CHAPTER I

INTRODUCTION

1. The present thesis is the outcome of the researches done by the author pertaining to the problem of absolute convergence of a Fourier series. A trigonometric series

$$(1.1) \qquad \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

is said to be a Fourier series if there exists a 2π-periodic and Lebesgue- integrable function f such that the coefficients a_n and b_n of the series (1.1) are related to the function f by the equations:

$$(1.2) \qquad a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt \, dt, \quad n = 0, 1, 2, \ldots$$

$$(1.3) \qquad b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt \, dt, \quad n = 1, 2, 3, \ldots$$

The equations (1.2) and (1.3) are some times called Euler-Fourier formulae. The fact that the series (1.1) is the Fourier series of f is represented symbolically as follows:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

2. It was S. Bernstein[1] who for the first time studied the problem of absolute convergence of a Fourier series in the year 1914. In order to state the result of Bernstein and its generalisations given by other mathematicians, it will be convenient to introduce some definitions and notations.

Let $\omega(\delta) = \omega(\delta, f) = \sup_{|x_2 - x_1| \leq \delta} |f(x_1) - f(x_2)|$,

for $x_1, x_2 \in [0, 2\pi)$.

The function $\omega(\delta)$ is called the modulus of continuity of the function f. We say that f satisfies a Lipschitz condition of order α, $0 < \alpha \leq 1$, in $(0, 2\pi)$, and write $f \in \mathrm{Lip}\,\alpha\,(0, 2\pi)$ if there exists a constant c, independent of δ and depending only on f, such that

$$\omega(\delta) \leq c\,\delta^{\alpha}.$$

A positive function $L(u)$, defined for $u > u_0$ is said to be a slowly increasing function if $L(u)\,u^{\delta}$ is an increasing function for every $\delta > 0$ and $L(u)\,u^{-\delta}$ is decreasing function for sufficiently large u. Bernstein's theorem, referred to above, is as follows:

---

1) Bernstein [1]

<u>Theorem I.1</u> (Bernstein): <u>If</u> $f \in \mathrm{Lip}\,\alpha(0,2\pi)$ <u>for</u> $\alpha > \frac{1}{2}$,

<u>then the series</u>

(1.4) $$\sum_{n=1}^{\infty}(|a_n| + |b_n|)$$

<u>is convergent.</u>

For $\alpha = \frac{1}{2}$, the series (1.4) may not converge as can be seen by considering the real and imaginary components of the series

$$\sum_{n=2}^{\infty} \frac{e^{in\log n}}{n} e^{inx}$$

which belong[1] to Lip $\frac{1}{2}$.

It is evident that the convergence of the series (1.4) implies the absolute convergence of the series (1.1). It may be remarked that the series (1.1) may converge absolutely at a point or even in an infinite set of points without the series (1.4) being convergent. An example is furnished by the series

$$\sum_{n=1}^{\infty} \sin n!x$$

which converges absolutely when x is a multiple of π. But $\sum_{n=1}^{\infty}(|a_n| + |b_n|)$ is divergent.

However, the absolute convergence of the series (1.1) in a set of positive measure implies the convergence of the series (1.4). This result is due to Denjoy[2] and Lusin[3].

---

1) Zygmund [48 p. 243] 2) Denjoy [7] 3) Lusin [22]

In the present thesis we shall mostly be concerned with the problem of investigating the conditions under which the series (1.4) converges.

The fact that the Fourier series of a function $f \in \text{Lip}\,\frac{1}{2}$ may not be absolutely convergent naturally led mathematicians to investigate additional conditions to be imposed on the function f in order to ensure the absolute convergence of its Fourier series. We give below some of the important results obtained by different authors in this regard.

In the year 1944 L. Neder[1)] proved the following theorem:

<u>Theorem 1.2</u> (Neder). <u>If there exists a constant</u> c <u>independent of</u> δ <u>and depending only on</u> f <u>such that</u>

$$\omega(\delta) \leq \frac{c\,\delta^{\frac{1}{2}}}{l_1(\delta)\,l_2(\delta)\cdots l_k^{1+\varepsilon}(\delta)},$$

<u>where</u> $\varepsilon > 0$ <u>and</u>

$$l_1(\delta) = \log(e + \delta^{-1}),$$

$$l_2(\delta) = \log\log(e^e + \delta^{-1}), \text{ etc.,}$$

<u>then the series</u> (1.4) <u>converges and consequently the Fourier series of</u> f <u>converges absolutely everywhere.</u>

---

1) Neder [32]

M. Tomic[1], who has studied the absolute convergence of the Fourier series of a function f under different conditions, proved the following theorem in the year 1960:

<u>Theorem I.3</u> (Tomic). Let $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

<u>If</u> (i) $a_n \geq - \frac{\omega(\frac{1}{n})\log n}{n}$, $n \geq 1$,

(ii) $\omega(1/x)$ <u>increases slowly</u>,

<u>then the convergence of the series</u>

$$\sum_{n=1}^{\infty} \frac{\omega(1/n)\log n}{n}$$

<u>implies the convergence of the series</u>

$$\sum_{n=1}^{\infty} | a_n |$$

M. Tomic also proved a similar theorem for sine series.

<u>Theorem I.4</u> (Tomic). Let $f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx$.

<u>If</u> (i) $b_n \geq - \frac{\omega(1/n)}{n}$, $n \geq 1$,

(ii) $\omega(1/x)$ <u>increases slowly</u>,

(iii) $\sum_{n=1}^{\infty} \omega(1/n)n^{-1} < \infty$,

<u>then</u> $\sum_{n=1}^{\infty} |b_n| < \infty$.

---

1) Tomic [38]

This theorem was further generalized in 1963 by Tomic[1] himself in the following form:

<u>Theorem I.5</u> (Tomic). <u>If the Fourier coefficients</u> a_n <u>and</u> b_n <u>of</u> f <u>are positive, then the convergence of the series</u>

$$\sum_{n=1}^{\infty} \frac{\omega(1/n)}{n}$$

<u>implies the convergence of the series</u> (1.4).

Given a Fourier series (1.1) it is instructive to consider the series

$$(1.6) \qquad \sum_{n=1}^{\infty} \left(|a_n|^{\beta} + |b_n|^{\beta}\right)$$

and to investigate the values of the exponent β which will make the series (1.6) convergent. O. Szász[2] proved in the year 1922 the following theorem:

<u>Theorem I.6</u> (Szász). <u>If</u> $f \in \operatorname{Lip} \alpha(0, 2\pi)$, $0 < \alpha \leq 1$, <u>then the series</u> (1.6) <u>converges for all</u> $\beta > \dfrac{2}{2\alpha + 1}$.

Theorem I.6 is not, in general, true when $\beta = \dfrac{2}{2\alpha + 1}$ as can be seen[3] by considering the series

$$\sum_{n=2}^{\infty} \frac{e^{in \log n}}{n^{\alpha + (1/2)}} e^{inx}, \quad 0 < \alpha < 1.$$

---

1) Tomic [39] 2) Szász [34] 3) Zygmund [49p. 243]

In the year 1945 A. C. Zaanen[1] established the following theorem:

<u>Theorem I.7</u> (Zaanen). <u>If for certain</u> $\varepsilon > 0$,

$$\omega(\delta) \leq \frac{c\,\delta^{\alpha}}{\left[l_1(\delta) l_2(\delta) \ldots l_k^{1+\varepsilon}(\delta)\right]^{\frac{2\alpha+1}{2}}},$$

<u>then</u> (1.5) <u>converges for</u> $\beta = \frac{2}{2\alpha+1}$.

It may be remarked that Bernstein's theorem I.1 becomes a special case of Szàsz's theorem if in the latter we take $\alpha > 1/2$. For, then $2(2\alpha+1)^{-1} < 1$ so that β could be taken to be greater than or equal to 1. It is well known that the Fourier series of a function of bounded variation and even that of an absolutely continuous function, may not converge absolutely. Indeed, the trigonometric series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{n \log n}$$

which does not converge absolutely, is the Fourier series of an absolutely continuous function[2].

In view of this situation, the following theorem proved by A. Zygmund[3] in the year 1928, which is closely related to Bernstein's theorem, should be considered quite important.

---

1) Zaanen [44] 2) Zygmund [48 p. 241] 3) Zygmund [46]

Theorem I.8 (Zygmund). If f is of bounded variation and belongs to Lip α $(0, 2\pi)$, $\alpha > 0$, then the Fourier series of f converges absolutely.

It may be noted that the above mentioned theorem of Zygmund holds true even if in its hypothesis the condition that $f \in$ Lip α for $\alpha > 0$ is replaced by the following weaker condition :

$$(1.6) \qquad \omega(\delta) = O(\log \delta^{-1})^{-2-\eta}, \qquad \eta > 0 .$$

In a recent paper Ching-tsun Loo[1] showed that the Fourier series of a function may not be absolutely convergent when $\eta = 0$ in the condition (1.6).

Zygmund's theorem was generalized by Z. Waraszkiewicz[2] and A. Zygmund[3] in the year 1929 as follows:

Theorem I.9 (Waraszkiewicz and Zygmund). If f is of bounded variation and belongs to Lip $\alpha(0, 2\pi)$, $\alpha > 0$, then the series (1.5) converges for all $\beta > 2(\alpha+2)^{-1}$, but not necessarily for $\beta = 2(\alpha+2)^{-1}$.

---

1) Loo [21] 2) Waraszkiewicz [42] 3) Zygmund [46]

In Chapter II of the present thesis we have investigated the absolute convergence of the series (1.5) and have established the following result[1]:

<u>Theorem I.10.</u> *If f is of bounded variation and satisfies the condition*

$$\omega(\delta) \leq \frac{c\,\delta^{\alpha}}{\left[l_1(\delta)\, l_2(\delta) \ldots l_k^{1+\varepsilon}(\delta)\right]^{\alpha+2}}, \quad \varepsilon > 0,$$

then the series (1.5) converges for $\beta = 2(\alpha+2)^{-1}$.

We have also generalised the Theorem I.5 of M. Tomic in Chapter II and have proved the following theorem[2]:

<u>Theorem I.11.</u> *Let $f(x)$ be continuous and periodic in $(0, 2\pi)$. If*

(i) $f(x) \in \mathrm{Lip}\,\alpha(0, 2\pi)$, $0 < \alpha \leq 1$,

(ii) a_n *and* b_n *are positive,*

then the series (1.5) converges for $\beta > (1+\alpha)^{-1}$.

<u>Theorem I.12.</u> *Let $g(x)$ be a periodic and continuous function in $(0, 2\pi)$ and*

$$g(x) \sim \frac{a_0'}{2} + \sum_{n=1}^{\infty} (a_n' \cos nx + b_n' \sin nx),$$

and denote by $\omega_1(\delta)$ the modulus of continuity of g.

---

1) Yadav and Goyal [9] 2) Goyal [12]

If

(i) $\omega_1(\delta) \leq \omega(\delta)$, $a_n' \geq -a_n$; $b_n' \geq -b_n$,

(ii) $f(x)$ satisfies the condition of Theorem I.11,

then

$$\sum_{n=1}^{\infty} (|a_n'|^{\beta} + |b_n'|^{\beta})$$

converges for $\beta > (1+\alpha)^{-1}$.

As we have noted earlier, the series (1.4), where a_n and b_n are Fourier coefficients of the function $f \in Lip\ \alpha$ may diverge if $\alpha \leq 1/2$. However, in accordance with Zygmund's Theorem I.6, the series (1.4) will converge if f is of bounded variation and $f \in Lip\ \alpha$, $\alpha > 0$. A natural question that arises in this connection is as to whether the series (1.4) corresponding to a function f of $Lip\ \alpha$, $\alpha \leq 1/2$ can be made to converge by introducing some convergent factors. A number of theorems have been proved in this regard by different authors and we shall mention those which are connected with our work.

G. H. Hardy[1] proved in the year 1916 the following theorems:

Theorem I.13 (Hardy). If $f \in Lip\ \alpha\ (0, 2\pi)$, $0 < \alpha \leq 1$, then

(1.7) $$\sum_{n=1}^{\infty} n^{\beta-\frac{1}{2}} (|a_n| + |b_n|) < \infty \text{ , for every } \beta < \alpha .$$

Theorem I.14 (Hardy). If $f \in Lip\ \alpha\ (0, 2\pi)$, and

1) Hardy [16]

f is of bounded variation, then

$$\sum_{n=1}^{\infty} n^{\beta/2}(|a_n| + |b_n|) < \infty, \text{ for } \beta < \alpha . \tag{1.8}$$

In the year 1945, A. C. Zaanen[1)] extended Theorem I.13 as follows:

<u>Theorem I.15</u> (Zaanen). *If for certain $\quad > 0$,*

$$\omega(\delta) \leq \frac{c\,\delta^{\alpha}}{l_1(\delta)\, l_2(\delta) \ldots l_k^{1+\varepsilon}(\delta)} ,$$

then (1.7) holds for $\beta = \alpha$.

In Chapter II we prove a theorem which is an extension of Theorem I.14. This extension is analogous to the extension of Theorem I.13 given by Zaanen. Our theorem is:

<u>Theorem I.16.</u> *If f is of bounded variation and satisfies the condition*

$$\omega(\delta) \leq \frac{c\delta^{\alpha}}{\left[l_1(\delta)\, l_2(\delta) \ldots l_k^{1+\varepsilon}(\delta)\right]^2} ,$$

then the conclusion (1.8) holds for $\beta = \alpha$.

We have also considered the conditions under which the conclusion (1.8) holds for values of $\beta > \alpha$. Here

---

1) Zaanen [44]

precisely we prove the following theorems[1] in Chapter II:

<u>Theorem I.17.</u> <u>Let</u> $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

<u>If</u> (i) $f(x) \in Lip\,\alpha$, $0 < \alpha \leq 1$,

(ii) $a_n \geq \frac{-\omega(1/n)\log n}{n}$, $n \geq 1$,

(iii) $\omega(1/x)$ <u>increases slowly, then</u>

$$\sum_{n=1}^{\infty} n^{\beta/2}(|a_n|) < \infty, \text{ for } \beta < 2\alpha .$$

An analogous result has been proved for sine series.

<u>Theorem I.18.</u> <u>Let</u> $f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx$.

<u>If</u> (i) $f(x) \in Lip\,\alpha$, $0 < \alpha \leq 1$,

(ii) $b_n \geq \frac{-\omega(1/n)}{n}$, $n \geq 1$,

(iii) $\omega(1/x)$ <u>is slowly increasing, then</u>

$$\sum_{n=1}^{\infty} n^{\beta-(1/2)}(|b_n|) < \infty , \text{ for } \beta < \alpha + \frac{1}{2} .$$

We have also proved in Chapter II the following theorem[2]:

Theorem I.19. <u>If</u> $f \in Lip\,\alpha$, $0 < \alpha \leq 1$,
<u>and</u> a_n , b_n <u>are positive, then the conclusion</u> (1.8) <u>holds for</u> $\beta < 2\alpha$.

---

1) Goyal [10] 2) Goyal [11]

It may be remarked that

$$\sum_{n=1}^{\infty} n^{\beta/2}(|a_n'| + |b_n'|) < \infty \text{ , for } \beta < 2\alpha,$$

if $f(x)$ satisfies the conditions of Theorem I.19 and $g(x)$ is a periodic and continuous function in $(0, 2\pi)$ with Fourier coefficients a_n', b_n' and modulus of continuity ω_1 such that

$$\omega_1(\delta) \leq \omega(\delta) \text{ , } a_n' \geq -a_n \text{ ,}$$

$$b_n' \geq -b_n \text{ .}$$

4. In Chapter II we also show that the condition of bounded variation in our Theorem I.10 and Theorem I.16 can be replaced by weaker condition of bounded r^{th} variation. A function $f(x)$ defined in an interval (a,b) is said to be of bounded r^{th} variation if for arbitrary numbers

$x_0, x_1, \ldots\ldots, x_n$, in arithmetic progression,

satisfying $a = x_0 < x_1 < \ldots\ldots < x_n = b$ and for every integral value of n,

$$\sum_{i=0}^{n-r} | \Delta_r f(x_i)| \leq M,$$

where

$$\Delta f(x_i) = f(x_{i+1}) - f(x_i) \text{ ,}$$

$$\ldots\ldots = \ldots\ldots\ldots\ldots$$

$$\Delta_q f(x_i) = \Delta_{q-1} f(x_{i+1}) - \Delta_{q-1} f(x_i)$$

and M is a constant.

Also we define $\omega_r(\delta, f)$, the generalised modulus of continuity of order r of the function f, as

$$\omega_r(\delta \text{ , } f) = \sup_{|h| \leq \delta} | \Delta_r f(x \text{ , } h)| \text{ ,}$$

where $\Delta_r f(x, h) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f(x+(r-2i)h)$, $r=1,2,3,\ldots$

Now we are in a position to state our theorems.

<u>Theorem I.20.</u> <u>If</u> (i) $f(x)$ <u>is of bounded</u> r^{th} <u>variation in</u> $(0, 2\pi)$ <u>and</u>

(ii) $$\omega_r(\delta) \leq \frac{c\,\delta^{\alpha}}{\left[l_1(\delta) l_2(\delta) \ldots l_k^{1+\epsilon}(\delta)\right]^{\alpha+2}}$$

<u>for</u> $\epsilon > 0$, <u>then the series</u> (1.5) <u>is convergent for</u> $\beta = 2(\alpha+2)^{-1}$.

<u>Theorem I.21.</u> <u>If</u> (i) $f(x)$ <u>is of bounded</u> r^{th} <u>variation in</u> $(0, 2\pi)$ <u>and</u>

(ii) $$\omega_r(\delta) \leq \frac{c\,\delta^{\alpha}}{\left[l_1(\delta) l_2(\delta) \ldots l_k^{1+\epsilon}(\delta)\right]^{2}},$$

<u>then the series</u> (1.8) <u>converges for</u> $\beta = \alpha$.

It is a well known fact that every function, which is of bounded variation, is also of bounded r^{th} variation, but the converse is not always true. In view of this remark we observe that Theorem I.20 and Theorem I.21 are more general than Theorem 1 and Theorem 4 respectively.

We also remark that for the convergence of the series (1.5) it is sufficient that f is of bounded r^{th} variation and

$$\int_0^1 h^{\beta-2} \omega_r^{\beta/2}(h)\, dh = O(1) .$$

For the convergence of series (1.8) it is sufficient that

f is of bounded r^{th} variation and

$$\int_0^1 h^{-1-(\beta/2)} \; \omega_n^{1/2}(h)\, dh = O(1).$$

5. A new class of functions, called Lip (α, p), $\alpha > 0$, $1 \leq p \leq 2$, which is more general than the class Lip α, was defined by Hardy and Littlewood[1] in the year 1928. A function f is said to belong to the class Lip(α, p), $0 < \alpha \leq 1$, $1 \leq p$, if

$$\left\{ \int_0^{2\pi} |f(x+h)-f(x)|^p \right\}^{1/p} = O(|h|^{\alpha}).$$

We remark that the conclusions of Bernstein's theorem I.1 and Szász's theorem I.6, are true if in the hypothesis of these theorems, the condition that $f \in$ Lip α is replaced by less stringent condition that $f \in$ Lip $(\alpha, 2)$.

In the year 1942 Min-Teh Cheng[2] proved the following theorem :

<u>Theorem I. 22</u> (Cheng). <u>If</u> $0 < \alpha \leq 1$, $1 < p \leq 2$, $h > 0$ <u>and</u>

$$\int_0^{2\pi} |f(x+h)-f(x)|^p dx = O\left\{ h(\log h^{-1})^{-1-\alpha p} \right\}$$

<u>then the series</u>

$$(1.9) \qquad \sum_{n=2}^{\infty} (|a_n| + |b_n|) \log^T n$$

converges for $T < \alpha + p^{-1} - 1$. Moreover the series (1.9) may not converge for $T = \alpha + p^{-1} - 1$.

---

1) Hardy and Littlewood [13] 2) Cheng [6]

In the year 1962 B. S. Yadav 1) gave a shorter and more direct proof of the theorem of Min-Teh Cheng and he also proved its generalization. In Chapter III we have also studied this kind of problem and have obtained more general results than that of Yadav. Our theorems are:

<u>Theorem I.23.</u> <u>If</u> $0 < \alpha \leq 1$, $1 < p \leq 2$, $h > 0$, $\varepsilon > 0$ <u>and</u>

$$\omega_p^{(m)}(f,h) = O\left\{h^{1/p}(\log h^{-1})^{-(p^{-1}+\alpha)}(\log\log h^{-1})^{-(1+\varepsilon)}\right\},$$

<u>where</u>

$$\omega_p^{(m)}(\delta) = \sup_{0 \leq t \leq \delta}\left\{(1/2\pi)\int_{-\pi}^{\pi}|\Delta_m f(x,t)|^p\right\}^{p^{-1}}$$

<u>then the series</u> (1.9) <u>converges for</u>

$T = \alpha + p^{-1} - 1.$

<u>Theorem I.24.</u> <u>If</u> $0 < \alpha \leq 1$, $1 < p \leq 2$, $h > 0$, $\varepsilon > 0$ <u>and</u>

$$\omega_p^{(m)}(f,h) = O\left\{h^{\delta/p}(\log h^{-1})^{-(\beta^{-1}+\alpha)}(\log\log h^{-1})^{-\left(\frac{1+\varepsilon}{\beta}\right)}\right\},$$

<u>where</u> $\delta = 1 + \dfrac{p(1-\beta)}{\beta}$,

<u>then the series</u>

$$\sum_{n=2}^{\infty}(|a_n|^{\beta} + |b_n|^{\beta})\log^T n$$

<u>converges for</u> $\beta = p(T+1)(1+\alpha p)^{-1}$.

More general results than that of Min-Teh Cheng have been established by different authors. In particular O. Szász 2)

---

1) Yadav [43] 2) Szász [35]

proved the following generalization of Theorem I.13 in the year 1928:

Theorem I. 25 (Szasz). If $f \in Lip(\alpha, p)$, $0 < \alpha < 1$, $1 \leq p \leq 2$, then the series (1.7) converges for $\beta < \alpha + (1/2)-(1/p)$, but not necessarily for $\beta = \alpha + (1/2)-(1/p)$.

In Chapter III, we prove a theorem which is an extension of Theorem I.25. Our theorem is:

Theorem I.26. Let $f(x) \in L^p$, $1 \leq p \leq 2$. If

$$\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}[\Delta_h(f,t)]^p\right\}^{p^{-1}} = O\left\{\frac{h^\alpha}{l_1(h)\ldots l_k^{1+\varepsilon}(h)}\right\},$$

then (1.7) holds for $\beta=\alpha+(1/2)-(1/p)$.

We have also given an extension of Theorem I.26.

6. In the year 1926 Hardy and Littlewood[1) considered the summability (c,1) of the series

(1.10)
$$\sum_{n=1}^{\infty}\left(\frac{S_n-S}{n}\right)$$

where S_n is the n^{th} partial sum at a point x of the Fourier series

$$\sum_{n=1}^{\infty} a_n \cos nx = \sum_{n=1}^{\infty} A_n$$

---

1) Hardy and Littlewood [17]

corresponding to a function f and s is a suitably chosen number independent of n. A. Zygmund[1] has given a necessary and sufficient condition for the convergence of the series (1.10). In the year 1956, R. Mohanty and S. Mohopatra[2] proved the following theorem:

For a given function f we write

$$\phi(t) = \frac{f(x+t)+f(x-t)-2s}{2},$$

$$\phi_1(t) = \frac{1}{t} \int_0^t \phi(u)du.$$

Theorem I. 27 (Mohanty and Mohopatra). If

(i) $\phi_1(t)\log(k/t)$ is of bounded variation in $(0,2\pi)$,

(ii) $\left|\frac{\phi_1(t)}{t}\right|$ is integrable in $(0,2\pi)$,

(iii) $(n^{\delta} A_n)$ is of bounded variation for δ, $0<\delta<1$,

then the series (1.10) is absolutely convergent.

In Chapter IV we prove the following theorems[3] by taking $s = 0$.

Theorem I.28. If

(i) $$\int_0^{\pi} \left|\frac{\phi(t)}{\sin(t/2)}\right| \log(2\pi/t)dt < \infty,$$

(ii) $(n^{\delta} \Delta A_n)$ is of bounded variation for some δ, $0 < \delta < 1$,

then the series (1.10) is absolutely convergent.

---

1) Zygmund [49 p. 61] 2) Mohanty and Mohopatra [28] 3) Goyal [13], [11]

Theorem I.29. If

(i) $\int_0^\pi \left|\frac{\phi(t)}{\sin(t/2)}\right| \log (2\pi/t)\, dt < \infty$,

(ii) $a_n = O(n^{-\alpha})$, $\alpha > \delta > 0$,

then the series (1.10) converges absolutely.

7. In Chapter V, we have studied the absolute convergence of a lacunary Fourier series.

Let the Fourier coefficients a_n, b_n of $f(x)$, vanish except for a strictly increasing sequence $\{n_k\}$ of positive integers. Noble[1] proved the following theorems on the absolute convergence of a Fourier series satisfying a certain gap condition :

Theorem I.30 (Noble). If

(i) $\lim_{k \to \infty} \frac{N_k}{\log n_k} = \infty$, where $N_k = \min \begin{Bmatrix} (n_{k+1} - n_k), \\ (n_k - n_{k+1}) \end{Bmatrix}$

(ii) $f(x)$ is of bounded variation in some interval $|x - x_0| \le \delta$,

(iii) $f(x) \in \mathrm{Lip}\,\alpha$, $0 < \alpha < 1$ in $|x - x_0| \le \delta$,

then the series (1.4) is convergent.

Theorem I.31 (Noble). If (i) $\lim_{k \to \infty} \frac{N_k}{\log n_k} = \infty$,

---

1) Noble [31]

(ii) <u>f(x) is of bounded variation in some interval</u> $|x - x_0| \leq \delta$,

(iii) $f(x) \in \mathrm{Lip}\,\alpha$, $0 < \alpha < 1$ in $|x - x_0| \leq \delta$,

<u>then the series (1.5) converges for</u> $\beta > \frac{2}{\alpha+2}$.

More powerful methods were employed by Paley and Wiener[1] for such kind of problems. P. B. Kennedy[2] used these methods to give a simple proof of Noble's Theorem I.30 under less restricted gap hypothesis. He proved the following theorem :

<u>Theorem I.32</u> (Kennedy). <u>If</u> (i) $n_{k+1} - n_k \to \infty$ <u>as</u> $k \to \infty$,

(ii) <u>f(x) is of bounded variation in some interval</u> $|x - x_0| \leq \delta$,

(iii) $f(x) \in \mathrm{Lip}\,\alpha$, $0 < \alpha < 1$ <u>in</u> $|x - x_0| \leq \delta$,

<u>then the series</u> (1.4) <u>is convergent.</u>

In Chapter V we prove the following theorems which are more general than Kennedy's theorem I.32:

<u>Theorem I.33.</u> <u>If</u> (i) $n_{k+1} - n_k \to \infty$ <u>as</u> $k \to \infty$,

(ii) <u>f(x) is of bounded second variation in some interval</u> $|x - x_0| \leq \delta$ <u>and</u>

(iii) $$\omega(h) \leq \frac{c}{\left[l_1(h)\,l_2(h)\ldots l_k^{1+\varepsilon}(h)\right]^2}, \quad \varepsilon > 0,$$

in $|x - x_0| \leq \delta$, <u>then the series</u> (1.4) <u>is convergent.</u>

---

1) Paley and Wiener [33] 2) Kennedy [23]

<u>Theorem I.34.</u> <u>If</u> (i) $n_{k+1}-n_k \to \infty$ <u>as</u> $k \to \infty$,

(ii) $f(x)$ <u>is of bounded second variation in some interval</u> $|x - x_0| \leq \delta$,

(iii) $$\omega(h) \leq \frac{c\, h^{\alpha}}{\left[l_1(h)\, l_2(h) \ldots l_k^{1+\varepsilon}(h)\right]^{\alpha+2}} , \quad \varepsilon > 0 ,$$

in $|x - x_0| \leq \delta$, <u>then the series</u> (1.5) <u>converges</u> for $\beta = 2/(\alpha+2)$.

Noble also has given the following extension of Theorem I.30:

<u>Theorem I.35</u> (Noble). <u>If</u> (i) $\lim_{k \to \infty} \frac{n_k}{\log n_k} = \infty$,

(ii) $f(x)$ <u>is of bounded variation in some interval</u> $|x - x_0| \leq \delta$,

(iii) $f(x) \in \mathrm{Lip}\, \alpha$, $0 < \alpha < 1$ <u>in</u> $|x - x_0| \leq \delta$, <u>then the series</u> (1.8) <u>converges for</u> $\beta < \alpha$.

We have also given an analogous extension of Theorem I.33 as follows:

<u>Theorem I.36.</u> <u>If</u> (i) $n_{k+1}- n_k \to \infty$ <u>as</u> $k \to \infty$,

(ii) $f(x)$ <u>is of bounded second variation in some interval</u> $|x - x_0| \leq \delta$,

(iii) $$\omega(h) \leq \frac{c\, h^{\alpha}}{\left[l_1(h) l_2(h) \ldots l_k^{1+\varepsilon}(h)\right]^2} , \quad \varepsilon > 0 ,$$

in $|x - x_0| \leq \delta$, then the series (1.8) converges for $\beta = \alpha$.

In Chapter V we also treat the Fourier series with a certain gap and satisfying some continuity condition at a point only instead of in some small interval. In the year 1955 Masako Sato[1] proved the following two theorems:

Theorem I. 37 (Masako Sato). Let $(1/2) < a < \alpha < 1$,

$$0 < \beta < \frac{2 - \alpha}{3}, (\beta/2) < \alpha - a \leq (2-\alpha-\beta)/4 ,$$

$$k^{(2\alpha-2a-\beta)^{-1}} < n_k < e^{2k(2+\alpha+\beta)^{-1}} ,$$

$$|n_{k+1} - n_k| > 4e\, K\, n_k^{\beta} .$$

If (i) $\frac{1}{h^{\beta}} \int_0^{h^{\beta}} |f(t) - f(t \pm h)|^2 \, dt = O(h^{2\alpha})$ as $h \to 0$,

(ii) $\frac{1}{\tau} \int_0^{\tau} |f(t) - f(t \pm h)|^2 \, dt = O(1)$ uniformly in $\tau > h^{\beta}$, then the series (1.4) is convergent.

Theorem I.38 (Sato). Let $(1/2) < a < \alpha < 1$,

$$0 < \beta < \frac{1 - \alpha}{2}, \quad \beta/2 < \alpha - a < \frac{1 + \beta}{4} ,$$

$$n_k = [k^{\gamma}], \ (k=1,2,3,\dots), \ \gamma > (2\alpha-2a-\beta)^{-1} .$$

---

1) Masako Sato [30]

If the conditions (i) and (ii) of Theorem I.37 are satisfied, then the series (1.4) is convergent.

We have extended the above mentioned theorems of Sato as follows[1]:

Theorem I.39. Let $\frac{1}{2} < \alpha \leq 1$.

Under the hypothesis of the theorem I.37 of Masako Sato, the series (1.5) converges for $\beta > 2/(2\alpha+1)$.

Theorem I.40. Under the hypothesis of Theorem I.39, the series (1.7) converges for $\beta < \alpha$.

We have also given extensions of Theorem I.39 and I.40 which are analogous to the extension given by Sato of Theorem I.37.

8. A function f is said to be a contraction of another function g if

$$|f(x_2) - f(x_1)| \leq |g(x_2) - g(x_1)|,$$

for any two points x_1, x_2 of their common domain.

In the year 1949 A. Beurling[2] established a test for the absolute convergence of a Fourier series. His theorem is:

---

1) Goyal [15] 2) Beurling [5]

Theorem I. 41 (Beurling). *Let f and g be continuous even functions of period 2π, with Fourier cosine coefficients a_n and c_n and let f be a contraction of g. If*

(i) $|c_n| \le \gamma_n$,

(ii) $\gamma_n \downarrow 0$

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$,

then $\sum_{n=1}^{\infty} |a_n| < \infty$.

Beurling's theorem was subsequently generalized by R.P. Boas Jr.[1] in the year 1960, who proved the following theorem:

Theorem I. 42 (Boas). *Let f be a contraction of g.*

If (i) $|c_n| \le \gamma_n$,

(ii) $\sum_{n=1}^{\infty} n^{-\frac{3}{2}} \left[\sum_{k=1}^{n} k^2 \gamma_k^2\right]^{1/2} + \sum_{n=1}^{\infty} n^{-1/2} \left[\sum_{k=n+1}^{\infty} \gamma_k^2\right]^{1/2} < \infty$,

then $\sum_{n=1}^{\infty} |a_n| < \infty$,

where a_n *and* c_n *are the Fourier cosine coefficients of* f *and* g *respectively.*

We have generalized the theorem of Boas in Chapter VI. Our theorem is as follows:

Theorem I.43. *Let f and g be continuous even functions,*

---

1) R. P. Boas Jr. [4]

each of period 2π with Fourier cosine coefficients a_n and c_n respectively. Let f be the contraction of the function g. If $0 < \beta \leq 1$, $0 < \gamma < \beta/2$ and

$$\sum_{n=1}^{\infty} n^{\gamma-\frac{3}{2}\beta} \left[\sum_{k=1}^{n} k^2 c_k^2\right]^{\beta/2} + \sum_{n=1}^{\infty} n^{\gamma-(\beta/2)} \left[\sum_{k=n+1}^{\infty} c_k^2\right]^{\beta/2} < \infty,$$

then

$$\sum_{n=1}^{\infty} n^{\gamma}|a_n|^{\beta} < \infty .$$

At the end of the thesis a bibliography is given. Part of the research work incorporated in this thesis has been published by the author in the form of eight research papers in Mathematical Journals. Reprints of these papers are contained in the appendix to the thesis.