CHAPTER - III

INVERSE SERIES RELATION - I

3.1 INTRODUCTION

In this chapter we obtain an inverse series relation of 2.1(5) analogous to the inversion formula of the explicit representation of $P_n(m,x,y,p,C)$ given by Gould ([9], p.707, Eq. (6.2)). Such relations play an important role in the expansion of arbitrary polynomials in terms of the polynomials under consideration. For orthogonal polynomials the orthogonal property provides an effective tool for obtaining the inversion formulas, but for non-orthogonal polynomials no systematic method is available. In a series of papers Gould (e.g. [8], [9] [10]) made use of some results from Combinatorial Analysis to obtain a number of pairs of inverse series relations, whereas Al - Salam [1], Dickinson [1] and Rainville [H] employed some other techniques to obtain such inversion formulas. In section 3.2 we follow the technique used by Gould to derive the desired inversion formula and discuss some of its particular cases in section 3.3.

The inversion formula derived in 3.2 gives rise to a series transform which we discuss in section 3.4 and prove therein an interesting theorem on the convolution of the said series transform. Our investigations related to the properties of this transform led us to an alternative proof of the theorem given in 3.2 which forms the subject matter of section 3.5.

3.2 THE INVERSE SERIES RELATION

In order to derive the proposed inversion formula we first observe that the pair of general inverse series relation proved by Gould [9] in the form of the theorem:

$$(1) \qquad F(n) = \sum_{k=0}^{[n/m]} \binom{p-n+mk}{k} f(n-mk)$$

if and only if

$$(2) \qquad f(n) = \sum_{k=0}^{[n/m]} (-1)^k \binom{p-n+k}{k} \frac{p+mk-n}{p+k-n} F(n-mk)$$

admits of the following mild, but quite useful extension in the form :

Theorem 2 :

$$(3) \qquad F(n) = \sum_{k=0}^{[n/m]} y^k \binom{p-\lambda n+\lambda mk}{k} f(n-mk)$$

if and only if

$$(4) \qquad f(n) = \sum_{k=0}^{[n/m]} (-y)^k \binom{p-\lambda n+k}{k} \frac{p+\lambda mk-\lambda n}{p+k-\lambda n} F(n-mk),$$

where p, and λ are arbitrary parameters.

For positive integral values of λ this extension is quite trivial, but for arbitrary values of λ it is not so. However, the proof of (3) and (4) for arbitrary values of λ runs parallel to that of (1) and (2) wherein use is made of the well-known addition formula (cf., e.g., Gould[3])

$$(5)\qquad \sum_{k=0}^{n} \binom{a+bk}{k}\binom{c+bn-bk}{n-k}\frac{c}{c+bn-bk} = \binom{a+c+bn}{n} ,$$

we, therefore, omit the details for reasons of brevity.

On comparing 2.1(5) with (3) we readily get the inversion formula of 2.1(5) as given below :

$$(6)\qquad x^n = \frac{1}{\gamma_n}\sum_{k=0}^{[n/m]}(-y)^k\binom{-c-nr+k}{k}\frac{mrk-nr-c}{k-nr-c}\, f^c_{n-mk}(x) ,$$

which can also be put in the alternative form

$$(7)\qquad x^n = \frac{1}{\gamma_n}\sum_{k=0}^{[n/m]}(-y)^k\frac{(1-c-nr)_{k-1}(mrk-c-nr)}{k!}\, f^c_{n-mk}(x) .$$

For integral values of r , positive or negative, an appeal to the formula

$$(a)_{n-r}\,(1-a-n)_r = (-1)^r (a)_n$$

would transform (7) to the elegant looking form

$$(8)\qquad x^n = \frac{(c)_{nr}}{\gamma_n}\sum_{k=0}^{[n/m]} y^k\frac{(c+nr-mrk)}{k!\,(c)_{nr-k+1}}\, f^c_{n-mk}(x) .$$

The inversion formula (6) or its equivalent forms (7) and (8) can be used to obtain the expansion of an arbitrary polynomial

$$(9) \qquad Q_s(x) = \sum_{n=0}^{s} A_n \, x^n$$

in a series of polynomials $\overset{c}{f}_n(x)$. For example, if we employ the formula (8), the corresponding expansion formula is

$$(10) \qquad Q_s(x) = \sum_{n=0}^{s} B_n \, \overset{c}{f}_n(x) \, ,$$

where

$$(11) \qquad B_n = \sum_{k=0}^{[(s-n)/m]} \frac{(c+nr)\,(c)_{nr+mrk} \; A_{n+mk}}{k!\,(c)_{nr+mrk-k+1} \; \nu_{n+mk}} \, y^k \, .$$

3.3 PARTICULAR CASES

By assigning appropriate values to the various parameters involved in the inversion formula (6) or its equivalent forms (7) and (8), one can easily obtain the inversion formulas for the polynomials included in the definition 2.1(3). In particular, we mention the following reducible cases :

(i) The particular case of (8) corresponding to the reducibility of $\overset{c}{f}_n(x)$ to $P_n(m,x,y,-\nu,1)$ as given in section 2.1 may be stated in the form

$$(1) \quad \frac{(mx)^n}{n!} = \sum_{k=0}^{[n/m]} \frac{(\nu - mk + n)y^k}{k!(\nu)_{n+1-k}} P_{n-mk}(m,x,y,-\nu,1) ,$$

which is an alternate form of the inversion formula given by Gould [9, p.707, Eq. (6.2)] and we prefer this form as it lends readily itself to further reduction to the standard forms of the inversion formulas for those classical polynomials which are included in $P_n(m,x,y,p,1)$.

(ii) Another special case of (8) worth mentioning would occur when we use the relation $f_n^c(x,-1,r,1) = g_n^c(x,r,1)$. The corresponding inversion formula is

$$(2) \quad x^n = \frac{(c)_{nr}}{\gamma_n} \sum_{k=0}^{n} (-1)^{n-k} \frac{(c + rk)}{(n-k)!(c)_{nr-n+k+1}} g_k^c(x,r,1).$$

The relation (2) can be further particularized by taking $r = 2$, and replacing x by $-4x$, as a result of which (2) would simplify to

$$(3) \quad x^n = \frac{(c)_{2n}}{2^{2n}\gamma_n} \sum_{k=0}^{n} (-1)^k \frac{(c + 2k)}{(n-k)!(c)_{n+k+1}} f_k(x),$$

where $f_n(x)$ are the polynomials considered by Rainville [H, p.137] given in 1.2(11).

The formula (2) is believed to be new while (3) is the known result 1.2(14) given by equation **(4) in** (Rainville[H], p.137), where it has been obtained by using series manipulations in the generating function for $f_n(x)$.

Similarly the expansion formulae 3.2(10) and 3.2(11) can also be particularized for the various polynomial system included in the definition of $f_n^c(x)$.

3.4 CONVOLUTION TRANSFORM

The relation 3.2(3) may be viewed as a series transform which we denote symbolically in the form

(1) $F(n) = S_p[f(n)]$.

This transform possesses an interesting property which we state in the form :

<u>Theorem 3</u> :

If

(2) $F(n) = S_p[f(n)]$,

(3) $G(n) = S_q[g(n)]$,

and if, $(F * G)(n)$ and $(f * g)(n)$ stand for the series convolutions defined by

(4) $(F * G)(n) = \sum_{j=0}^{n} F(j)\, G(n-j)$,

(5) $(f * g)(n) = \sum_{j=0}^{n} f(j)\, g(n-j)$,

then

(6) $(F * G)(n) = S_{p+q}[(f * g)(n)]$,

that is, the convolution of the series transforms is the series transform of the convolution.

To prove this theorem, we first observe that

$$(7) \quad \sum_{n=0}^{\infty} F(n)t^n = (1 + yt^m)^p \sum_{n=0}^{\infty} f(n)z^n ,$$

and

$$(8) \quad \sum_{n=0}^{\infty} G(n)t^n = (1 + yt^m)^q \sum_{n=0}^{\infty} g(n)z^n ,$$

where

$$(9) \quad z = \frac{t}{(1 + yt^m)^{\lambda}} .$$

Now it is easy to see that

$$\sum_{n=0}^{\infty} (F * G)(n)t^n = \sum_{n=0}^{\infty} G(n)t^n \sum_{j=0}^{\infty} F(j)t^j ,$$

$$= (1 + yt^m)^{p+q} \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} f(j)\, g(n-j),$$

$$= (1 + yt^m)^{p+q} \sum_{n=0}^{\infty} z^n (f * g)\ (n) ,$$

$$= \sum_{n=0}^{\infty} S_{p+q} \left[(f * g)\ (n)\right] t^n ,$$

whence it follows that

$$(F * G)\ (n) = S_{p+q} \left[(f * g)\ (n)\right] .$$

If the various parameters involved in the series transform 3.4(1) are particularized suitably, so as to yield the explicit representation of the generalized Humbert polynomials $P_n(m,x,y,p,C)$ introduced by Gould [9], the relation (6) would simplify to

$$(10) \quad \sum_{j=0}^{n} P_j(m,x,y,p,C)\, P_{n-j}(m,x,y,q,C) = P_n(m,x,y,p+q,C),$$

whereas the simplified form of (6), in the case of Laguerre polynomials, is the well known relation (Rainville [H], p.209, Eq. (3))

$$(11) \quad \sum_{j=0}^{n} L_j^{(\alpha)}(x)\, L_{n-j}^{(\beta)}(x) = L_n^{1+\alpha+\beta}(2x).$$

3.5 INVERSE TRANSFORM

In order to obtain the inverse transform of 3.2(3), we begin with the relation 3.4(7) namely,

$$\sum_{n=0}^{\infty} f(n)z^n = (1+yt^m)^{-p} \sum_{n=0}^{\infty} F(n)t^n ,$$

$$(1) \qquad = \sum_{n=0}^{\infty} F(n)z^n\,(1+yt^m)^{\lambda n-p} ,$$

which in view of the well known Lagrange Expansion (Pólya and Szegö [G], p.146, Problem 212) given by 2.4(8) takes the form

$$(2)\quad \sum_{n=0}^{\infty} f(n)\, z^n = \sum_{n=0}^{\infty} F(n) z^n$$

$$\cdot \sum_{k=0}^{\infty} \frac{\lambda n-p}{\lambda n-p+mk\lambda} \binom{\lambda n-p+\lambda mk}{k} (yz^m)^k$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} F(n-mk) \; \frac{\lambda n-\lambda mk-p}{\lambda n-p} \binom{\lambda n-p}{k} y^k z^n \;.$$

Thus it follows that,

$$(3)\quad f(n) = \sum_{k=0}^{[n/m]} y^k \quad \frac{p-\lambda n+\lambda mk}{p-\lambda n} \binom{\lambda n-p}{k} F(n-mk).$$

The relation (3) may also be stated in the form

$$(4)\quad (f * g)(n) = \sum_{k=0}^{[n/m]} y^k \; \frac{p+q-\lambda n+\lambda mk}{p+q-\lambda n} \binom{\lambda n-p-q}{k}$$

$$(F * G)\;(n - mk).$$

It is also worth mentioning here that if we start with (3) and do analogous analysis we shall get 3.2(3) as the inverse of (3). Thus the technique of this section provides us with an alternative proof of the Theorem 2 given in section 3.2.