CHAPTER - IV

INVERSE SERIES RELATION - II

4.1 INTRODUCTION

As mentioned in Chapter II, the work of Rainville [H] on the class of polynomials $\{f_n(x)\}$ defined by the generating relation 1.2(11) and the related work of Chandel ([1], [2]) and Jain [1] inspired Rekha Panda [1] to initiate the study of the class of polynomials $\{g_n^c(x,r,s)\}$. Amongst the results on $g_n^c(x,r,s)$ incorporated in (Panda [1], Srivastava R. [1]) one can find the analogues of all the results for $f_n(x)$ given by Rainville [H] except the interesting expansion 1.2(14) of x^n in terms of $f_n(x)$ which may be viewed as the inverse of the explicit representation of $f_n(x)$. Because of the fact that the special case $s = 1$ of the polynomials $g_n^c(x,r,s)$ is included in the polynomials $f_n^c(x)$, the corresponding expansion of x^n in terms of $g_n^c(x,r,1)$ is obtainable from 3.3(2). But the natural problem of expressing x^n in terms of $g_n^c(x,r,s)$ or equivalently the problem of finding the inversion formula of the explicit representation (Panda [1], p. 105, Eq.(4))

$$(1) \quad g_n^c(x,r,s) = \sum_{k=0}^{[n/s]} \frac{(c+rk)_{n-sk}}{(n-sk)!} \gamma_k x^k ,$$

does not have its solution in the theorem proved in

Chapter III. Our attempts in this direction leads us to some interesting results incorporated in the subsequent sections of this chapter.

4.2 THE INVERSE RELATION

With a view to obtain the desired inversion formula of 4.1(1) we prove here the following Theorem:

Theorem 4:

If

$$(1)\qquad F(n) = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \binom{p+qsk-sk}{n-sk} f(k),$$

then

$$(2)\qquad f(n) = \sum_{k=0}^{ns} \frac{p+qk-k}{p+qsn-k} \binom{p+qsn-k}{sn-k} F(k),$$

where p and q are arbitrary parameters and s is a positive integer.

Writing (1) and (2) in the forms

$$F(n) = \sum_{k=0}^{[n/s]} F(n,k)\, f(k), \quad f(n) = \sum_{k=0}^{sn} f(n,k)\, F(k),$$

it is easy to observe that the validity of the above theorem is established if the following orthogonal relation holds true.

$$(3)\qquad \delta(n,m) = \sum_{k=sm}^{sn} f(n,k)\cdot F(k,m) = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}.$$

In order to prove (3), we employ the method which runs parallel to the method given in the book by Riordan [I].

We first note that the expression for $\delta(n,m)$ may be written as

$$(4) \quad \delta(n,m) = \sum_{k=sm}^{sn} (-1)^{k-sm} \binom{p+qsm-sm}{k-sm} \cdot \left\{ \binom{p+qsn-k}{sn-k} - q \binom{p+qsn-k-1}{sn-k-1} \right\},$$

which we abbreviate as

$$(5) \quad \delta(n,m) = I(n,m) + q\, J(n,m),$$

with

$$(6) \quad I(n,m) = \sum_{k=sm}^{sn} (-1)^{k-sm} \binom{p+qsm-sm}{k-sm} \binom{p+qsn-k}{sn-k}$$

$$= \sum_{j=0}^{sn-sm} (-1)^{j} \binom{A+(sn-sm)q-j}{sn-sm-j} \binom{A}{j}, \quad A = p+qsm-sm.$$

Now, in view of the relation

$$\binom{-n}{m} = (-1)^{m} \binom{n+m-1}{m},$$

and the Vandermonde's Convolution

$$\binom{n}{m} = \sum_{k=0} \binom{n-p}{m-k} \binom{p}{k},$$

the last expression for I(n,m) can successively be put as

(7) $$I(n,m) = \sum_{j=0}^{sn-sm} (-1)^{sn-sm} \binom{-1-A-(q-1)(sn-sm)}{sn-sm-j} \binom{A}{j}$$

(8) $$I(n,m) = (-1)^{sn-sm} \binom{-1-(q-1)(sn-sm)}{sn-sm}$$

(9) $$I(n,m) = \binom{qsn - qsm}{sn-sm}.$$

Likewise, the expression

(10) $$J(n,m) = \sum_{k=sm}^{sn-1} (-1)^{k-sm+1} \binom{p+qsm-sm}{k-sm} \binom{p+qsn-k-1}{sn-k-1}$$

can be simplified to give

(11) $$J(n,m) = (-1) \binom{qsn - qsm - 1}{sn-sm-1};$$

so that

(12) $$qJ(n,m) = (-1) \binom{qsn - qsm}{sn-sm}.$$

Equations (9) and (12), when combined with (5), yield

$$\delta(n,m) = 0 \ , \text{ for } \ n \neq m$$

whereas, for n = m, (4) evidently gives

$$\delta(n,m) = 1.$$

This completes the proof of the theorem.

An alternative form of (1) and (2) is

If

$$(13) \qquad F^*(n) = \sum_{k=0}^{[n/s]} \binom{p+qsk-sk}{n-sk} f^*(k) ,$$

then

$$(14) \qquad f^*(n) = \sum_{k=0}^{sn} (-1)^{sn-k} \frac{p+qk-k}{p+qsn-k} \binom{p+qsn-k}{sn-k} F^*(k) .$$

This is the result of the substitution
$F^*(n) = (-1)^n F(n)$, $f^*(n) = (-1)^{sn} f(n)$
in (1) and (2).

Yet another variation of (13) and (14) may be obtained by multiplying $F^*(n)$ by $p+qn-n$ and $f^*(n)$ by $p+qsn-sn$; we thus get :

If

$$(15) \qquad F^*(n) = \sum_{k=0}^{[n/s]} \left\{ \binom{p+qsk-sk}{n-sk} + q \binom{p+qsk-sk}{n-sk-1} \right\} f^*(k) ,$$

then

$$(16) \qquad f^*(n) = \sum_{k=0}^{sn} (-1)^{sn-k} \binom{p+qsn-k}{sn-k} F^*(k).$$

In view of the Theorem 2 which we proved in Chapter III, it is quite natural to ask if the converse of Theorem 4 holds. The answer to this question, though in affirmative, doesn't seen to be establishable by the method used in this section.

Therefore, in the next section, we invoke the alternative discussed in Chapter III to prove the Theorem 4 in an extended form together with its converse.

4.3 EXTENSION OF THEOREM 4

The first theorem that we prove in this section is in the form:

Theorem 5:

If (as before)

$$(1)\quad F(n) = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \binom{p+qsk-sk}{n-sk} f(k),$$

then

$$(2)\quad f(n) = \sum_{k=0}^{sn} \frac{p+qk-k}{p+qsn-k} \binom{p+qsn-k}{sn-k} F(k),$$

and

$$(3)\quad \sum_{k=0}^{n} \frac{p+qk-k}{p+qn-k} \binom{p+qn-k}{n-k} F(k) = 0, \quad n \neq ms, \quad m=1,2,\ldots$$

To prove the above theorem, we note that

$$(4)\quad \sum_{n=0}^{\infty} F(n)t^n = (1-t)^p \sum_{n=0}^{\infty} f(n)z^n$$

where

$$(5)\quad z = \frac{t^s}{(1-t)^{(1-q)s}}.$$

Therefore,

$$\sum_{n=0}^{\infty} f(n)z^n = (1-t)^{-p} \sum_{n=0}^{\infty} F(n)t^n$$

$$(6) \qquad = \sum_{n=0}^{\infty} F(n)z^{n/s} (1-t)^{-p+\overline{1-q}n} ,$$

which on making use of Lagrange's expansion formula (Pólya and Szegö[G],p.146, Problem 212) 2.4(8) yields

$$(7) \quad \sum_{n=0}^{\infty} f(n)z^n = \sum_{n=0}^{\infty} F(n)z^{n/s} \cdot \sum_{k=0}^{\infty} \frac{-p+(1-q)n}{-p+(1-q)(n+k)} \binom{-p+(1-q)(n+k)}{k} (-1)^k \; z^{k/s}$$

$$(8) \quad = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{-p+(1-q)(n-k)}{n-p-qn} (-1)^k \binom{n-p-qn}{k} z^{n/s} F(n-k).$$

Now on equating the coefficients of z^n , we obtain

$$(9) \quad f(n) = \sum_{k=0}^{ns} \frac{-p+(1-q)(ns-k)}{ns-p-qns} (-1)^k \binom{ns-p-qns}{k} F(ns-k) ,$$

which on making use of

$$(10) \quad \sum_{k=0}^{n} A(k,n) = \sum_{k=0}^{n} A(n-k,n)$$

takes the form,

$$(11) \qquad f(n) = \sum_{k=0}^{ns} \frac{-p+(1-q)k}{-p+ns-qns} \binom{-p+ns-qns}{ns-k} (-1)^{ns-k} F(k).$$

Further, in view of the relation,

$$(12)\qquad (-1)^k \binom{-\alpha}{k} = \binom{\alpha+k-1}{k} = \frac{\alpha}{\alpha+k} \binom{\alpha+k}{k},$$

$$(-1)^{ns-k} \binom{-p+ns-qns}{ns-k}$$ can be put as

$$\frac{p+qns-ns}{p+qns-k} \binom{p+qns-k}{ns-k}$$

and thus (11) yields

$$(13)\qquad f(n) = \sum_{k=0}^{ns} \frac{p+qk-k}{p+qns-k} \binom{p+qns-k}{ns-k} F(k).$$

On the other hand, the comparison of the coefficients of $z^{n/s}$ in the equation (8) for $n \neq ms$, $m=1,2,\ldots$ gives

$$(14)\qquad 0 = \sum_{k=0}^{n} \frac{-p+(1-q)(n-k)}{n-p-qn} (-1)^k \binom{n-p-qn}{k} F(n-k),$$

which on using (10) and (12) leads us to

$$(15)\qquad \sum_{k=0}^{n} \frac{p+qk-k}{p+qn-k} \binom{p+qn-k}{n-k} F(k)=0\ ;\ n \neq ms,\quad m=1,2,3,\ldots,$$

hence the Theorem 5.

Next, if we let

$$(16)\quad \bar{f}(n) = \sum_{k=0}^{n} \frac{p+qk-k}{p+qn-k} \binom{p+qn-k}{n-k} F(k) ,$$

then we have

$$\sum_{n=0}^{\infty} \bar{f}(n)t^{n} = \sum_{k=0}^{\infty} F(k)t^{k} \sum_{n=0}^{\infty} \frac{p+qk-k}{p+qk-k+qn} \binom{p+(q-1)k+qn}{n} t^{n} ,$$

which in view of 2.4(8) can be expressed in the form

$$\sum_{n=0}^{\infty} \bar{f}(n)t^{n} = \sum_{k=0}^{\infty} F(k)t^{k} x^{p+qk-k} , \quad x = 1 + tx^{q}$$

$$= x^{p} \sum_{k=0}^{\infty} F(k)u^{k} , \quad u = tx^{q-1} = (x-1)x^{-1} .$$

Therefore,

$$\sum_{n=0}^{\infty} F(n)u^{n} = x^{-p} \sum_{n=0}^{\infty} \bar{f}(n)t^{n}$$

$$= \sum_{n=0}^{\infty} \bar{f}(n)u^{n}(1-u)^{p+qn-n}$$

$$= \sum_{n=0}^{\infty} \bar{f}(n)u^{n} \sum_{k=0}^{\infty} \binom{p+qn-n}{k} (-u)^{k}$$

$$(17)\quad = \sum_{n=0}^{\infty} u^{n} \sum_{k=0}^{n} (-1)^{k} \binom{p+q(n-k)-(n-k)}{k} \bar{f}(n-k),$$

which on equating the coefficients of u^{n} leads us to

$$(18)\qquad F(n) = \sum_{k=0}^{n} (-1)^{k} \binom{p+q(n-k)-(n-k)}{k} \bar{f}(n-k).$$

The use of (10) transforms (18) to

$$(19)\qquad F(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{p+qk-k}{n-k} \bar{f}(k).$$

Now, if we assume the relations (2) and (3), then in view of (2)

$$(20)\qquad \bar{f}(ns) = f(n) ,$$

and in view of (3)

$$(21)\qquad \bar{f}(n) = 0, \text{ for } n \neq ms,\ m = 1, 2, \ldots$$

Thus on making use of (21), (19) reduces to

$$(22)\qquad F(n) = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \binom{p+qsk-sk}{n-sk} \bar{f}(sk) ,$$

which in view of (20) gives us

$$(23)\qquad F(n) = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \binom{p+qsk-sk}{n-sk} f(k),$$

We, thus have the following converse of Theorem 5:

Theorem 6:

If

$$(24)\qquad f(n) = \sum_{k=0}^{sn} \frac{p+qk-k}{p+qsn-k} \binom{p+qsn-k}{sn-k} F(k),$$

and

$$(25)\qquad \sum_{k=0}^{n} \frac{p+qk-k}{p+qn-k} \binom{p+qn-k}{n-k} F(k) = 0,\ n \neq ms,\ m = 1,2,\ldots$$

then

$$(26)\qquad F(n) = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \binom{p+qsk-sk}{n-sk} f(k).$$

Theorem 5 and Theorem 6 may be combined into the following:

Theorem 7:

$$(27)\qquad F(n) = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \binom{p+qsk-sk}{n-sk} f(k),$$

if and only if

$$(28)\qquad f(n) = \sum_{k=0}^{sn} \frac{p+qk-k}{p+qsn-k} \binom{p+qsn-k}{sn-k} F(k),$$

and

$$(29)\qquad \sum_{k=0}^{n} \frac{p+qk-k}{p+qn-k} \binom{p+qn-k}{n-k} F(k) = 0,\ n \neq ms,\ m = 1,2,\ldots$$

It is also worth mentioning here that if we denote the series transform (1) symbolically in the form

(30) $F(n) = T_p \left[f(n) \right]$,

then the following theorem on convolution may be easily obtained by following the method used in Theorem 3.

Theorem 8 :

If

(31) $F(n) = T_p \left[f(n) \right]$,

(32) $G(n) = T_q \left[g(n) \right]$,

and if, $(F * G)(n)$ and $(f * g)(n)$ stand for the series convolutions defined as

(33) $$(F * G)(n) = \sum_{j=0}^{n} F(j)\, G(n-j),$$

(34) $$(f * g)(n) = \sum_{j=0}^{n} f(j)\, g(n-j),$$

then

(35) $$(F * G)(n) = T_{p+q} \left[(f * g)(n) \right],$$

that is, the convolution of the T- series transforms of two functions f and g is the T- series transform of the convolution of f and g.

4.4 EXPANSION OF x^n IN TERMS OF $g_n^c(x,r,s)$

The explicit representation 4.1(1) can be written in the form

$$(1) \qquad g_n^c(x,r,s) = \sum_{k=0}^{[n/s]} \binom{-c-rk}{n-sk} (-1)^{n-sk}\, \gamma_k x^k$$

which on being compared with 4.3(1), with $p = c$, $q = -(r-s)/s$, readily yields the expansion (or the inversion formula of 4.1(1)) formula

$$(2) \qquad x^n = \frac{1}{\gamma_n} \sum_{k=0}^{sn} \frac{c+rk/s}{(c+(r-s)n+k)} \binom{-c-(r-s)n-k}{sn-k} g_k^c(x,r,s).$$

The inversion formula (2) may also be expressed in the following alternative forms:

$$(3) \qquad x^n = \frac{1}{\gamma_n} \sum_{k=0}^{sn} \frac{(-c-rk/s)}{(sn-k)!} (1-c-rn)_{sn-k-1} \; g_k^c(x,r,s) \; ,$$

$$(4) \qquad x^n = \frac{1}{\gamma_n} \sum_{k=0}^{sn} \frac{(-1)^{sn-k} (c+rk/s)(c)_{rn}}{(sn-k)!(c)_{(r-s)n+k+1}} \; g_k^c(x,r,s).$$

In view of 4.3 (3), we also have

$$(5) \qquad \sum_{k=0}^{n} \frac{(-1)^{n-k} (c+rk/s)(c)_{rn/s}}{(n-k)!\;(c)_{(\frac{r-s}{s})n+k+1}} \; g_k^c(x,r,s) = 0 \; ,$$

for $n \neq ms$, $m = 1,2,3,\ldots$

When $s = 1$, (4) evidently reduces to 3.3(2)

$$(6) \qquad x^n = \frac{(c)_{rn}}{\gamma_n} \sum_{k=0}^{n} \frac{(-1)^{n-k} (c+rk)}{(n-k)!(c)_{(r-1)n+k+1}} \; g_k^c(x,r,1),$$

which can be further particularized by taking $r = 2$ and replacing x by $-4x$; (6) would thus simplify to 1.2(14).