CHAPTER IV

ON THE ABSOLUTE CONVERGENCE OF A SERIES ASSOCIATED WITH A FOURIER SERIES

Let f be L- integrable in $(0,2\pi)$, periodic with period 2π and let its Fourier series be

$$f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx = \sum_{n=1}^{\infty} A_n .$$

Let $$S_n = \sum_{k=1}^{n} A_k.$$

In this chapter we shall study the absolute convergence of the series

(1) $$\sum_{n=1}^{\infty} \frac{S_n - S}{n} \text{, at a point } x,$$

where S is a suitably choosen number independent of n. We write

$$\phi(t) = \frac{f(x+t)+f(x-t)-2S}{2}$$

and

$$\phi_1(t) = (1/t)\int_0^t \phi(u)\,du .$$

Hardy and Littlewood[1)] were the first to consider the question of summability (c,1) of the series (1).

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1) Hardy and Littlewood [17]

A. Zygmund[1] obtained a necessary and sufficient condition for the convergence of this series.

R. Mohanti and G. Mohapatra[2] investigated conditions under which the series (1) is either strongly summable or is absolutely convergent. In fact, Mohanti and Mohapatra proved the following result regarding the absolute convergence of the series (1):

<u>Theorem E.</u> <u>If</u>

(i) $\phi_1(t) \log (K/t)$ <u>is of bounded variation in</u> $(0, 2\pi)$

(ii) $\left|\frac{\phi(t)}{t}\right|$ <u>is integrable in</u> $(0, 2\pi)$

(iii) $(n^{\delta} A_n)$ <u>is of bounded variation for</u> $\delta, 0 < \delta < 1$,

<u>then the series</u> (1) <u>is absolutely convergent.</u>

Taking $S = 0$, we have examined the question of absolute convergence of the series (1) under conditions which are different from those of Theorem E. We first prove the following theorem:

<u>Theorem 17.</u> <u>If</u>

(i) $$\int_0^{\pi} \left| \frac{\phi(t)}{\sin(t/2)} \right| \log (2\pi/t)\, dt < \infty,$$

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1) Zygmund [49, p.61] 2) Mohanti and Mohapatra [28]

(ii) $n^{\delta}\Delta(A_n)$ *is of bounded variation for some* $\delta > 0$, $\delta < 1$,

where $\Delta(A_n) = A_n - A_{n-1}$,

then the series (1) *converges absolutely.*

An analogous result for sine series is :

Theorem 18. Let $f(t) \sim \sum_{n=1}^{\infty} b_n \sin nt = \sum_{n=1}^{\infty} B_n$.

If (i) $\int_0^{\pi} \left|\frac{\phi(t)}{\sin(t/2)}\right| \log (2\pi/t)\, dt < \infty$,

(ii) n^{δ} (B_n) *is of bounded variation for some* δ, $0 < \delta < 1$,

then the series

$$\sum_{n=1}^{\infty} \frac{T_n}{n}$$

is absolutely convergent, where $T_n = \sum_{k=1}^{n} B_k$.

Proof of Theorem 17. We have

$$\frac{S_n(x)}{n} = \frac{1}{2n\pi} \int_0^{\pi} \frac{\phi(t) \sin(n+\frac{1}{2})t\, dt}{\sin(t/2)}$$

$$= \frac{1}{2n\pi} \int_0^{\pi/n^{\delta'}} \frac{\phi(t) \sin(n+\frac{1}{2})t\, dt}{\sin(t/2)}$$

$$+ \frac{1}{2n\pi} \int_{\pi/n^{\delta'}}^{\pi} \frac{\phi(t) \sin(n+\frac{1}{2})t\, dt}{\sin(t/2)}$$

where $\delta' = \delta_n'$ is chosen such that $n^{1-\delta'}$ is an integer and $\delta' = (\delta/4) + O\left(\frac{1}{n^{1-(\delta/4)}\log n}\right)$ [1].

We write

$$2\pi \sum_{n=1}^{\infty} \left|\frac{S_n}{n}\right| = \sum_{n=1}^{\infty} (1/n)\left|\int_0^{\pi} \frac{\sin(n+\frac{1}{2})t\ \phi(t)\ dt}{\sin(t/2)}\right|$$

$$\leq \sum_{n=1}^{\infty} (1/n)\left|\int_0^{\pi/n^{\delta'}} \frac{\sin(n+\frac{1}{2})t\ \phi(t)dt}{\sin(t/2)}\right|$$

$$+ \sum_{n=1}^{\infty} (1/n)\left|\int_{\pi/n^{\delta'}}^{\pi} \frac{\sin(n+\frac{1}{2})t\ \phi(t)dt}{\sin(t/2)}\right|$$

$$= I + J.$$

Now,

$$I = \sum_{n=1}^{\infty} (1/n)\left|\int_0^{\pi/n^{\delta'}} \frac{\sin(n+\frac{1}{2})t\ \phi(t)dt}{\sin(t/2)}\right|$$

$$\leq \sum_{n=1}^{\infty} (1/n) \int_0^{\pi/n^{\delta'}} \left|\frac{\phi(t)}{\sin(t/2)}\right| dt$$

$$\leq \sum_{n=1}^{\infty} (1/n) \sum_{k=n^{\delta'}}^{\infty} \int_{\pi/k+1}^{\pi/k} \left|\frac{\phi(t)}{\sin(t/2)}\right| dt$$

$$\leq A \sum_{k=1}^{\infty} \int_{\pi/k+1}^{\pi/k} \left|\frac{\phi(t)}{\sin(t/2)}\right| \sum_{n=1}^{k^{1/\delta'}} \frac{1}{n}\ dt$$

$$\leq A \sum_{k=1}^{\infty} \int_{\pi/k+1}^{\pi/k} \left|\frac{\phi(t)}{\sin(t/2)}\right| \log k\ dt$$

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1) Izumi [20]

$$\leq A \sum_{k=1}^{\infty} \int_{\pi/k+1}^{\pi/k} \left| \frac{\phi(t)}{\sin(t/2)} \right| \log(2\pi/t)\, dt$$

since $\log k < \log(2\pi/t)$ for $(\pi/k+1) \leq t \leq (\pi/k)$.

Therefore

$$I \leq A \int_0^{\pi} \left| \frac{\phi(t)}{\sin(t/2)} \right| \log(2\pi/t)\, dt .$$

Thus it follows from condition (i) of the hypothesis that

$$I < \infty .$$

Since

$$\phi(t) \sim \sum_{k=1}^{\infty} a_k \cos kx \cos kt$$

We have

$$J_n = \int_{\pi/n\delta'}^{\pi} \frac{\sin(n+\frac{1}{2})t\, \phi(t)\, dt}{\sin(t/2)}$$

$$= \sum_{k=1}^{\infty} a_k \cos kx \int_{\pi/n\delta'}^{\pi} \frac{\sin(n+\frac{1}{2})t \cos kt\, dt}{\sin(t/2)}$$

$$= \sum_{k=1}^{\infty} a_k \cos kx \int_{\pi/n\delta'}^{\pi} \left\{ \frac{\sin(n+\frac{1}{2})t}{\sin(t/2)} \right\} \left\{ \frac{\sin(k-\frac{1}{2})t - \sin(k+\frac{1}{2})t}{-2\sin(t/2)} \right\}$$

$$= \sum_{k=1}^{\infty} \Delta(A_k) \int_{\pi/n\delta'}^{\pi} \frac{\sin(n+\frac{1}{2})t \sin(k+\frac{1}{2})t\, dt}{2\sin^2(t/2)}$$

(2)

$$+ a_1 \cos x \int_{\pi/n\delta'}^{\pi} \frac{\sin(n+\frac{1}{2})t\, dt}{2\sin^2(t/2)}$$

since

$$\lim_{k \to \infty} A_k \int_{\pi/n\delta}^{\pi} \frac{\sin(n+\frac{1}{2})t \sin(k+\frac{1}{2})t\, dt}{2\sin^2(t/2)} = 0 .$$

Now

$$\sum_{k=1}^{\infty} \Delta(\Lambda_k) \int_{\pi/n^\delta}^{\pi} \frac{\sin(n+\frac{1}{2})t \ \sin(k+\frac{1}{2})t \ dt}{2\sin^2(t/2)}$$

$$= \sum_{k=1}^{\infty} k^\delta \Delta(\Lambda_k) \int_{\pi/n^\delta}^{\pi} \frac{\sin(n+\frac{1}{2})t \ \sin(k+\frac{1}{2})t \ dt}{2\sin^2(t/2) \quad k^\delta}$$

$$= \sum_{k=1}^{\infty} \Delta(k^\delta \Delta(\Lambda_k)) \int_{\pi/n^\delta}^{\pi} \frac{\sin(n+\frac{1}{2})t}{2\sin^2(t/2)} \left(\sum_{j=1}^{k} \frac{\sin(j+\frac{1}{2})t}{j^\delta}\right) dt$$

$$+ \ C \sum_{j=1}^{\infty} \int_{\pi/n^\delta}^{\pi} \frac{\sin(n+\frac{1}{2})t \ \sin(j+\frac{1}{2})t \ dt}{2\sin^2(t/2) \quad j^\delta} \ , \text{ where}$$

$$\lim_{k\to\infty} k^\delta \Delta(\Lambda_k) = C$$

(3) $$= J_{n_1} + J_{n_2} \ .$$

Now

$$\sum_{n=1}^{\infty} \left| \frac{J_{n_1}}{n} \right|$$

$$\leq \sum_{n=1}^{\infty} (1/n) \sum_{k=1}^{\infty} \left| \Delta(k^\delta \Delta(\Lambda_k)) \right| \left| \int_{\pi/n^\delta}^{\pi} \frac{\sin(n+\frac{1}{2})t}{2\sin^2(t/2)} \left(\sum_{j=1}^{k} \frac{\sin(j+\frac{1}{2})t}{j^\delta}\right) dt \right|$$

$$\leq \sum_{k=1}^{\infty} \left|\Delta(k^\delta \Delta(\Lambda_k))\right| \sum_{n=1}^{\infty} (1/n) \sum_{j=1}^{k} \left| \int_{\pi/n^\delta}^{\pi} \frac{\sin(n+\frac{1}{2})t \ \sin(j+\frac{1}{2})t \ dt}{j^\delta \ 2\sin^2(t/2)} \right|$$

We can prove that

$$\sum_{n=1}^{\infty} (1/n) \sum_{j=1}^{k} \frac{1}{j^{\delta}} \left| \int_{\pi/n^{\delta'}}^{\pi} \frac{\sin(n+\frac{1}{2})t \ \sin(j+\frac{1}{2})t}{2 \sin^2(t/2)} dt \right| < \infty,$$

since the left hand side is

$$< A \sum_{j=1}^{\infty} \frac{1}{j^{\delta}} \sum_{n=1}^{\infty} \frac{1}{n} \ \frac{n^{2\delta'}}{|n-j|}$$

$$\leq A \sum_{j=1}^{\infty} \frac{1}{j^{\delta}} \left(\sum_{n=1}^{j/2} + \sum_{j/2+1}^{2j} + \sum_{2j+1}^{\infty} \right) \frac{1}{n^{1-2\delta'} |n-j|}$$

$$\leq A_1 \sum_{j=1}^{\infty} \frac{\log j}{j^{1+\delta-2\delta'}}$$

$$< \infty.$$

Thus $\displaystyle\sum_{n=1}^{\infty} \left| \frac{J_{n_1}}{n} \right| < \infty .$

Similarly we can prove that

$$\sum_{n=1}^{\infty} \left| \frac{J_{n_2}}{n} \right| < \infty .$$

Hence it follows from (3) that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| \sum_{k=1}^{\infty} \Delta(A_k) \int_{\pi/n^{\delta'}}^{\pi} \frac{\sin(n+\frac{1}{2})t \ \sin(k+\frac{1}{2})t \ dt}{2 \sin^2(t/2)} \right| < \infty .$$

As $\operatorname{cosec}^2(t/2)$ is decreasing in $(\pi/n^{\delta'}, \pi)$, we see that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| a_1 \cos x \int_{\pi/n^{\delta'}}^{\pi} \frac{\sin(n+\frac{1}{2})t \ dt}{2 \sin^2(t/2)} \right|$$

$$\leq A \sum_{n=1}^{\infty} \frac{n^{2\delta'}}{n} \left| \int_{\pi/n^{\delta'}}^{\pi} \sin(n+\tfrac{1}{2})t \ dt \right|$$

$$\leq A_1 \sum_{n=1}^{\infty} \frac{1}{n^{1-2\delta'}} \left\{ \frac{1}{(n+\frac{1}{2})} \right\}$$

$$< \infty .$$

Thus it follows from (2) that

$$\sum_{n=1}^{\infty} \left|\frac{J_n}{n}\right| < \infty$$

which leads to the conclusion that $J < \infty$.

This completes the proof of Theorem 17.

We prove below another theorem for the absolute convergence of the series (1).

Theorem 19. If

(i) $$\int_0^{\pi} \left|\frac{\phi(t)}{\sin(t/2)}\right| \log (2\pi/t)\, dt < \infty,$$

(ii) $a_n = O(n^{-\alpha})$, $\alpha > \delta > 0$,

then the series (1) converges absolutely.

Proof. We have

$$\frac{S_n}{n} = \frac{1}{2n\pi} \int_0^{\pi} \frac{\phi(t)}{\sin(t/2)} \sin(n+\tfrac{1}{2})t\, dt$$

$$= \frac{1}{2n\pi} \int_0^{\pi/n^{\delta}} \frac{\phi(t)}{\sin(t/2)} \sin(n+\tfrac{1}{2})t\, dt$$

$$+ \frac{1}{2n\pi} \int_{\pi/n^{\delta}}^{\pi} \frac{\phi(t)}{\sin(t/2)} \sin(n+\tfrac{1}{2})t\, dt,$$ where $\delta > 0$.

Now,

$$2\pi \sum_{n=1}^{\infty} \left| \frac{S_n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^{\pi} \frac{\sin(n+\frac{1}{2})t \;\phi(t)}{\sin(t/2)} \, dt \right|$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^{\pi/n^{\delta}} \frac{\sin(n+\frac{1}{2})t \;\phi(t)}{\sin(t/2)} \, dt \right|$$

(4)

$$+ \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_{\pi/n^{\delta}}^{\pi} \frac{\sin(n+\frac{1}{2})t \;\phi(t)}{\sin(t/2)} \, dt \right|$$

By following an analysis similar to the proof of Theorem 17, it follows that

5) $$\sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^{\pi/n^{\delta}} \frac{\sin(n+\frac{1}{2})t \;\phi(t)}{\sin(t/2)} \, dt \right| < \infty .$$

Since

$$\phi(t) \sim \sum_{k=1}^{\infty} a_k \cos kx \cos kt ,$$

we have

$$J_n \equiv \int_{\pi/n^{\delta}}^{\pi} \frac{\phi(t) \sin(n+\frac{1}{2})t}{\sin(t/2)} \, dt$$

$$= \sum_{k=1}^{\infty} a_k \cos kx \int_{\pi/n^{\delta}}^{\pi} \frac{\sin(n+\frac{1}{2})t \cos kt}{\sin(t/2)} \, dt$$

$$= \sum_{k=1}^{\infty} a_k \cos kx \int_{\pi/n^{\delta}}^{\pi} \frac{\sin(k+n+\frac{1}{2})t - \sin(k-n-\frac{1}{2})t}{2 \sin(t/2)} \, dt$$

As cosec $(t/2)$ is decreasing in the interval $(\pi/n^{\delta}, \pi)$, by using the second mean value theorem, we have

$$\left| \int_{\pi/n^{\delta}}^{\pi} \frac{\sin(k+n+\frac{1}{2})t - \sin(k-n-\frac{1}{2})t}{2 \sin(t/2)} \, dt \right|$$

$$\leq c\, n^{\delta} \left| \int_{\pi/n^{\delta}}^{\pi} \sin(k+n+\tfrac{1}{2})t - \sin(k-n-\tfrac{1}{2})t \;\; dt \right|$$

$$\leq c_1 n^\delta \left| \frac{1}{k+n+\frac{1}{2}} - \frac{1}{k-n-\frac{1}{2}} \right|$$

$$\leq c_2 \frac{n^\delta}{|k-n-\frac{1}{2}|}$$

and hence

$$|J_n| \leq A \sum_{k=1}^{\infty}{}' |a_k| |\cos kx| \frac{n^\delta}{|k-n-\frac{1}{2}|} + O(n^{-\alpha})$$

where ' denotes that the term k=n is omitted.

In virtue of the second hypothesis of the theorem, we get

$$|J_n| \leq A \sum_{k=1}^{\infty}{}' k^{-\alpha} \frac{n^\delta}{|k-n-\frac{1}{2}|} + O(n^{-\alpha})$$

$$\leq A \sum_{k=1}^{n-1} k^{-\alpha} \frac{n^\delta}{|k-n-\frac{1}{2}|}$$

$$+ A \sum_{k=n+1}^{\infty} k^{-\alpha} \frac{n^\delta}{|k-n-\frac{1}{2}|} + O(n^{-\alpha})$$

$$= R_1 + R_2 + R_3 .$$

Now,

$$R_1 < B \frac{n^\delta}{\frac{1}{2}n} \sum_{k=1}^{[\frac{n}{2}]} k^{-\alpha} + B_1 n^\delta (n/2)^{-\alpha} \sum_{k=[\frac{n}{2}]+1}^{n-1} \frac{1}{n-k-\frac{1}{2}}$$

$$= O(n^{\delta-\alpha}) + O(n^{\delta-\alpha} \log n) .$$

Similarly,

$$R_2 < C n^{\delta-\alpha} \sum_{k=n+1}^{2n} \frac{1}{k-n-\frac{1}{2}} + C_1 n^\delta \sum_{k=2n+1}^{\infty} \frac{k^{-\alpha}}{k/2}$$

$$= O(n^{\delta-\alpha} \log n) + O(n^{\delta-\alpha}) .$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| \int_{\pi/n^{\delta}}^{\pi} \frac{\sin(n+\frac{1}{2})t \;\phi(t)}{\sin(t/2)}\, dt \right|$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} |J_n|$$

$$< A \sum_{n=1}^{\infty} \frac{\log n}{n^{1+\alpha-\delta}} + B_2 \sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha-\delta}}$$

$$+ C_2 \sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha}}$$

(6) $< \infty$, for $\alpha > \delta$.

Hence it follows from (4), (5) and (6) that

$$\sum_{n=1}^{\infty} \frac{|S_n|}{n} < \infty .$$

This completes the proof of the theorem.