CHAPTER - VI

BIORTHOGONAL POLYNOMIALS

6.1 INTRODUCTION

Besides the classical orthogonal polynomials and their various generalizations discussed in earlier chapters, the class of polynomials $\left\{f_n^c(x)\right\}$ (Chapter II, eqn.2.1(3)) embraces yet another interesting set of polynomials in its folds of generality. This set occurs in our study of biorthogonal pair of polynomials associated with Jacobi weight function. As mentioned in Chapter I, the notion of biorthogonal pair of polynomials was introduced by Konhauser[1] according to whom two sets of polynomials $\left\{R_m(x) \mid R_m(x)\right.$ being a polynomial of degree m in the polynomial $\left.r(x)\right\}$ and $\left\{S_n(x) \mid S_n(x)\right.$ being a polynomial of degree n in the polynomial $\left.s(x)\right\}$ are said to form a biorthogonal pair of polynomials over the interval (a,b) with respect to an admissible weight function $p(x)$ and basic polynomials $r(x)$ and $s(x)$ if

$$(1) \qquad \int_a^b p(x)\, R_m(x)\, S_n(x)\, dx \begin{cases} = 0, & m,n=0,1,2,\ldots,\ m \neq n, \\ \neq 0, & m = n. \end{cases}$$

Among those who contributed to the study of biorthogonal polynomials associated with the weight function

$x^{\alpha} e^{-x}$, the names of Konhauser ([1], [2]), Carlitz [6] and Srivastava ([3], [12]) are worth mentioning. Prabhakar and Kashyap [1] studied the biorthogonal polynomials suggested by Legendre polynomials and the particular case $P_n^{(\alpha,0)}(x)$ of Jacobi polynomials.

In this Chapter we present our results related to study of the polynomial sets in powers of basic polynomials $r(x) = (\frac{1-x}{2})^k$ and $s(x) = (\frac{1-x}{2})$, which we denote by $W_n^{(\alpha,\beta)}(x;k)$ and $X_n^{(\alpha,\beta)}(x;k)$ respectively.

These polynomials satisfy the biorthogonality condition:

$$(2) \quad J_{n,m} = \int_{-1}^{1} (\tfrac{1-x}{2})^{\alpha} (\tfrac{1+x}{2})^{\beta} W_n^{(\alpha,\beta)}(x;k) \; X_m^{(\alpha,\beta)}(x;k)dx$$

$$= 0 \, , \; m \neq n$$

$$\neq 0 \, , \; m = n,$$

which quality them to be called as the pair of biorthogonal polynomials over $(-1,1)$ with respect to the weight function $(\frac{1-x}{2})^{\alpha} (\frac{1+x}{2})^{\beta}$, $Re(\alpha), Re(\beta) > -1$ and the basic polynomials $(\frac{1-x}{2})^k$ and $(\frac{1-x}{2})$.

This chapter also contains the evaluation of an n^{th} order determinant

$$\left| \frac{(a_j)_{i-1}}{(a_j+b)_{i-1}} \right|, \quad i,j = 1,2,\dots,n,$$

which we required during the course of our study.

6.2 THE POLYNOMIALS IN $\left(\frac{1-x}{2}\right)^k$

In terms of the factorial notation

$$(1) \quad \begin{cases} (a)_n = a(a+1) \dots (a+n-1) = \dfrac{\Gamma(a+n)}{\Gamma(a)}, \ a \neq 0 \\ (a)_0 = 1, \end{cases}$$

we consider the n^{th} degree polynomials in $\left(\frac{1-x}{2}\right)^k$

$$(2) \quad W_n^{(\alpha,\beta)}(x;k) = \frac{(1+\alpha)_n}{n!} \sum_{j=0}^{n} \frac{(-n)_j \ (1+\alpha+\beta+n)_{kj}}{j! \ (1+\alpha)_{kj}} \left(\frac{1-x}{2}\right)^{kj},$$

as members of one set of a biorthogonal pair associated with the admissible weight function $p(x) = \left(\frac{1-x}{2}\right)^{\alpha}\left(\frac{1+x}{2}\right)^{\beta}$, $Re(\alpha) > -1$, $Re(\beta) > -1$ on the finite interval $(-1,1)$, k being a positive integer.

This definition is suggested by that of the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, to which (2) would reduce when $k = 1$. $W_n^{(\alpha,\beta)}(x;k)$ may also be written in hypergeometric form as

$$(3)\quad W_n^{(\alpha,\beta)}(x;k) = \frac{(1+\alpha)_n}{n!}\,{}_{k+1}F_k\left[\begin{matrix}-n, & \Delta(k,1+\alpha+\beta+n); \\ & \Delta(k,1+\alpha);\end{matrix}\quad \left(\frac{1-x}{2}\right)^k\right]$$

with $\Delta(m;\lambda)$ as before denoting the set of parameters

$$\frac{\lambda}{m}, \frac{\lambda+1}{m}, \ldots, \frac{\lambda+m-1}{m} .$$

In order to prove that the set of polynomials $W_n^{(\alpha,\beta)}(x;k)$ and the $X_n^{(\alpha,\beta)}(x;k)$ satisfy the biorthogonality condition 6.1(2), it is sufficient to show that

$$(4)\quad \int_{-1}^{1}\left(\frac{1-x}{2}\right)^{\alpha+i}\left(\frac{1+x}{2}\right)^{\beta} W_n^{(\alpha,\beta)}(x;k)dx \begin{cases} = 0, & i=0,1,\ldots,n-1, \\ \neq 0, & i = n,\end{cases}$$

and

$$(5)\quad \int_{-1}^{1}\left(\frac{1-x}{2}\right)^{\alpha+ki}\left(\frac{1+x}{2}\right)^{\beta} X_n^{(\alpha,\beta)}(x;k)dx \begin{cases} = 0, & i=0,1,\ldots,n-1, \\ \neq 0, & i=n.\end{cases}$$

We first prove the condition (4). Denoting the left hand side of (4) by $I_{i,n}$ and substituting the representation

(2) for $W_n^{(\alpha,\beta)}(x;k)$ in (4), we obtain,

$$I_{i,n} = \frac{(1+\alpha)_n}{n!} \sum_{j=0}^{n} \frac{(-n)_j\ (1+\alpha+\beta+n)_{kj}}{j!(1+\alpha)_{kj}} \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{\alpha+i+kj} \left(\frac{1+x}{2}\right)^{\beta} dx,$$

$$= \frac{2\,\Gamma(\beta+1)\ (1+\alpha)_n}{n!\ (1+\alpha+\beta)_n} \sum_{j=0}^{n} \frac{(-n)_j\ (1+\alpha+\beta)_{n+kj}\ \Gamma(\alpha+i+kj+1)}{j!(1+\alpha)_{kj}\ \Gamma(\alpha+i+kj+\beta+2)}$$

$$= \frac{2\,\Gamma(1+\alpha+n)\ \Gamma(1+\beta)}{n!\ \Gamma(1+\alpha+\beta+n)} \sum_{j=0}^{n} \frac{(-n)_j}{j!} (\alpha+kj+1)_i (\alpha+\beta+kj+i+2)_{n-i-1}$$

$$= \frac{2\,\Gamma(1+\alpha+n)\ \Gamma(1+\beta)}{n!\ \Gamma(1+\alpha+\beta+n)} \sum_{j=0}^{n} \frac{(-n)_j}{j!} E_\alpha^j (\alpha+1)_i (\alpha+\beta+i+2)_{n-i-1},$$

where

E_α denotes the shift operator

$$E_\alpha\, f(\alpha) = f(\alpha + k).$$

When $i < n$, the expression within the last summation represents the n^{th} difference of a polynomial of degree $n-1$ in α, hence $I_{i,n} = 0$ for $i < n$. For $i = n$, $I_{i,n}$ is obviously non-zero. To determine $I_{n,n}$ we observe that the expression for $I_{n,n}$ may be written in the form

$$I_{n,n} = \frac{2\,\Gamma(1+\alpha+n)\ \Gamma(1+\beta)}{n!\ \Gamma(1+\alpha+\beta+n)} \sum_{j=0}^{n} \frac{(-n)_j\ (\alpha+kj+1)_n}{j!\ (\alpha+\beta+1+kj+n)}\ .$$

Now since $(\alpha+\beta+1+kj+n)$ is a factor of $\left[(\alpha+kj+1)_n - (-1)^n (\beta+1)_n\right]$, the summation involved in $I_{n,n}$ may be split into two parts, one of which representing the n^{th} difference of a polynomial of degree n-1 in α, vanishes and so $I_{n,n}$ reduces to

$$I_{n,n} = \frac{2\,\Gamma(1+\alpha+n)\,\Gamma(1+\beta+n)\,(-1)^n}{n!\,(1+\alpha+\beta+n)} \sum_{j=0}^{n} \frac{(-n)_j}{j!\,(\alpha+\beta+1+kj+n)}$$

$$= \frac{2(-1)^n\,\Gamma(1+\alpha+n)\,\Gamma(1+\beta+n)}{n!\,\Gamma(1+\alpha+\beta+n)} \int_0^1 t^{\alpha+\beta+n}(1-t^k)^n\,dt$$

$$= \frac{2(-1)^n\,\Gamma(1+\alpha+n)\,\Gamma(1+\beta+n)}{k\,\Gamma(1+\alpha+\beta+n)\left(\frac{\alpha+\beta+n+1}{k}\right)_{n+1}}.$$

We thus get

$$(6)\qquad \int_{-1}^{1}\left(\frac{1-x}{2}\right)^{\alpha+n}\left(\frac{1+x}{2}\right)^{\beta} W_n^{(\alpha,\beta)}(x;k)\,dx$$

$$= \frac{2(-1)^n\,\Gamma(1+\alpha+n)\,\Gamma(1+\beta+n)}{k\,\Gamma(1+\alpha+\beta+n)\left(\frac{\alpha+\beta+n+1}{k}\right)_{n+1}}.$$

For k = 1, (6) leads to the known formula (Rainville[H], Eq. (15), p. 261)

$$(7)\qquad \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{\alpha+n} \left(\frac{1+x}{2}\right)^{\beta} P_n^{(\alpha,\beta)}(x)dx = 2(-1)^n B(1+\alpha+n, 1+\beta+n),$$

where for $\beta = 0$ it reduces to (Prabhakar and Kashyap [1], Eq.(2.14))

$$(8)\qquad \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{\alpha+n} V_n^{\alpha}(x;k)dx = \frac{2(-1)^n n!}{k\left(\frac{\alpha+n+1}{k}\right)_{n+1}} .$$

The proof of the other part of the biorthogonality condition viz. (5), would require the determination of the polynomials $X_n^{(\alpha,\beta)}(x;k)$, which we do in section 6.5 of this Chapter. We therefore defer its proof till the determination of the polynomials $X_n^{(\alpha,\beta)}(x;k)$.

6.3 INVERSE SERIES OF 6.2(2) AND THE RECURRENCE RELATIONS FOR $W_n^{(\alpha,\beta)}(x;k)$

Making use of the definition 6.2(2) it is easy to see that the polynomials $W_n^{(\alpha,\beta)}(x;k)$ are generated by the relation

$$(1)\qquad \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)_n}{(1+\alpha)_n} W_n^{(\alpha,\beta)}(x;k)t^n = (1-t)^{-1-\alpha-\beta}\, G\left[\frac{-v^k t}{(1-t)^{k+1}}\right],$$

where $v = \frac{1-x}{2}$, and

(2) $$G[z] = \sum_{j=0}^{\infty} \frac{(1+\alpha+\beta)_{(k+1)j}}{(1+\alpha)_{kj}\ j!} z^j .$$

A comparison of (1) with the generating relation 2.1(3) for the general class of polynomials $\left\{ f_n^c(x) \right\}$, reveals the fact that by choosing the parameters involved in the definition of $f_n^c(x)$ appropriately, one can obtain the polynomials $W_n^{(\alpha,\beta)}(x;k)$ as particular case of $f_n^c(x)$. This fact together with the known relation 3.2(8) leads us to the inverse series of 6.2(2) in the form

(3) $$\left(\frac{1-x}{2}\right)^{kn}$$

$$= (1+\alpha)_{kn} n! \sum_{j=0}^{n} \frac{(-1)^j\ (1+\alpha+\beta+kj+j)}{(n-j)!\ (1+\alpha)_j (1+\alpha+\beta+j)_{kn+1}} W_j^{(\alpha,\beta)}(x;k).$$

For $\beta = 0$, (3) would give the expansion of $\left(\frac{1-x}{2}\right)^{kn}$ in terms of the polynomials $V_j^{\alpha}(x;k)$ studied by Prabhakar and Kashyap [1] .

For the derivation of differential and mixed recurrence relation, we put

(4) $$\Phi(v,t) = (1-t)^{-c}\ G\left[\frac{-tv^s}{(1-t)^r}\right] .$$

It is readily seen that $\Phi(v,t)$ satisfies the differential equation

$$(5)\qquad \frac{v}{s}(1-t+rt)\frac{\partial \Phi}{\partial v} - t(1-t)\frac{\partial \Phi}{\partial t} = - ct\,\Phi ,$$

which in view of (1) yields the following differential recurrence relations.

$$(6)\qquad (x-1)\left[(\alpha+\beta+n)\, D\, W_n^{(\alpha,\beta)}(x;k) + k(\alpha+n)\, D\, W_{n-1}^{(\alpha,\beta)}(x;k)\right]$$

$$= k(\alpha+\beta+n)\left[n\, W_n^{(\alpha,\beta)}(x;k) - (\alpha+n)\, W_{n-1}^{(\alpha,\beta)}(x;k)\right],$$

$$(7)\qquad (x-1)D\, W_n^{(\alpha,\beta)}(x;k) - kn\, W_n^{(\alpha,\beta)}(x;k)$$

$$= \frac{(1+\alpha)_n}{(1+\alpha+\beta)_n}\left[-\sum_{j=0}^{n-1}\frac{(1+\alpha+\beta)_j}{(1+\alpha)_j}\left\{k(\alpha+\beta+1)\, W_j^{(\alpha,\beta)}(x;k)\right.\right.$$

$$\left.\left. + (x-1)\,(k+1)\, D\, W_j^{(\alpha,\beta)}(x;k)\right\}\right]$$

and

$$(8)\qquad (x-1)D\, W_n^{(\alpha,\beta)}(x;k) - kn\, W_n^{(\alpha,\beta)}(x;k)$$

$$= \frac{(1+\alpha)_n}{(1+\alpha+\beta)_n}\sum_{j=0}^{n-1}\frac{(-k)^{n-j}(1+\alpha+\beta+\overline{k+1}\,j)}{(1+\alpha)_j}(1+\alpha+\beta)_j\, W_j^{(\alpha,\beta)}(x;k).$$

Direct differentiation of $W_n^{(\alpha,\beta)}(x;k)$ and use of the relation

$$(\nu)_{n-k} = \frac{(\nu)_n}{(-1)^k(1-\nu-n)_k}$$

give,

(9) $\quad (x-1)\ D\ W_n^{(\alpha,\beta)}(x;k)$

$$= \frac{k(\alpha+n)\ (1+\alpha+\beta+n)_k\ (-1)^{k+1}}{(1-\alpha-k-n)_k} \left(\frac{1-x}{2}\right)^k\ W_{n-1}^{(\alpha+k,\beta+1)}(x;k)\ ,$$

which could otherwise very well be obtained from equation (6) also. For $k = 1$, (9) reduces to the well knwon relation for Jacobi polynomials (Rainville [H], p.263, Eq. (2))

(10) $$D\ P_n^{(\alpha,\beta)}(x) = \tfrac{1}{2}(1+\alpha+\beta+n)\ P_{n-1}^{(\alpha+1,\beta+1)}(x).$$

Further, combination of (6) with 6.2(2) leads to

(11) $\quad (x-1)\ (\alpha+\beta+n)\ D\ W_n^{(\alpha,\beta)}(x;k)$

$$= \frac{k}{(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n-1}{j-1} \frac{(1+\alpha)_n}{(1+\alpha)_{kj}} (\alpha+\beta+n)_{kj+1} \left(\frac{1-x}{2}\right)^{kj} .$$

Multiplying both sides of (11) by $\left(\frac{1-x}{2}\right)^{\alpha}$ and then differentiating k times we get

(12) $\quad D^k \left[(x-1)\ \left(\frac{1-x}{2}\right)^{\alpha}\ D\ W_n^{(\alpha,\beta)}(x;k)\right]$

$$= \left(\frac{1-x}{2}\right)^{\alpha} (-1)^{k+1}\ k(\alpha+n)\ (1+\alpha+\beta+n)_k\ W_{n-1}^{(\alpha,\beta+k+1)}(x;k)/2^k .$$

From 6.2(2) it can be easily seen that

(13) $$\int \left(\frac{1-x}{2}\right)^{\alpha+\beta+n} D\, W_n^{(\alpha,\beta)}(x;k)\, dx$$

$$= \frac{(-k)\ (1+\alpha+\beta+n)_{k-1}}{(1+\alpha+n)_{k-1}} \left(\frac{1-x}{2}\right)^{\alpha+\beta+n+k} W_{n-1}^{(\alpha+k,\beta)}(x;k) \ ,$$

which for $\beta = 0$ simplifies to

(14) $$\int \left(\frac{1-x}{2}\right)^{\alpha+n} D\, V_n^{\alpha}(x;k)dx = (-k) \left(\frac{1-x}{2}\right)^{\alpha+n+k} V_{n-1}^{\alpha+k}(x;k) \ ,$$

and for $k=1$, reduces to

(15) $$\int\left(\frac{1-x}{2}\right)^{\alpha+\beta+n} D\, P_n^{(\alpha,\beta)}(x)dx = (-1)\left(\frac{1-x}{2}\right)^{\alpha+\beta+n+1} P_{n-1}^{(\alpha+1,\beta)}(x).$$

As $s(x) = \left(\frac{1-x}{2}\right)^k = [r(x)]^k$, by an application of the theorem (Konhauser [1], Th. 2.6), it follows that there exist pure recurrence relations of the form

$$\left(\frac{1-x}{2}\right)^k W_n^{(\alpha,\beta)}(x;k) = \sum_{i=n-k}^{n+1} b_{n,i}\ W_i^{(\alpha,\beta)}(x;k) \ ,$$

and

$$\left(\frac{1-x}{2}\right)^k X_n^{(\alpha,\beta)}(x;k) = \sum_{i=n-1}^{n+k} a_{n,i}\ X_i^{(\alpha,\beta)}(x;k) \ ,$$

where the coefficients $b_{n,i}$ and $a_{n,i}$ are functions of n and independent of x , which can be obtained by substituting the representation of the polynomials and equating the powers of $\left(\frac{1-x}{2}\right)^{ki}$, $i = 1+1, n, \ldots$ in case of $b_{n,i}$ and that of $\left(\frac{1-x}{2}\right)^{k+i}$, $i = n, n-1, \ldots$ in case of $a_{n,i}$.

6.4 EVALUATION OF THE DETERMINANT $\left|\frac{(a_j)_{i-1}}{(a_j+b)_{i-1}}\right|$

Before we determine the polynomials $X_n^{(\alpha,\beta)}(x;k)$, we prove here a lemma giving the value of the determinant $\left|\frac{(a_j)_{i-1}}{(a_j+b)_{i-1}}\right|$,which we shall require in the study of $X_n^{(\alpha,\beta)}(x;k)$.

<u>LEMMA</u>

If

(1) $$D_n = \left|\frac{(a_j)_{i-1}}{(a_j+b)_{i-1}}\right| , \quad i,j = 1,2,\ldots,n$$

stands for the n^{th} order determinant whose (i,j) element is

$$\frac{(a_j)_{i-1}}{(a_j+b)_{i-1}} ,$$

then

(2) $$D_n = \prod_{i>j}^{n} (a_i-a_j) \prod_{j=1}^{n-1} (b)_j \Big/ \prod_{j=1}^{n} (a_j+b)_{n-1} .$$

Proof :

Regarding a_j' s and b as (n+1) independent variables, D_n in its expanded form can be put over a common denominator $\prod_{j=1}^{n} (a_j+b)_{n-1}$, which is of degree n(n-1) in the (n+1) variables. As each individual term of the determinant is of degree zero, the value of D_n should also be of degree zero. Therefore the numerator must be a polynomial of degree n^2-n in a_j' s and b.

If $a_i = a_j$, obviously D_n vanishes, it, therefore, follows that the numerator for the value of D_n , must contain a factor of the form $\prod_{i > j}^{n} (a_i - a_j)$ which is a term of degree $\frac{n^2 - n}{2}$.

Further denoting the successive rows of D_n by $R_1, R_2, \ldots, R_n$ we observe that

$$\Delta^m R_r = \left[\frac{(-1)^m (b)_m (a_1)_{r-1}}{(a_1+b)_{r+m-1}} , \ldots , \frac{(-1)^m (b)_m (a_n)_{r-1}}{(a_n+b)_{r+m-1}} \right]$$

where

$$\Delta R_r = R_{r+1} - R_r = (E - 1) R_r ;$$

and since

$$\Delta^m R_r = (E - 1)^m R_r$$

$$= R_{m+r} - \binom{m}{1} R_{m+r-1} + \dots + (-1)^m R_r ,$$

the operation

$$R_{m+r} - \binom{m}{1} R_{m+r-1} + \dots + (-1)^m R_r$$

on the $(m+r)^{th}$ row of D_n, which is equivalent to adding to $(m+r)^{th}$ row an appropriate linear combination of upper m rows thus leaving the value of the determinant unchanged, changes the $(m+r)^{th}$ row into $\Delta^m R_r$.

Performing the operation described above, by taking $r = 1$ and $m = n-1, n-2, \dots, 1$ successively, the determinant D_n becomes

$$(3) \quad D_n = (-1)^{\frac{n(n-1)}{2}} \left[\prod_{j=1}^{n-1} (b)_j\right] \left|\frac{1}{(a_j+b)_{i-1}}\right|, \quad i,j=1,2,\dots,n.$$

(3) shows that $\prod_{j=1}^{n-1} (b)_j$ is also a factor of the numerator of D_n. Since the degree of $\prod_{i>j}^{n} (a_i-a_j) \prod_{j=1}^{n} (b)_j$ is n^2-n, we can write

$$(4) \quad D_n = c_n \prod_{i>j}^{n} (a_i - a_j) \prod_{j=1}^{n-1} (b)_j \Big/ \prod_{j=1}^{n} (a_j + b)_{n-1} \ ,$$

where c_n is a constant. To determine the constant c_n we note that when $b = 1$, D_n becomes

$$a_1 \, a_2 \, \ldots \, a_n \left| \frac{1}{a_j + i - 1} \right| , \; i,j = 1,2,\ldots,n \ ,$$

which in view of the known result (c.f. Davis [A], Lemma 11.3.1)

$$(5) \quad \left| \frac{1}{a_i + b_j} \right| , \; i,j = 1,2, \ldots, n$$

$$= \prod_{i>j}^{n} (a_i - a_j) \prod_{i>j}^{n} (b_i - b_j) \Big/ \prod_{i,j=1}^{n} (a_i + b_j) \ ,$$

has the value

$$\prod_{i>j}^{n} (a_i - a_j) \prod_{j=1}^{n-1} (j)! \Big/ \prod_{j=1}^{n} (a_j + 1)_{n-1} \ .$$

It therefore follows that the constant c_n in (4) is equal to unity. Hence the lemma.

(3) together with (2) leads us to the

COROLLARY:

If

$$(6) \quad D_n^* = \left| \frac{1}{(a_j + b)_{i-1}} \right| , i,j = 1,2,\ldots,n \ ,$$

then

(7) $$D_n^* = (-1)^{\frac{n(n-1)}{2}} \prod_{i>j}^{n} (a_i - a_j) \Big/ \prod_{j=1}^{n} (a_j + b)_{n-1} .$$

6.5 DETERMINATION OF $X_n^{(\alpha,\beta)}(x;k)$

Let

$$\Phi_{i,j} = \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{\alpha+i+kj} \left(\frac{1+x}{2}\right)^{\beta} dx$$

$$= \frac{2\,\Gamma(\alpha+i+kj+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+i+kj+2)},$$

and consider the polynomials

(1) $$\Psi_n^{(\alpha,\beta)}(x;k) = \begin{vmatrix} \Phi(0,0) & \Phi(0,1) & \cdots & \Phi(0,n-1) & 1 \\ \Phi(1,0) & \Phi(1,1) & \cdots & \Phi(1,n-1) & \frac{1-x}{2} \\ \vdots & & & & \\ \Phi(n,0) & \Phi(n,1) & \cdots & \Phi(n,n-1) & \left(\frac{1-x}{2}\right)^n \end{vmatrix} .$$

Multiplying both sides of (1) with $\left(\frac{1-x}{2}\right)^{\alpha+kj} \left(\frac{1+x}{2}\right)^{\beta}$ and integrating with respect to x between the limits -1 to 1, we get

$$\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{\alpha+kj}\left(\frac{1+x}{2}\right)^{\beta}\Psi_n^{(\alpha,\beta)}(x;k)dx$$

$$= \begin{vmatrix} \Phi(0,0) & \Phi(0,1) & \dots & \Phi(0,n-1) & \Phi(0,j) \\ \Phi(1,0) & \Phi(1,1) & \dots & \Phi(1,n-1) & \Phi(1,j) \\ \vdots & & & & \\ \Phi(n,0) & \Phi(n,1) & \dots & \Phi(n,n-1) & \Phi(n,j) \end{vmatrix}$$

$$\begin{aligned} &= 0 \text{ for } j=0,1,\dots,n-1 \\ &\neq 0 \text{ for } j = n. \end{aligned}$$

Thus 6.2(5) is seen to be satisfied by $\Psi_n^{(\alpha,\beta)}(x;k)$, so $\Psi_n^{(\alpha,\beta)}(x;k)$ is $X_n^{(\alpha,\beta)}(x;k)$ except for a constant multiplicative factor.

Being guided by the explicit representation of Jacobi polynomials to which $X_n^{(\alpha,\beta)}(x;k)$ would correspond, when k=1, we set

$$(2) \quad X_n^{(\alpha,\beta)}(x;k) = \frac{\left(\frac{\alpha+1}{k}\right)_n}{\Delta_{n,0} \quad n!} \quad \Psi_n^{(\alpha,\beta)}(x;k) \ ,$$

where $\Delta_{n,0}$ is the cofactor of the (1,n+1) element of the determinant representation of $\Psi_n^{(\alpha,\beta)}(x;k)$.

Now if we expand the determinant for $\Psi_n^{(\alpha,\beta)}(x;k)$ in terms of the elements of its last column, substitute the values of $\Phi(i,j)$ and make the necessary simplification, we get $X_n^{(\alpha,\beta)}(x;k)$ in the form

$$(3)\quad X_n^{(\alpha,\beta)}(x;k) = \sum_{r=0}^{n} C_{n,r} \left(\frac{1-x}{2}\right)^r ,$$

where

$$(4)\quad C_{n,r} = \left(\frac{\alpha+1}{k}\right)_n / n! \; \frac{D_{n,r}}{D_{n,0}} ,$$

$D_{n,r}$ being the co-factor of $(r+1,n+1)$ element of the $(n+1)^{th}$ order determinant

$$\left| \frac{(\alpha+1+kj-k)_{i-1}}{(\alpha+\beta+2+kj-k)_{i-1}} \right| , \quad i,j = 1,2,\ldots,n+1.$$

When $\beta = 0$, $D_{n,r}$ and $D_{n,0}$ which are given by

$$(5)\quad D_{n,r} = k^n \left(\frac{\alpha+1}{k}\right)_n (-1)^{n+r} \left| \frac{1}{\alpha+i+kj-k} \right| , \quad \begin{array}{l} i=1,2,\ldots,n+1, i\neq r+1 \\ j=1,2,\ldots,n \end{array}$$

and

$$(6)\quad D_{n,0} = k^n \left(\frac{\alpha+1}{k}\right)_n (-1)^n \left| \frac{1}{\alpha+i+1+kj-k} \right| , i,j=1,2,\ldots,n$$

can be evaluated with the help of the result 6.4(5). We shall thus get

$$(7)\quad X_n^{(\alpha,0)}(x;k) = \frac{1}{n!} \sum_{j=0}^{n} \frac{(-n)_j}{j!} \left(\frac{\alpha+j+1}{k}\right)_n \left(\frac{1-x}{2}\right)^j ,$$

which corresponds to the polynomials $U_n^{\alpha}(x;k)$ studied by Prabhakar and Kashyap [1].

For non-zero values of β and $k > 1$, a closed form expression for $D_{n,r}$, $r \neq 0,n$, does not seem to be obtainable. However $D_{n,n}$ and $D_{n,0}$ can be evaluated with the help of the determinant given in the previous section. Omitting the details, which are quite straight forward, we give below the values of $D_{n,n}$ and $D_{n,0}$.

$$D_{n,n} = \left| \frac{(\alpha+1+jk-k)_{i-1}}{(\alpha+\beta+2+jk-k)_{i-1}} \right| , \quad i,j = 1,2,\ldots,n$$

$$= k^{\frac{n(n-1)}{2}} \prod_{j=1}^{n-1} (\beta+1)_j \, (j)! \Big/ \prod_{j=1}^{n} (\alpha+\beta+2+jk-k)_{n-1} ,$$

and

$$D_{n,0} = (-1)^n \left| \frac{(\alpha+1+kj-k)_i}{(\alpha+\beta+2+kj-k)_i} \right| , \quad i,j = 1,2,\ldots,n$$

$$= \left(\frac{\alpha+1}{k}\right)_n (-1)^n \, k^{\frac{n(n-1)}{2}} \prod_{j=1}^{n-1} (\beta+1)_j (j)! \; \left(\frac{\alpha+\beta+2}{k}\right)_n \prod_{j=1}^{n} (\alpha+\beta+3+kj-k)_{n-1} ,$$

so that the leading coefficient $C_{n,n}$ in $X_n^{(\alpha,\beta)}(x;k)$ becomes

$$(8) \qquad C_{n,n} = \frac{(-1)^n}{n!} \left(\frac{\alpha+\beta+n+1}{k}\right)_n .$$

After having determined $C_{n,n}$ we can now evaluate the integral

$$J_{n,n} = \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{\alpha} \left(\frac{1+x}{2}\right)^{\beta} W_n^{(\alpha,\beta)}(x;k) \; X_n^{(\alpha,\beta)}(x;k)dx.$$

Substituting right hand side of (3) for $X_n^{(\alpha,\beta)}(x;k)$ and making use of 6.2(4) and 6.2(6) we get

$$J_{n,n} = C_{n,n} \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{\alpha+n} \left(\frac{1+x}{2}\right)^{\beta} W_n^{(\alpha,\beta)}(x;k)dx$$

$$= \frac{(-1)^n}{n!} \left(\frac{\alpha+\beta+n+1}{k}\right)_n \frac{2(-1)^n \Gamma(1+\alpha+n) \; \Gamma(1+\beta+n)}{k\Gamma(1+\alpha+\beta+n) \; \left(\frac{\alpha+\beta+n+1}{k}\right)_{n+1}},$$

so that

$$(9) \quad J_{n,n} = \frac{2\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}{n! \; \Gamma(1+\alpha+\beta+n)(\alpha+\beta+n+1+kn)} .$$

Next, we evaluate the integral

$$\int_{-1}^{1} \left(\frac{1-x}{2}\right)^{\alpha+kn} \left(\frac{1+x}{2}\right)^{\beta} X_n^{(\alpha,\beta)}(x;k)dx ,$$

for which both members of 6.3(3) are first multiplied by $\left(\frac{1-x}{2}\right)^{\alpha} \left(\frac{1+x}{2}\right)^{\beta} W_n^{(\alpha,\beta)}(x;k)$ and then integrated with respect to x over the interval $[-1,1]$. Thereafter an application of 6.1(2) and (9) gives us

$$(10) \quad \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{\alpha+kn} \left(\frac{1+x}{2}\right)^{\beta} X_n^{(\alpha,\beta)}(x;k)dx$$

$$= \frac{2(-1)^n \; n! \; \Gamma(1+\beta+n) \; \Gamma(1+\alpha+kn)}{n! \; \Gamma(2+\alpha+\beta+\overline{k+1}n)} .$$

It is worth mentioning here that the problem of constructing a pair of biorthogonal polynomials suggested by Jacobi polynomials has also been investigated by Madhekar and Thakare[1], but our approach of introducing the polynomials $W_n^{(\alpha,\beta)}(x;k)$ and $X_n^{(\alpha,\beta)}(x;k)$ is entirely different from that of Madhekar and Thakare. It is also worthy of note that Madhekar and Thakare's analogue of $X_n^{(\alpha,\beta)}(x;k)$ which they denote by $K_n(\alpha,\beta,k,x)$ is also representable in the alternative form

$$(11) \quad K_n(\alpha,\beta,k,x) = \frac{1}{n!} \sum_{r=0}^{n} \left(\frac{1-x}{2}\right)^r$$

$$\cdot \sum_{s=0}^{r} (-1)^r \binom{n-s}{r-s} \binom{\beta+n}{s} \Delta^s f(\alpha) ,$$

where

$$f(\alpha) = \left(\frac{1+\alpha}{k}\right)_n$$

and

$$\Delta f(\alpha) = f(\alpha+1) - f(\alpha).$$

When $\beta = 0$, the coefficient of $\left(\frac{1-x}{2}\right)^r$ in (11) can be successively simplified as below :

$$\frac{1}{n!} \sum_{s=0}^{r} (-1)^r \binom{n-s}{r-s} \binom{n}{s} \Delta^s f(\alpha)$$

$$= \frac{(-1)^r}{n!} \binom{n}{r} \sum_{s=0}^{r} \binom{r}{s} \Delta^s f(\alpha)$$

$$= \frac{(-1)^r}{n!} \binom{n}{r} f(\alpha + r)$$

$$= \frac{(-1)^r}{n!} \binom{n}{r} \left(\frac{1+\alpha+r}{k}\right)_n ,$$

which makes $K_n(\alpha,0,k,x)$ a constant multiple of $U_n^{\alpha}(x;k)$ (Prabhakar and Kashyap[1]); but Madhekar and Thakare fail to take note of the above fact which leads them to make the erroneous remark regarding the polynomials studied by Prabhakar and Kashyap.

We conclude this Chapter with the remark that as basic polynomials may be chosen differently, it is possible to have more than one biorthogonal pair associated with a given weight function, for example, in the case of Jacobi weight function, basic polynomials could also be chosen as $r(x) = \frac{1+x}{2}$; $s(x) = \left(\frac{1+x}{2}\right)^k$ for which similar analysis can be carried out.