x + \delta + h]$ respectively. This is possible as f(t) is continuous in the interval $[x - \delta , x - \delta + h]$ and $[x + \delta , x + \delta + h]$ as the function of t.

Now for $x \in [C + \delta , d - \delta]$ and $o < | h | < \delta_{1x} < \delta < \frac{\varepsilon}{4k}$

We have,

$$|G_k(x + h) - G_k(x) = | \int_a^b h_k^2 (t, x + h)\ \rho(t)\ dt$$

$$- \int_a^b h_k^2 (t, x)\ \rho(t)\ dt |$$

$$= |(\int_a^{x+h-\delta} + \int_{x+h+\delta}^b) f^2 (t) F_k^2(t, x+h)$$

$$\rho(t)\ dt - (\int_a^{x-\delta} + \int_{x+\delta}^b)$$

$$f^2(t)\, F_k^{\,2}(t, x)\, \varrho(t)\, dt \,|$$

Let us put,

$E = [a, x - \delta + h] \cup [x + \delta , b] \cap [C, d]$ then the continuity of $F_k(t, x)$ is true for any $t \in E$ and all x.

Hence,

$$| G_k(x + h) - G_k(x) |$$

$$= | (\int_E f^2(t)\, F_k^{\,2}(t, x + h)\, \varrho(t)\, dt - \int_{x+\delta}^{x+\delta+h} f^2(t) F_k^{\,2}(t, x + h)\, \varrho(t)\, dt) - (\int_E f^2(t)\, F_k^{\,2}(t, x)\, \varrho(t)\, dt - \int_{x-\delta}^{x-\delta+h} f^2(t)\, F_k^{\,2}(t, x)\, \varrho(t)\, dt) |$$

$$\leqslant | \int_E f^2(t)\, F_k^{\,2}(t, x + h)\, \varrho(t)\, dt - \int_E f^2(t) F_k^{\,2}(t,x)\, \varrho(t)\, dt | + | \int_{x-\delta}^{x-\delta+h} f^2(t)\, F_k^{\,2}(t, x)\, \varrho(t) dt - \int_{x+\delta}^{x+\delta+h} f^2(t)\, F_k^{\,2}(t, x + h)\, \varrho(t)\, dt |$$

$$\leq \int_E f^2(t) \mid F_k^2(t, x+h) - F_k^2(t, x) \mid \rho(t)\, dt$$

$$+ \int_{x-\delta}^{x-\delta+h} f^2(t)\, F_k^2(t, x)\, \rho(t)\, dt$$

$$+ \int_{x+\delta}^{x+\delta+h} f^2(t)\, F_k^2(t, x+h)\, \rho(t)\, dt$$

$$< M_1^2 \frac{\varepsilon}{2M_1^2(d - C)} \mid E \mid + 2k \mid h \mid < \frac{\varepsilon}{2} + 2k \frac{\varepsilon}{4k} = \varepsilon$$

Hence it follows that $G_k(x)$ is uniformly continuous. Now, we proceed to prove that $C_n(x)$ is also uniformly continuous in $[C + \delta, d - \delta]$. As noted above, since for each t and for every $x \in [a, b]$, $x \neq t$, $F_k(t, x)$ is continuous as a function of x only.

For given $\varepsilon > 0$ $\exists$ $\delta_{2x} > 0$, $0 < \mid h \mid < \delta_{2x} < \delta$ $\ni$

$$\mid F_k(t, x+h) - F_k(t, x) \mid < \frac{\varepsilon}{M_1 \mid E \mid^{\frac{1}{2}}}.$$

Now, for $x \in [C + \delta, d - \delta]$ and $\mid h \mid < \delta_{2x}$, by Cauchy's inequality,

$$\mid C_n(x+h) - C_n(x) \mid = \mid \int_a^b h_k(t, x+h)\, \phi_n(t)\, \rho(t)\, dt - \int_a^b h_k(t, x)\, \phi_n(t)\, \rho(t)\, dt \mid$$

$$\leq \left[\int_a^b \left\{h_k(t, x+h) - h_k(t,x)\right\}^2 \rho(t)dt \int_a^b \phi_n^2(t)\,\rho(t)dt\right]^{\frac{1}{2}}$$

$$= \left[\int_E f^2(t)\,(F_k(t,x+h) - F_k(t,x))^2 \rho(t)\, dt\, \right]^{1/2}$$

$$= \left[M_1^2 \quad \frac{\varepsilon^2}{M_1^2 |E|} \quad |E|\ \right]^{1/2}$$

Hence, it follows that $C_n(x)$ is uniformly continuous on $[c+\delta\ ,\ d-\delta\ ]$. Consequently, it follows that from (5.1.13) and by Lemma 4 that the series

$$\sum_{v=-i}^{\infty} C^2_{v+i}\,(x)$$

Converges uniformly and therefore, it follows that the sequence $\left\{C^2_{v+i}\,(x)\right\}$ converges uniformly to zero as $n \to \infty$ and this implies the n^{th} $(\bar{N}, P_n)$ means of $\left\{C^2_{v+i}(x)\right\}$ converges uniformly to zero. i.e.

$$\frac{1}{P_n} \sum_{v=o}^{n} P_v \left\{\int_a^b h_k(t,x)\ \phi_{v+i}(t)\ \rho(t)\ dt\right\}^2 = o(1).$$

Thus, it follows from (5.1.12) that

$$\left(\int_a^{x-\delta} + \int_{x+\delta}^{b}\right) f(t)\ \bar{N}_n(t,x)\ \ \rho(t)\ dt = O(1).$$

Since, this relation is true for any $L^2_{\rho(t)}$ - integrable function continuous in $[c, d]$, in particular taking $f(t) = 1,\ t \in [a,b]$, We have,

(5.1.14) $$\left(\int_a^{x-\delta} + \int_{x+\delta}^b \right) \bar{N}_n(t,x)\ \rho(t)\ dt = O(1).$$

Now,

$$\bar{N}_n(t,x) = \frac{1}{P_n} \sum_{v=o}^{n} p_v k_v(t,x)$$

$$= \frac{1}{P_n} \sum_{v=o}^{n} p_v \sum_{m=o}^{n} \phi_m(t)\ \phi_m(x).$$

$$= \frac{1}{P_n} \sum_{m=o}^{n} \phi_m(t)\ \phi_m(x) \sum_{v=m}^{n} p_v\ .$$

Hence it follows from the constant preserving polynomial property of the system $\{\phi_n(x)\}$ that

$$\int_a^b \bar{N}_n(t,x)\, \rho(t)\ dt = \frac{1}{P_n}\int_a^b \left\{ \sum_{m=o}^{n} \phi_m(t)\ \phi_m(x) \sum_{v=m}^{n} p_v \right\} \rho(t)\ dt.$$

$$= \frac{1}{P_n} \sum_{m=o}^{n} \frac{\phi_m(x)}{\phi_o(x)} \sum_{v=m}^{n} p_v \int_a^b \phi_m(t)\phi_o(x)\rho(t)\, dt$$

$$= \frac{1}{P_n} \sum_{v=o}^{n} p_v$$

$$= 1.$$

Consequently, it follows from (5.1.14) that

$$(5.1.15) \qquad \int_{x-\delta}^{x+\delta} \bar{N}_n(t,x) \quad \rho(t) \ dt = 1 + o(1)$$

Thus it follows from (5.1.14) and (5.1.15) that the relation (5.1.3) is uniformly satisfied in the $[c+\delta, d-\delta]$ with $\psi_n(t,x) = \bar{N}_n(t,x)$.

Further for $x \in [c+\delta, d-\delta]$ and $t \in [a, x-\delta] \cup [x+\delta, b]$

$$|\bar{N}_n(t,x)| = \left| \frac{1}{P_n} \sum_{v=0}^{n} p_v k_v(t,x) \right|$$

$$= \left| \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{k=1}^{v} F_k(t, x) \sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(v)} \cdot \right.$$

$$\left. \phi_{v+i}(t) \ \phi_{v+j}(x) \right|$$

$$< \frac{1}{P_n} \sum_{k=1}^{v} \sum_{i,j=-p}^{p} |F_k(t,x)| \ \left| \sum_{v=0}^{n} p_v \ | \gamma_{i,j,k}^{(v)} \right| \cdot$$

$$| \phi_{v+i}(t) | \ | \phi_{v+j}(x) |$$

$$= O(1) \frac{1}{P_n} \sum_{k=1}^{v} \sum_{i,j=-p}^{p} |F_k(t,x)| \ P_n$$

$$= O(1) \sum_{k=1}^{v} |F_k(t, x)| = O\left(\frac{1}{|t-x|}\right)$$

$$= O\left(\frac{1}{\delta}\right)$$

Hence,

$$|\bar{N}_n(t, x)| < \phi(\delta)$$

i.e. the relation (5.1.4) is uniformly satisfied in the interval $[c+\delta, d-\delta]$ with $\psi_n(t,x) = \bar{N}_n(t, x)$.

In otherwords we have proved that the $(\bar{N}, p_n)$ means of the expansion

$$f(x) \sim \sum_{n=0}^{\infty} C_n \phi_n(x)$$

are uniformly singular in the interval $[c+\delta, d-\delta]$. Also from Theorem 1 the validity of the relation

$$\bar{Q}_n(x) = O(1)$$

follows for every subinterval $[c+\delta, d-\delta]$ of $[c, d]$.

Consequently it follows from Lemma 3 that,

$$I_n(f,x) = \bar{T}_n(x) = \frac{1}{P_n} \sum_{v=0}^{n} p_v S_v(x)$$

$$= \frac{1}{P_n} \sum_{k=0}^{n} C_k\phi_k(x) \sum_{v=k}^{n} p_v$$

$$= \frac{1}{P_n} \sum_{v=o}^{n} p_v \sum_{k=o}^{v} C_k \phi_k(x)$$

$$= \frac{1}{P_n} \sum_{k=o}^{n} \left(\int_a^b f(t)\, \phi_k(t)\, \rho(t)\, dt \;\; \phi_k(x) \sum_{v=k}^{n} p_v \right)$$

$$= \int_a^b f(t) \left\{ \frac{1}{P_n} \sum_{k=o}^{n} \phi_k(t)\, \phi_k(x) \sum_{v=k}^{n} p_v \right\} \rho(t)\, dt$$

$$= \int_a^b f(t) \left\{ \frac{1}{P_n} \sum_{v=o}^{n} p_v \sum_{k=o}^{v} \phi_k(t)\, \phi_k(x) \right\} \rho(t)\, dt$$

$$= \int_a^b f(t) \left\{ \frac{1}{P_n} \sum_{v=o}^{n} p_v k_v(t,x) \right\} \rho(t)\, dt$$

$$= \int_a^b f(t) \;\; \bar{N}_n(t,x) \;\; \rho(t)\, dt.$$

converges to $f(x)$ uniformly in $[c+\delta\,,\, d-\delta\,]$ with this the theorem is proved.

<u>Proof of Theorem 3</u> :- Proof follows on the same line as of theorem 1.

<u>Proof of Theorem 4</u> :— Proof follows on the same line as of theorem 2.