CHAPTER - VIII

CERTAIN MISCELLANEOUS RESULTS

8.1 INTRODUCTION

As the title indicates, this concluding Chapter embodies certain miscellaneous results associated with the polynomials $f_n^c(x)$ and other related polynomials. In section 2 of this Chapter we present an expansion formula for the product of a finite number of polynomials belonging to the class $\{f_n^c(x)\}$. Section 8.3 deals with a generalization of the generating functions given in Chapter II, and in the last section we introduce yet another extension of Boas and Buck's generating relation 7.1(3), 7.1(5)-7.1(7), and derive certain recurrence relations for the polynomials defined by that generating function.

8.2 EXPANSION FORMULA

Following the work of Erdelyi [1], who obtained a multiple variable hypergeometric representation 5.1(5) for the connecting coefficients in the expansion of the product

$$L_{n_1}^{\alpha_1}(a_1x) \ldots L_{n_p}^{\alpha_p}(a_px)$$

in terms of $L_n(x)$, we obtain here an analogous representation in terms of generalized multiple hypergeometric series due to Srivastava and Daoust [1], for the connecting coefficients in the expansion of the product

$$f_{n_1}^{c_1}(a_1x, y_1, r_1, m) \ldots f_{n_p}^{c_p}(a_px, y_p, r_p, m).$$

In the derivation of the derived result we shall assume that λ_n occuring in the definition 2.1(4) of $f_n^{c}(x,y,r,m)$ is representable in the form

$$(1) \qquad \lambda_n = \frac{\prod_{i=1}^{B} (b_i)_n}{\prod_{i=1}^{E} (e_i)_n \, n!}$$

and we shall also make use of the notations given by 7.2(5). The generalized multiple hypergeometric series of Srivastava and Daoust, that we shall employ here is defined as below (Srivastava and Daoust [1], Eq.(4.1))

$$(2)\left\{\begin{aligned}&F^{A:B';\ldots;B^{(n)}}_{G:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:\ [(b'):\phi'];\ldots;\\ [(c):\psi',\ldots,\psi^{(n)}]:\ [(d'):\delta'];\ldots;\end{matrix}\right.\\ &\left.\begin{matrix}[(b^{(n)}):\phi^{(n)}]:\\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\quad x_1,\ldots,x_n\right)\\ &=\sum_{m_1=0}^{\infty}\ldots\sum_{m_n=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+\ldots+m_n\theta_j^{(n)}}\prod_{j=1}^{B'}(b_j')_{m_1\phi_j'}\ldots}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j^{(1)}+\ldots m_n\psi_j^{(n)}}\prod_{j=1}^{D'}(d_j')_{m_1\delta_j'}\ldots}\\ &\quad\frac{\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}\ \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}\ .\end{aligned}\right.$$

Now we begin by considering the product

$$f_{n_1}^{c_1}(a_1x,\ y_1,\ r_1,\ m)\ldots\ f_{n_p}^{c_p}(a_px,\ y_p,\ r_p,\ m)$$

which for the sake of brevity, is denoted by Ω, In view of 2.1(5) and 3.2(7), we can express Ω in the form

$$(3)\ \Omega = \sum_{k_1=0}^{n_1^*} \cdots \sum_{k_p=0}^{n_p^*} \prod_{j=1}^{p} \left[\frac{(c_j+r_jn_j-mr_jk_j)_{k_j}(-y_j)^{k_j} a_j^{n_j-mk_j} \gamma_{n_j-mk_j}}{k_j\ !} \right]$$

$$\cdot \frac{1}{\gamma_{N-mK}} \sum_{s=0}^{N^*-K} \frac{(-y)^s(-c-rN+rmK+rms)(1-c-rN+rmK)_{s-1}}{s!}$$

$$\cdot f^c_{N-mK-ms}(x),$$

which can be further simplified to

$$(4)\quad \Omega = \sum_{q=0}^{N^*} D_q^{(n_1,\ldots,n_p)} f^c_{N-mq}(x),$$

where the coefficients $D_q^{(n_1,\ldots,n_p)}$ are given by

$$(5)\quad D_q^{(n_1,\ldots,n_p)} = \sum_{k_1=0}^{n_1^*} \cdots \sum_{k_p=0}^{n_p^*}$$

$$\cdot \prod_{j=1}^{p} \left[\frac{(c_j+r_j\ n_j-r_jmk_j)_{k_j}}{k_j!}(-y_j)^{k_j} a_j^{n_j-mk_j} \gamma_{n_j-mk_j} \right]$$

$$\cdot \frac{(-y)^{q-K}(-c-rN+rmq)(1-c-rN+rmK)_{q-K-1}}{(q-K)!\ \gamma_{N-mK}} .$$

Now, if we replace γ_n by the expression given by (1) and make use of the identities

$$(6)\qquad (c+n-mk)_k = \frac{(-1)^k(1-c-n)_{mk}}{(1-c-n)_{(m-1)k}} ,$$

and

$$(7)\qquad (\alpha+mk)_{n-k} = \frac{(\alpha)_n(\alpha+n)_{(m-1)k}}{(\alpha)_{mk}},$$

the expression (5) for the coefficients $D_q^{(n_1,\ldots,n_p)}$ takes the form

$$(8)\left\{\begin{aligned} D_q^{(n_1,\ldots,n_p)} &= C \sum_{k_1=0}^{n_1^*} \cdots \sum_{k_p=0}^{n_p^*} \frac{(-q)_K(-c-rN+q)_{(rm-1)K}}{(-N)_{mK}(1-c-rN)_{rmK}} \\ &\cdot \frac{\prod_{i=1}^{B}(1-b_i-N)_{mK}}{\prod_{i=1}^{E}(1-e_i-N)_{mK}} \prod_{j=1}^{p}\left[\frac{(1-c_j-r_j\,n_j)_{r_j\,mk_j}\,(-n_j)_{mk_j}}{(1-c_j-r_j\,n_j)_{(r_jm-1)k_j}\,k_j!}\right] \\ &\cdot \frac{\prod_{i=1}^{E}(1-e_i-n_j)_{mk_j}}{\prod_{i=1}^{B}(1-b_i-n_j)_{mk_j}} \left(\frac{y_j}{y\,a_j^m}\right)^{k_j} , \end{aligned}\right.$$

where C is given by

$$(9)\qquad C = \frac{(-c-rN+rmq)(1-c-rN)_{q-1}\,(-y)^q N!\,\prod_{i=1}^{E}(e_i)_N}{q\,!\,\prod_{i=1}^{B}(b_i)_N} \cdot \prod_{j=1}^{p} \frac{\prod_{i=1}^{B}(b_i)_{n_j}\,a_j^{n_j}}{\prod_{i=1}^{E}(e_i)_{n_j}\,n_j!} .$$

The above representation for $D_q^{(n_1,\ldots,n_p)}$ when interpreted in terms of the definition (2), leads us to the desired formula

$$
(10)\quad \begin{cases}
D_q^{(n_1,\ldots,n_p)} \\
= C\ F^{B+2:E+2;\ldots;E+2}_{E+2:B+1;\ldots;B+1}\Bigg([-q : 1,\ldots,1], \\
\qquad [-N : m,\ldots,m], \\
[-c-rN+q : rm-1,\ldots,rm-1],\ [(1-b-N):m,\ldots,m]: \\
[-c-rN+1 : rm,\ldots,rm],\ [(1-e-N):m,\ldots,m]: \\
[-n_1:m],\ [1-c_1-r_1n_1:mr_1],\ [(1-e-n_1):m];\ \ldots ; \\
\qquad [1-c_1-r_1n_1 : mr_1-1],\ [(1-b-n_1):m];\ \ldots ; \\
[-n_p:m],\ [1-c_p-r_p n_p : mr_p],\ [(1-e-n_p):m]; \\
\qquad [1-c_p-r_p n_p : mr_p-1],\ [(1-b-n_p):m]; \\
\dfrac{y_1}{ya_1^m} \cdots \dfrac{y_p}{ya_p^m}\Bigg).
\end{cases}
$$

In (10), if we set $B = E$ and $b_i = e_i$ for $i=1,2,\ldots,B$, we shall get the multiple hypergeometric series representation for the coefficients involved in the expansions of the product

$$\Gamma_{n_1}^{c_1}(a_1x, y_1, r_1, m) \ldots \Gamma_{n_p}^{c_p}(a_px, y_p, r_p, m) \qquad \text{where}$$

$\Gamma_n^c(x,y,r,m)$ denotes the particular case of $f_n^c(x,y,r,m)$ discussed in Chapter V. This resulting expansion can be further particularized to yield the formula 5.1(4)-5.1(5) derived by Erdelyi [1].

8.3 A GENERAL GENERATING FUNCTION

A few years ago, in an attempt to present a unified treatment of the various polynomial systems introduced into analysis from time to time Srivastava and Buschman [1] gave the following :

Theorem (Srivastava and Buschman [1], p.366)

Corresponding to the power series

$$(1)\qquad \Psi(u) = \sum_{n=0}^{\infty} \gamma_n u^n, \quad \gamma_0 \neq 0,$$

let

$$(2)\qquad S_{n,q}^{(\alpha,\beta)}(\lambda;x) = \sum_{k=0}^{[n/q]} \frac{(-n)_{qk}(1+\alpha+(\beta+1)n)_{\lambda k}}{(1+\alpha+\beta n)_{(\lambda+q)k}} \gamma_k x^k,$$

and

$$(3)\qquad \theta(n,q;\ \alpha,\beta,\gamma,\lambda;\ u)$$

$$= \sum_{k=0}^{\infty} \frac{\gamma}{\gamma+(\beta+1)qk} \binom{\alpha-\gamma+\lambda k}{n} \binom{n+qk+\gamma/(\beta+1)}{n}^{-1} \gamma_k u^k,$$

where α,β,γ and λ are arbitrary complex numbers, q is a positive integer, and $n = 0,1,2,\ldots$

then

$$(4) \quad \sum_{n=0}^{\infty} \frac{\nu}{\nu+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} S_{n,q}^{(\alpha,\beta)}(\lambda;x)t^n$$

$$=(1+w)^{\alpha} \phi(x(-w)^q (1+w)^{\lambda}, -w/(1+w)),$$

where, for convenience,

$$(5) \quad \phi(u,v) = \sum_{n=0}^{\infty} \theta(n,q;\alpha,\beta,\nu,\lambda;u)v^n ,$$

and

$$(6) \qquad w = t(1+w)^{\beta+1}, \; w(0) = 0 .$$

This theorem provides an extension of the theorem given earlier by Mittal [2] and is also capable of yielding a number of several known as well as new generating relations for the various particular cases of the polynomials $S_n^{(\alpha,\beta)}(\lambda;x)$.

Having been inspired by the above theorem, we prove here the following analogue of the above mentioned theorem involving our polynomials $f_n^c(x)$.

Theorem 9:

Corresponding to the power-series $\Psi(u)$ given by (1) let

$$(7)\quad f_n^c(x,y,r,m) = \sum_{k=0}^{[n/m]} \frac{(rn-rmk+c)_k}{k!} (-y)^k \gamma_{n-mk}\, x^{n-mk}$$

and

$$(8)\quad \chi_n(m,c,\sigma,r,u) = \sum_{k=0}^{\infty} \frac{\gamma}{\gamma-\sigma k} \binom{-c-\gamma-rk}{n} \binom{n+k/m-\gamma/(\sigma m)}{n}^{-1} \gamma_k u^k ,$$

where c,γ,σ are arbitrary complex numbers, r is any integer, m is a positive integer and $n = 0,1,2,\ldots$

Then

$$(9)\quad \sum_{n=0}^{\infty} \frac{\gamma}{\gamma-\sigma n} f_n^{c+\sigma n}(x,y,r,m)t^n$$

$$=(1+y\zeta^m)^{-c}\, \Phi\left(\frac{x\zeta}{(1+y\zeta^m)^r} , \frac{-y\zeta^m}{(1+y\zeta^m)} \right) ,$$

where

$$(10)\quad \zeta = t(1+y\zeta^m)^{-\sigma}$$

and, for convenience

$$(11)\quad \Phi(u,v) = \sum_{n=0}^{\infty} \chi_n(m,c,\sigma,r,u)v^n.$$

In order to prove this theorem, we first observe that $f_n^c(x,y,r,m)$, written briefly as $f_n^c(x)$, may be expressed in the form

$$(12)\quad f_n^c(x) = \sum_{k=0}^{[n/m]} \binom{-c-rn+rmk}{k} y^k \gamma_{n-mk}\, x^{n-mk} ,$$

so that

$$(13) \qquad \sum_{n=0}^{\infty} \frac{\nu}{\nu-\sigma n} f_n^{c+\sigma n}(x)t^n$$

$$= \sum_{n=0}^{\infty} \frac{\nu}{\nu-\sigma n}(xt)^n \gamma_n \sum_{k=0}^{\infty} \cdot \frac{\nu-\sigma n}{\nu-\sigma n-\sigma mk} \cdot \binom{-c-\sigma n-rn-\sigma mk}{k} (yt^m)^k .$$

The inner series may be transformed by making use of Gould's identity [5, p.196]

$$(14) \qquad \sum_{k=0}^{\infty} \frac{a}{a+bk} \binom{c+bk}{k} t^k$$

$$=(1+w)^c \sum_{k=0}^{\infty} (-1)^k \binom{c-a}{k} \binom{k+a/b}{k}^{-1} \left(\frac{w}{1+w}\right)^k ,$$

where $w = t(w+1)^b$, as a result the right hand member of (13) becomes

$$(15) \qquad (1+y\xi^m)^{-c} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\nu}{\nu-\sigma n} \binom{-c-\nu-rn}{k} \binom{k+n/m-\nu/(\sigma m)}{k}^{-1}$$

$$\left(\frac{x\xi}{(1+y\xi^m)^r}\right)^n \left(-\frac{y\xi^m}{1+y\xi^m}\right)^k \gamma_n ,$$

whero from the theorem follows, at once .

When $\sigma \to 0$, it is easy to see that $\chi_n \to 0$ for $n = 1,2,\ldots$ and $\chi_0 \to \Psi(u)$, therefore the generating relation (9)-(11) would reduce to 2.1(3)-2.1(4) when $\sigma = 0$.

On the other hand, we observe that

$$\chi_n \to (m\sigma)^n \Psi(u) \text{ and } \Phi(u,v) \to \frac{\Psi(u)}{1-m\sigma v},$$

when $\nu \to \infty$. Consequently, the particular case $\nu \to \infty$ of (9)-(11) would correspond to 2.3(1)-2.3(2).

Further, if we express the binomial coefficient occuring in (12) in the form

$$(16) \qquad \binom{-c-rn+rmk}{k} = (-1)^k \binom{c+rn-rmk+k-1}{k},$$

and follow an analysis similar to Theorem 9 we shall get another theorem in the form :

<u>Theorem 10</u>:

In terms of power series $\Psi(u)$ given by (1), let

$$(17) \qquad \mathcal{J}_n(m,c,\sigma,r,u) = \sum_{k=0}^{\infty} \frac{\nu}{\nu+(\sigma+\frac{1}{m})k} \binom{c-\nu-1+rk-k/m}{n} \cdot \binom{n+k/m+\nu/(m\sigma+1)}{n}^{-1} \gamma_k u^k,$$

where, as before, c, ν, σ are arbitrary complex numbers, r is any integer, m is a positive integer and $n=0,1,2,\ldots$ then

$$(18)\quad \sum_{n=0}^{\infty} \frac{\nu}{\nu+(\sigma+1/m)n} f_n^{c+\sigma n}(x)t^n$$

$$= (1+w)^{c-1} K(xt(1+w)^{r+\sigma}, -w/(1+w)) ,$$

where

$$(19)\quad w = -yt^m(w+1)^{\sigma m+1}, \; w(0)=0 ,$$

and

$$(20)\quad K(u,v) = \sum_{n=0}^{\infty} J_n(m,c,\sigma,r,u)v^n$$

and as before $f_n^c(x,y,r,m)$ is given by (7).
When $\nu = c-1$, $r = 1/m$, the generating relation (18)-(20) would reduce to 2.4(1)-2.4(3) derived in Chapter II.

8.4 GENERAL RECURRENCE RELATIONS

In this section we consider the following variant of the Boas and Buck's generating relation 7.1(3), 7.1(5) - 7.1(7).

(1) $A(t)\,\Psi(xH(t)) = \sum_{n=0}^{\infty} p_n(x)t^n$,

(2) $\Psi(t) = \sum_{n=0}^{\infty} \gamma_n t^n , \qquad \gamma_0 \neq 0$

(3) $A(t) = \sum_{n=0}^{\infty} a_n t^{mn} , \qquad a_0 \neq 0$

(4) $H(t) = \sum_{n=0}^{\infty} h_n t^{mn+s} , \qquad h_0 \neq 0$.

This generating relation, besides providing an extension of Boas and Buck's generating relation, also includes the generating relation 2.1(1) - 2.1(2) studied by Panda[1]. In order to verify that $p_n(x)$, as defined by (1)-(4), is a polynomial, we, following the work of Boas and Buck [1], proceed as below :

Let

$$p_n(x) = \sum_{k=0}^{\infty} Q(k,n,m,s)x^k ,$$

so that

$$A(t)\Psi(xH(t)) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} Q(k,n,m,s)x^k t^n ,$$

wherein p- differentiations partially with respect to x yield

$$A(t)\,[H(t)]^p\,\Psi^{(p)}(xH(t))$$
$$= \sum_{n=0}^{\infty}\sum_{k=p}^{\infty} p!\binom{k}{p} Q(k,n,m,s)x^{k-p}\,t^n ,$$

which on putting $x = 0$ simplifies to

$$A(t)\,[H(t)]^{p}\,\Psi^{(p)}(0) = \sum_{n=0}^{\infty} p!\; Q(p,n,m,s)t^{n} .$$

Now in view of (2), (3) and (4)

$$A(t)[H(t)]^{p}\,\Psi^{(p)}(0) = \gamma_p\, p! \left[\sum_{n=0}^{\infty} a_n t^{mn}\right]\left[\sum_{n=0}^{\infty} h_n t^{mn+s}\right]^{p}$$

$$= a_o h_o^{p}\, p!\, \gamma_p t^{ps} \left[\sum_{n=0}^{\infty} \frac{a_n}{a_o}\, t^{mn}\right]\left[\sum_{n=0}^{\infty} \frac{h_n}{h_o}\, t^{mn}\right]^{p} .$$

The right hand member of the above expression may be put in the form

$$a_o h_o^{p}\, p!\, \gamma_p t^{ps} + \sum_{n=1}^{\infty} D(p,n,m,s)t^{mn+sp} ,$$

so that

$$Q(p,n,m,s) = 0 \text{ , for } n < sp$$
$$Q(p,sp,m,s) = a_o\, h_o^{p} \gamma_p .$$

Hence $Q(k,n,m,s) = 0$ for $k > \frac{n}{s}$, therefore, it follows that $p_n(x)$ is a polynomial of degree $\leq n^*$ where n^* stands for $[n/s]$.

Moreover, we also have

$$Q(n^*,n,m,s) = \gamma_{n^*}\, h_o^{n^*} a_o \neq 0 \quad \text{if and only if} \quad \gamma_{n^*} \neq 0 .$$

The above discussion may be summed up into the following

Theorem 11:

If $p_n(x)$ is defined by (1)-(4) then $p_n(x)$ is a polynomial in x of degree precisely n^* if and only if $\gamma_{n^*} \neq 0$.

We now derive certain recurrence relations for $p_n(x)$, for which we note that the equations (1)-(4), when subjected to usual analysis for deriving recurrence relations, yield the equation

$$(5) \qquad \frac{xtH'(t)}{H(t)} \cdot \frac{\partial F}{\partial x} - t\,\frac{\partial F}{\partial t} = -t\,\frac{A'(t)}{A(t)}\,F\,,$$

where F stands for the left hand member of (1).

Now in view of the nature of the functions $A(t)$ and $H(t)$, it is easy to verify that there exist sequences of numbers α_k and β_k such that

$$(6) \qquad t\,\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n\, t^{mn+m}\,,$$

$$(7) \qquad t\,\frac{H'(t)}{H(t)} = s + \sum_{n=0}^{\infty} \beta_n\, t^{mn+m}\,.$$

Substitution of (6) and (7) in (5) followed by a slight simplification leads us to

$$\sum_{n=0}^{\infty}\left\{sxp_n'(x) - np_n(x)\right\}t^n + \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]} x\beta_k\, p_{n-mk}'(x)t^{n+m}$$

$$= - \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]} \alpha_k\, p_{n-mk}(x)t^{n+m} ,$$

wherein a shift of index gives us

$$\sum_{n=0}^{m-1}\left\{s\ x\ p_n'(x) - n\ p_n(x)\right\}t^n + \sum_{n=m}^{\infty}\left\{s\ x\ p_n'(x) - n\ p_n(x)\right\}t^n$$

$$= - \sum_{n=m}^{\infty}\sum_{k=0}^{[n/m]-1} x\ \beta_k\ p_{n-mk-m}'(x)t^n$$

$$- \sum_{n=m}^{\infty}\sum_{k=0}^{[n/m]-1} \alpha_k\ p_{n-mk-m}(x)t^n ,$$

from which the recurrence relation

$$(8) \qquad s\ x\ p_n'(x) - n\ p_n(x) = - \sum_{k=0}^{[n/m]-1} \alpha_k\ p_{n-mk-m}(x)$$

$$- x \sum_{k=0}^{[n/m]-1} \beta_k\ p_{n-mk-m}'(x) \quad , \ n \geq m$$

follows at once.

If we express equation (5) in the form

$$xt\ \frac{H'(t)}{H(t)}\ A(t)\frac{\partial F}{\partial x} - tA(t)\ \frac{\partial F}{\partial t} = - t\ A'(t)\ F ,$$

and make use of the fact that there would exist a sequence of numbers δ_k such that

$$t\,A(t)\,\frac{H'(t)}{H(t)} = a_0\, s + \sum_{k=0}^{\infty} \delta_k\, t^{mk+m} ,$$

then we shall get the recurrence relation

$$(9)\quad a_0\left\{s\,x\,p_n'(x) - n\,p_n(x)\right\}$$

$$= - \sum_{k=0}^{[n/m]-1} x p'_{n-mk-m}(x)\delta_k + \sum_{k=0}^{[n/m]-1} a_{k+1}(n-2mk-2m)p_{n-mk-m}(x),$$

$$n \geq m .$$

Yet another recurrence relation would follow if we start with the following alternative form of (5).

$$x\,t\,H'(t)\,\frac{\partial F}{\partial x} - H(t)t\,\frac{\partial F}{\partial t} = - t\,H(t)\,\frac{A'(t)}{A(t)}\,F ,$$

and use the representation

$$t\,H(t)\,\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty} \sigma_n t^{mn+m+s} .$$

The resulting recurrence relation will be

$$(10)\quad h_0\left\{s\,x\,p_n'(x) - n\,p_n(x)\right\}$$

$$= - \sum_{k=0}^{[n/m]-1} x\,p'_{n-mk-m}(x)\,h_{k+1}\,(mk + m + s)$$

$$+ \sum_{k=0}^{[n/m]-1} \left\{(n-mk-m)h_{k+1} - \delta_k\right\} p_{n-mk-m}(x),\ n \geq m .$$

$$(11)\quad s\,x\,p_n'(x) = n\,p_n(x),\quad n = 0, 1, \ldots, m - 1.$$

If we choose

$$A(t) = (1+yt^m)^{-c} ,$$

and

$$H(t) = \frac{t^s}{(1+yt^m)^r} ,$$

then it can be easily verified that the sequences α_k , β_k , δ_k and σ_k occurring above are given by

$$\alpha_k = cm(-y)^{k+1} ,$$

$$\beta_k = rm(-y)^{k+1} ,$$

$$\delta_k = \frac{(c+1)_k}{k!} (-y)^{k+1} s \left[\frac{c+k+1}{k+1} + (\frac{rm}{s} -1) \right],$$

$$\sigma_k = cm \frac{(c+1)_k}{k!} (-y)^{k+1} .$$

Consequently the corresponding particular cases of (8), (9) and (10) may be stated in the form

$$(12) \quad s\, x\, D_x \left\{ f_n^c(x,y,r,m,s) \right\} - n\, f_n^c(x,y,r,m,s)$$

$$= c\, my \sum_{k=0}^{[n/m]-1} (-y)^k f^c_{n-mk-m}(x,y,r,m,s)$$

$$+ rmxy \sum_{k=0}^{[n/m]-1} (-y)^k D_x \left\{ f^c_{n-mk-m}(x,y,r,m,s) \right\} , \quad n \geq m ,$$

(13) $s\, x\, D_x \left\{ \overset{c}{f}_n(x,y,r,m,s) \right\} - n\, \overset{c}{f}_n(x,y,r,m,s)$

$$= - \sum_{k=0}^{[n/m]-1} x \frac{(c+1)_k}{k!} (-y)^{k+1} s \left\{ \frac{c+k+1}{k+1} + \left(\frac{rm}{s} - 1\right) \right\}$$

$$D_x \left\{ \overset{c}{f}_{n-mk-m}(x,y,r,m,s) \right\} + \sum_{k=0}^{[n/m]-1} \frac{(c)_{k+1}}{(k+1)!} (-y)^{k+1}$$

$$(n-2mk-2m)\, \overset{c}{f}_{n-mk-m}(x,y,r,m,s) \;.$$

(14) $s\, x\, D_x \left\{ \overset{c}{f}_n(x,y,r,m,s) \right\} - n\, \overset{c}{f}_n(x,y,r,m,s)$

$$= - \sum_{k=0}^{[n/m]-1} x(mk+m+s) \frac{(r)_{k+1}}{(k+1)!} (-y)^{k+1} D_x \left\{ \overset{c}{f}_{n-mk-m}(x,y,r,m,s) \right\}$$

$$+ \sum_{k=0}^{[n/m]-1} \frac{(r+1)_k}{k!} (-y)^{k+1} \left\{ \frac{r}{k+1} (n-mk-m) - cm \right\} \overset{c}{f}_{n-mk-m}(x,y,r,m,s),$$

where $\overset{c}{f}_n(x,y,r,m,s)$ stands for the polynomial generated by

$$(1+yt^m)^{-c}\, \Psi \left[\frac{xt^s}{(1+yt^m)^r} \right] = \sum_{n=0}^{\infty} \overset{c}{f}_n(x,y,r,m,s)\, t^n .$$

The recurrence relations (12) - (14) can be further particularized to the corresponding results given in Chapter II, when s=1, and to the recurrence relations for $\overset{c}{g}_n(x,r,s)$ derived by Panda [1] .