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This proves that

$$\sum_{n=1}^{\infty} n^{\alpha/2} (| a_n | + | b_n |) < \infty .$$

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Therefore,

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} P_n^{\beta} = \sum_{\nu=\nu_0}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} P_n^{\beta}$$

$$\leqslant c_3 \sum_{\nu=\nu_0}^{\infty} \frac{1}{(\nu-2)\log^{1+\varepsilon}(\nu-2)} < \infty;$$

since $|a_n|^{\beta}$, as also $|b_n|^{\beta}$, does not exceed P_n^{β}, it follows that

$$\sum_{n=1}^{\infty} \left(|a_n|^{\beta} + |b_n|^{\beta} \right) < \infty.$$

This completes the proof of the theorem.

4. *PROOF OF THEOREM 2.* We shall prove this theorem also for $k = 2$. From (10) we have

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} P_n^2 \leqslant c_4\, \omega\left(\frac{\pi}{2^\nu}\right) 2^{-\nu},$$

hence by (7) we get

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} P_n \leqslant \frac{c_5 2^{-\nu} 2^{-\nu\alpha}}{\left[l_1\left(\frac{\pi}{2^\nu}\right) l_2^{1+\varepsilon}\left(\frac{\pi}{2^\nu}\right)\right]^2}$$

$$\leqslant \frac{c_6 2^{-\nu(1+\alpha)}}{\left[(\nu-2)\log^{1+\varepsilon}(\nu-2)\right]^2}.$$

By Schwarz's inequality

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} P_n \leqslant \left(\Sigma P_n^2\right)^{1/2} 2^{\nu/2} \leqslant \frac{c_7 2^{-\nu\alpha/2}}{(\nu-2)\log^{1+\varepsilon}(\nu-2)}.$$

Therefore,

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} n^{\alpha/2} P_n \leqslant \frac{c_7}{(\nu-2)\log^{1+\varepsilon}(\nu-2)},$$

and hence

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} n^{\alpha/2} P_n = \sum_{\nu=\nu_0}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} n^{\alpha/2} P_n$$

$$\leqslant c_7 \sum_{\nu=\nu_0}^{\infty} \frac{1}{(\nu-2)\log^{1+\varepsilon}(\nu-2)} < \infty.$$

Hence

$$\sum_{n=1}^{N} P_n^2 \sin^2 \frac{n\pi}{2N} \leqslant \omega\left(\frac{\pi}{N}\right) V N^{-1}.$$

Putting $N = 2^\nu$, where ν is an integer $\geqslant \nu_0 \geqslant (\log 2)^{-2} + 3$ and taking into accouut only the terms with indices n exceeding $\frac{1}{2}$ N, we have from the last inequality

$$(10) \quad \sum_{n=2^{\nu-1}+1}^{2^\nu} P_n^2 \leqslant 2\,\omega\left(\frac{\pi}{2^\nu}\right) V 2^{-\nu}.$$

Hence by (4)

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} P_n^2 \leqslant \frac{2 V 2^{-\nu} c(2^{-\nu}\pi)^\alpha}{\left[l_1\left(\frac{\pi}{2^\nu}\right) l_2^{1+\varepsilon}\left(\frac{\pi}{2^\nu}\right)\right]^{\alpha+2}}$$

$$\leqslant C_1 \frac{2^{-\nu(1+\alpha)}}{\left[l_1\left(\frac{\pi}{2^\nu}\right) l_2^{1+\varepsilon}\left(\frac{\pi}{2^\nu}\right)\right]^{\alpha+2}}.$$

As shown in [6],

$$l_1\left(\frac{\pi}{2^\nu}\right) > (\nu-2)\log 2$$

and $$l_2\left(\frac{\pi}{2^\nu}\right) > \tfrac{1}{2}\log(\nu-2);$$

hence,

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} P_n^2 \leqslant \frac{c_2\, 2^{-\nu(1+\alpha)}}{\left[(\nu-2)\log^{1+\varepsilon}(\nu-2)\right]^{\alpha+2}}.$$

Applying Hölder's inequality we get

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} P_n^p \leqslant \left(\sum_{n=2^{\nu-1}+1}^{2^\nu} P_n^2\right)^{\beta/2} \left(\sum_{n=2^{\nu-1}+1}^{2^\nu} 1\right)^{1-\beta/2}$$

$$\leqslant C_3 \frac{2^{-\nu(1+\alpha)\beta/2}}{\left[(\nu-2)\log^{1+\varepsilon}(\nu-2)\right]^{(\alpha+2)\beta/2}} \cdot 2^{\nu(1-\beta/2)}$$

$$= \frac{c_3}{(\nu-2)\log^{1+\varepsilon}(\nu-1)}, \text{ for } \beta = \frac{2}{\alpha+2}.$$

Therefore

$$(8) \qquad \frac{1}{\pi}\int_0^{2\pi}[f(x+h)-f(x-h)]^2\,dx = 4\sum_{n=1}^{\infty} P_n^2 \sin^2 nh,$$

where $\quad P_n^2 = a_n^2 + b_n^2.$

Now $\quad \frac{1}{\pi}\int_0^{2\pi}[f(x+h)-f(x-h)]^2\,dx \leqslant \frac{1}{\pi}\,\omega(h)\int_0^{2\pi}|f(x+h)-f(x-h)|\,dx,$

where ω denotes the modulus of continuity of $f(x)$. Putting $h=\frac{\pi}{2N}$, we have

$$\frac{1}{\pi}\int_0^{2\pi}\left[f\left(x+\frac{\pi}{2N}\right)-f\left(x-\frac{\pi}{2N}\right)\right]^2 dx$$

$$(9) \qquad \leqslant \frac{1}{\pi}\,\omega\left(\frac{\pi}{N}\right)\int_0^{2\pi}\left|f\left(x+\frac{\pi}{2N}\right)-f\left(x-\frac{\pi}{2N}\right)\right|dx;$$

and $\quad \int_0^{2\pi}\left|f\left(x+\frac{\pi}{2N}\right)-f\left(x+\frac{\pi}{2N}\right)\right|dx = \int_0^{2\pi}\left|f\left(x+\frac{\pi}{N}\right)-f(x)\right|dx$

$$\leqslant \sum_{\nu=0}^{2N-1}\int_{\nu\pi/N}^{(\nu+1)\pi/N}\left|f\left(x+\frac{\pi}{N}\right)-f(x)\right|dx$$

$$\leqslant \int_0^{\pi/N}\sum_{\nu=0}^{2N-1}\left|f\left(x+\frac{\nu+1}{N}\pi\right)-f\left(x+\frac{\nu}{N}\pi\right)\right|dx.$$

Since $\sum_{\nu=0}^{2N-1}\left|f\left(x+\frac{\nu+1}{N}\pi\right)-f\left(x+\frac{\nu}{N}\pi\right)\right| \leqslant V,$

where V is the total variation of $f(x)$ in $(0, 2\pi)$, we have

$$\int_0^{2\pi}\left|f\left(x+\frac{\pi}{2N}\right)-f\left(x-\frac{\pi}{N}\right)\right|dx \leqslant \frac{\pi V}{N}.$$

Therefore it follows from (8) and (9) that

$$\sum_{n=1}^{\infty} P_n^2 \sin^2\frac{n\pi}{2N} \leqslant \omega\left(\frac{\pi}{N}\right)\frac{V}{N}.$$

THEOREM F. If

$$|f(x+h)-f(x)| \leqslant \frac{ch^{\alpha}}{\left[l_1(h)\, l_2(h) \ldots l_k^{1+\varepsilon}(h)\right]^{\frac{2\alpha+1}{2}}},$$

then (3) is true for $\beta = \frac{2}{2\alpha+1}$.

The aim of this paper is to make a similar addition to Theorem D. More precisely we intend to prove the following

THEOREM 1. If $f(x)$ is of bounded variation and satisfies the condition

$$(4) \quad |f(x+h)-f(x)| \leqslant \frac{ch^{\alpha}}{\left[l_1(h)\, l_2(h) \ldots l_k^{1+\varepsilon}(h)\right]^{\alpha+2}},$$

then (3) holds for $\beta = \frac{2}{\alpha+2}$.

If we put $\alpha = 0$ and $k = 1$, then this theorem reduces to the more general form of Zygmund's Theorem B [Remark; (7)].

2. G. H. Hardy [2] proved that, if $f(x) \in \text{Lip}\ \alpha$, $0 < \alpha \leqslant 1$, then

$$(5) \quad \sum_{n=1}^{\infty} n^{\beta-\frac{1}{2}} (|a_n| + |b_n|) < \infty,$$

for $\beta < \alpha$. If $f(x)$ is, in addition, of bounded variation, then

$$(6) \quad \sum_{n=1}^{\infty} n^{\beta/2} (|a_n| + |b_n|) < \infty,$$

for $\beta < \alpha$. A. C. Zaanen [6 ; Theorem 3] has proved the following addition:

If $|f(x+h)-f(x)| \leqslant \frac{ch^{\alpha}}{l_1(h)\, l_2(h) \ldots l_k^{1+\varepsilon}(h)}$, then (5) holds for $\beta = \alpha$. In this connection, we shall prove the following

THEOREM 2. If $f(x)$ is of bounded variation and satisfies the condition

$$(7) \quad |f(x+h)-f(x)| \leqslant \frac{ch^{\alpha}}{\left[l_1(h)\, l_2(h) \ldots l_k^{1+\varepsilon}(h)\right]^{2}},$$ then (6) holds for $\beta = \alpha$.

3. *PROOF OF THEOREM 1.* We shall prove the theorem for $k=2$. From (1) it follows that

$$f(x+h)-f(x-h) \sim 2 \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \sin nh.$$

ON THE ABSOLUTE CONVERGENCE OF FOURIER SERIES

B. S. Yadav and O. P. Goyal

1. Let $f(x)$ be L-integrable in $(0, 2\pi)$ and periodic outside with period 2π and let

$$(1) \qquad f(x) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

S. Bernstein [1] and A. Zygmund [7] proved the following two theorems respectively :

THEOREM A. If $f(x) \in \text{Lip}\, \alpha$, $\alpha > \frac{1}{2}$, then the Fourier series of $f(x)$ converges absolutely, that is

$$(2) \qquad \sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty.$$

For $\alpha = \frac{1}{2}$, this is no longer true [9].

THEOREM B. If $f(x)$ is of bounded variation and belongs to Lip α, $\alpha > 0$, then (2) holds.

These theorems were generalized by O. Szász [4] and Waraszkiewicz [5] and A. Zygmund [8, 9] respectively to the following theorems :

THEOREM C. If $f(x) \in \text{Lip}\, \alpha$, $0 < \alpha \leqslant 1$, then

$$(3) \qquad \sum_{n=1}^{\infty} \left(|a_n|^{\beta} + |b_n|^{\beta} \right) < \infty,$$

for $\beta > \dfrac{2}{2\alpha + 1}$, but not necessarily for $\beta = \dfrac{2}{2\alpha + 1}$.

THEOREM D. If $f(x)$ is, in addition, of bounded variation in Theorem C, then (3) holds for $\beta > \dfrac{2}{\alpha + 2}$, but not necessarily for $\beta = \dfrac{2}{\alpha + 2}$.

L. Neder [3] and A. C. Zaanen [6] have respectively made the following additions to the Theorems A and C :

THEOREM E. If for $h > 0$,

$$l_1(h) = \log(e + h^{-1})$$
$$l_2(h) = \log\log(e^e + h^{-1})$$

etc.,

and if for certain $\varepsilon > 0$,

$$|f(x+h) - f(x)| \leqslant \frac{ch^{\alpha}}{l_1(h)\, l_2(h) \ldots l_k^{1+\varepsilon}(h)},$$

then (2) holds also for $\alpha = \frac{1}{2}$.

ON THE ABSOLUTE CONVERGENCE OF FOURIER SERIES

O. P. Goyal

1. Let $f(x)$ be L-integrable in $(0, 2\pi)$ and periodic outside with period 2π, and let

$$(1) \quad f(x) \sim \frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x).$$

G. H. Hardy [1] proved that, if $f(x) \in \text{Lip}\, \alpha$, $0 < \alpha \leqslant 1$, then

$$(2) \quad \sum_{n=1}^{\infty} n^{\beta-\frac{1}{2}} (|a_n| + |b_n|) < \infty, \text{ for } \beta < \alpha.$$

If $f(x)$ is, in addition, of bounded variation, then

$$(3) \quad \sum_{n=1}^{\infty} n^{\beta/2} (|a_n| + |b_n|) < \infty, \text{ for } \beta < \alpha.$$

A. C. Zaanen [4] has proved the following addition :

If $\quad |f(x+h) - f(x)| \leqslant \dfrac{Ch^\alpha}{l_1(h)\, l_2(h) \ldots l_k^{1+\varepsilon}(h)}, \quad \varepsilon > 0, \quad$ where

$l_1(h) = \log\left(e + \dfrac{1}{h}\right), l_2(h) = \log\log\left(e^e + \dfrac{1}{h}\right), \quad \ldots \quad \ldots \quad \ldots$

then (2) holds for $\beta = \alpha$.

Recently, B. S. Yadav and O. P. Goyal [3] have proved the following addition :

If $f(x)$ is of bounded variation and satisfies the condition

$$|f(x+h) - f(x)| \leqslant \frac{Ch^\alpha}{\left(l_1(h)\, l_2(h) \ldots l_k^{1+\varepsilon}(h)\right)^2}, \ \varepsilon > 0, \text{ then (3) holds}$$

for $\beta = \alpha$.

In this connection, we shall prove the following theorems :

Let $f(x) \sim \dfrac{a_0}{2} + \displaystyle\sum_{\nu=1}^{\infty} a_\nu \cos \nu x,$

and ω denote the modulus of continuity of the function $f(x)$.

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Theorem 1

If (i) $f(x) \in \text{Lip}\,\alpha,\ 0 < \alpha \leq 1$

(ii) $a_n \geq \dfrac{-\omega\left(\frac{1}{n}\right)\log n}{n},\ n \geq 1,$

and (iii) $\omega\left(\frac{1}{x}\right)$ is increasing slowly, then

$$\sum_{n=1}^{\infty} n^{\beta-\frac{1}{2}}(\,|a_n|\,) < \infty, \text{ for } \beta < \alpha + \tfrac{1}{2}.$$

We have the analoguous theorem for sine series also.

Theorem 2

If (i) $f(x) \in \text{Lip}\,\alpha,\ 0 < \alpha \leq 1$

(ii) $b_n \geq \dfrac{-\omega\left(\frac{1}{n}\right)}{n},\ n \geq 1,$

and (iii) $\omega\left(\frac{1}{x}\right)$ is increasing slowly, then

$$\sum n^{\beta-\frac{1}{2}}(\,|b_n|\,) < \infty, \text{ for } \beta < \alpha + \tfrac{1}{2}.$$

We state here the two lemmas which we shall need in the proof of our theorems.

Lemma 1:

If $f(x) \in \text{Lip}\,\alpha$, $0 < \alpha \leq 1$, and $\omega(\delta)$ be its modulus of continuity, then

$$\sum_{1}^{\infty} \frac{\omega\left(\frac{1}{n}\right)\log n}{n} < \infty.$$

Lemma 2 (Tomic [2]):

Under the hypothesis of the theorem 1,

if $m = \left[\frac{n}{2}\right]$, $\lambda_\nu = \omega\left(\frac{1}{\nu}\right)\log \nu$, $\nu > 1$, $\lambda_0, \lambda_1 > 0$;

$B_m = \dfrac{1}{2\pi m}\displaystyle\sum_{\nu=p}^{m} \nu\left(a_\nu + \frac{\lambda_\nu}{\nu}\right)$, p being sufficiently large but fixed and $< m$, then

$$(4)\quad B_m = O\left(\omega\left(\frac{1}{n}\right)\log n\right) \text{ and also}$$

$$(5)\quad \frac{1}{2\pi}\left(a_m + \frac{\lambda_m}{m}\right) < B_m - B_{m-1} + \frac{B_{m-1}}{m-1}.$$

Lemma 2 follows from lemma 1 [2].

Proof of Theorem 1:

Taking into account that $\lambda_\nu\ (\nu \geqslant p)$ is a sequence increasing slowly, we shall have in vritue of (5)

$$|a_\nu| - \frac{\lambda_\nu}{\nu} \leqslant \left|a_\nu + \frac{\lambda_\nu}{\nu}\right|$$

$$< 2\pi\left[B_\nu - B_{\nu-1} + \frac{B_{\nu-1}}{\nu-1}\right]$$

Therefore,

$$\nu^{\beta-\frac{1}{2}}\left[\,|a_\nu| - \frac{\lambda_\nu}{\nu}\right] < 2\pi\left[\nu^{\beta-\frac{1}{2}}B_\nu - \nu^{\beta-\frac{1}{2}}B_{\nu-1} + \nu^{\beta-\frac{1}{2}}\frac{B_{\nu-1}}{\nu-1}\right]$$

$$< 2\pi\left[\nu^{\beta-\frac{1}{2}}B_\nu - (\nu-1)^{\beta-\frac{1}{2}}B_{\nu-1} + \nu^{\beta-\frac{1}{2}}\frac{B_{\nu-1}}{\nu-1}\right].$$

Hence,

$$\sum_{\nu=p+1}^{m} \nu^{\beta-\frac{1}{2}}|a_\nu| - \sum_{\nu=p+1}^{m} \nu^{\beta-\frac{1}{2}}\frac{\lambda_\nu}{\nu} \leqslant 2\pi \sum_{\nu=p+1}^{m} \nu^{\beta-\frac{1}{2}}B_\nu$$

$$- 2\pi \sum_{\nu=p+1}^{m} (\nu-1)^{\beta-\frac{1}{2}}B_{\nu-1} + 2\pi \sum_{\nu=p+1}^{m} \nu^{\beta-\frac{1}{4}}\frac{B_{\nu-1}}{\nu-1}$$

$$\leqslant 2\pi\left(m^{\beta-\frac{1}{2}}B_m + \sum_{\nu=p+1}^{m} \nu^{\beta-\frac{1}{2}}\frac{B_{\nu-1}}{\nu-1}\right)$$

$$\sum_{\nu=p+1}^{m} \nu^{\beta-\frac{1}{2}}|a_\nu| \leqslant 2\pi\, m^{\beta-\frac{1}{2}}B_m + 2\pi \sum_{\nu=p+1}^{m} \nu^{\beta-\frac{1}{2}}\frac{B_{\nu-1}}{\nu-1}$$

$$+ \sum_{\nu=p+1}^{m} \nu^{\beta-\frac{1}{2}}\frac{\lambda_\nu}{\nu},$$

$$\underset{m\to\infty}{\mathrm{Lt}} \sum_{\nu=p+1}^{m} \nu^{\beta-\frac{1}{2}}(|a_\nu|) \leqslant 2\pi \underset{m\to\infty}{\mathrm{Lt}}\, m^{\beta-\frac{1}{2}}B_m$$

$$+ \underset{n\to\infty}{\mathrm{Lt}}\, 2\pi \sum_{\nu=p+1}^{m} \nu^{\beta-\frac{1}{2}}\frac{B_\nu - 1}{\nu-1} + \underset{m\to\infty}{\mathrm{Lt}} \sum_{\nu=p+1}^{m} \nu^{\beta-\frac{1}{2}}\frac{\lambda_\nu}{\nu}$$

$$\sum_{\nu=p+1}^{\infty} \nu^{\beta-\frac{1}{2}}(|a_\nu|) \leqslant 2\pi \sum_{\nu=p+1}^{\infty} \nu^{\beta-\frac{1}{2}}\frac{B_{\nu-1}}{\nu-1} + \sum_{\nu=p+1}^{\infty} \nu^{\beta-\frac{1}{2}}\frac{\lambda_\nu}{\nu};$$

Since $\underset{m\to\infty}{\mathrm{Lt}}\, m^{\beta-\frac{1}{2}}B_m \leqslant \underset{n\to\infty}{\mathrm{Lt}}\, An^{\beta-\frac{1}{2}}\,\omega\left(\frac{1}{n}\right)\log n$ [in virtue of (4)]

$$= \underset{n \to \infty}{Lt} A \frac{\omega\left(\frac{1}{n}\right) \log n}{n^{\frac{1}{2}-\beta}} \leqslant \underset{n \to \infty}{Lt} A \frac{\log n}{n^{\alpha+\frac{1}{2}-\beta}} = 0,$$

for $\beta < \alpha + \frac{1}{2}$.

Therefore, in virtue of (4), we have,

$$\sum_{\nu=p+1}^{\infty} \nu^{\beta-\frac{1}{2}} (|a_\nu|) \leqslant 2\pi A \sum_{\nu=p}^{\infty} (\nu+1)^{\beta-\frac{1}{2}} \frac{\omega\left(\frac{1}{\nu}\right) \log \nu}{\nu} + \sum_{\nu=p+1}^{\infty} \nu^{\beta-\frac{1}{2}} \frac{\omega\left(\frac{1}{\nu}\right) \log \nu}{\nu}$$

$$\leqslant C \sum_{\nu=p+1}^{\infty} \nu^{\beta-\frac{1}{2}} \frac{\omega\left(\frac{1}{\nu}\right) \log \nu}{\nu}$$

$$\leqslant C' \sum_{\nu=p+1}^{\infty} \nu^{\beta-\frac{1}{2}} \frac{\log \nu}{\nu \cdot \nu^{\alpha}} \left[\text{since } \omega\left(\frac{1}{\nu}\right) < B \frac{1}{\nu^{\alpha}}\right],$$

$$\leqslant C' \sum_{\nu=p+1}^{\infty} \frac{\log \nu}{\nu^{1+\alpha+\frac{1}{2}-\beta}}.$$

$< \infty$, for $\alpha + \frac{1}{2} - \beta > 0$.

Therefore,

$$\sum_{\nu=1}^{\infty} \nu^{\beta-\frac{1}{2}} (|a_\nu|) < \infty, \text{ for } \beta < \alpha + \tfrac{1}{2}.$$

Proof of Theorem 2:

By taking $T_n' = \frac{1}{2} + \sum_{\nu=1}^{n+1} \left(1 - \frac{\nu}{n+1}\right) \cos \nu x$ and $\lambda_\nu = \omega\left(\frac{1}{\nu}\right)$, we can prove the Theorem 2 just as the Theorem 1 is proved.

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(8) $f(x, y)=A(x, y)-A(y, x),$ if $C=(-1)^n$;

(9) $f(x, y)=A(x, y)+A(y, x),$ if $C=(-1)^{n+1}$;

where $B_n(t)$, $B_{n-1}(t), \ldots, B_1(t)$ and $A(x, y)$ are arbitrary continuous functions.

(7) is the general continuous solution.

(8) and (9) are particular solutions.

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ON THE ABSOLUTE CONVERGENCE OF A SERIES ASSOCIATED WITH A FOURIER SERIES

O. P. Goyal

(Presented 26 February 1965)

Let $f(x)$ be L-integrable in $(0,2\pi)$ and periodic out side with period 2π, and let its Fourier series is

(1.1) $$f(x) \sim \sum_{n=1}^{\infty} a_n \cos n x=\sum_{n=1}^{\infty} A_n .$$

We shall be concerned in this note with the series

(1.2) $\sum_{n=1}^{\infty} \frac{S_n-S}{n}$, where S_n is the n^{th} partial sum of (1.1)) i.e.

(1.3) $S_n=\sum_{k=1}^{n} A_k$ and S is an appropriate number independent of n.

We shall take $S=0$.

We write

(1.4) $$\Phi(t)=\{f(x+t)+f(x-t)\}/2$$

and $$\Phi_1(t)=\frac{1}{t} \int_0^t \Phi(u)\, du .$$

Hardy and Littlewood [1] have considered the [c, 1] summability of the series (1.2) and Zygmund [3] has obtained the necessary and sufficient condition for its convergence. Recently R. Mohanty and S. Mohopatra [2] proved the following theorem:

T h e o r e m A. If (I) $\Phi_1(t) \log \frac{K}{t}$ is of bounded variation in $(0,2\pi)$,

(II) $\left|\dfrac{\Phi_1(t)}{t}\right|$ is integrable in $(0,2\pi)$,

(III) $(n^\delta An)$ is of bounded variation for $\delta > 0$,

then (1.2) is absolutely convergent.

We intend to prove the following theorem:

Theorem 1. If (I) $\int_0^\pi \left|\dfrac{\Phi(t)}{\sin{}^t/_2}\right| \log \dfrac{2\pi}{t}\, dt < \infty$,

and (II) $a_n = 0\,(n^{-\alpha})$, $\alpha > \delta > 0$,

then $\sum_{n=1}^{\infty} \dfrac{S_n}{n}$ is absolutely convergent.

Proof. We have

$$\frac{S_n}{n} = \frac{1}{2n\pi}\int_0^\pi \frac{\Phi(t)\sin(n+1/2)\,t\,dt}{\sin{}^t/_2}$$

$$= \frac{1}{2n\pi}\int_0^{\pi/n^\delta} \frac{\Phi(t)\sin(n+1/2)\,t\,dt}{\sin{}^t/_2} + \frac{1}{2n\pi}\int_{\pi/n^\delta}^{\pi} \frac{\Phi(t)\sin(n+1/2)\,t\,dt}{\sin{}^t/_2},\quad \delta > 0.$$

$$\Rightarrow\ 2\pi\sum_{n=1}^{\infty}\left|\frac{S_n}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}\left|\int_0^\pi \frac{\sin(n+1/2)\,t}{\sin{}^t/_2}\,\Phi(t)\,dt\right|$$

$$\leqslant \sum_{n=1}^{\infty}\frac{1}{n}\left|\int_0^{\pi/n^\delta}\right| + \sum_{n=1}^{\infty}\frac{1}{n}\left|\int_{\pi/n^\delta}^{\pi}\right|$$

$$= I + J.$$

Now

$$I \leqslant \sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\pi/n^\delta}\left|\frac{\Phi(t)}{\sin{}^t/_2}\right| dt$$

$$\leqslant \sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=n^\delta}^{\infty}\int_{\pi/k+1}^{\pi/k}\left|\frac{\Phi(t)}{\sin{}^t/_2}\right| dt$$

$$\leqslant A\sum_{k=1}^{\infty}\int_{\pi/k+1}^{\pi/k}\left|\frac{\Phi(t)}{\sin{}^t/_2}\right|\sum_{n=1}^{k^{1/\delta}}\frac{1}{n}\,dt$$

$$\leqslant A \sum_{k=1}^{\infty} \int_{\pi/k+1}^{\pi/k} \left| \frac{\Phi(t)}{\sin t/2} \right| \log k \, dt$$

$$\leqslant A \sum_{k=1}^{\infty} \int_{\pi/k+1}^{\pi/k} \log \frac{2\pi}{t} \left| \frac{\Phi(t)}{\sin t/2} \right| dt, \quad \left(\text{since } \frac{\pi}{k+1} \leqslant t \leqslant \pi/k, \right.$$

$$\left. t \leqslant 2\pi/k, \ \log k \leqslant \log \frac{2\pi}{t} \right).$$

$$\leqslant A \int_0^{\pi} \log \frac{2\pi}{t} \left| \frac{\Phi(t)}{\sin t/2} \right| dt$$

$$< \infty .$$

Since $\Phi(t) \sim \sum_{k=1}^{\infty} a_k \cos k x \cos k t$, we have the Parseval formula

$$\vartheta_n = \int_{\pi/n^{\delta}}^{\pi} \frac{\Phi(t) \sin(n+1/2)\, t\, dt}{\sin t/2} = \sum_{k=1}^{\infty} a_k \cos k x \int_{\pi/n^{\delta}}^{\pi} \frac{\sin(n+1/2)\, t \cos k t\, dt}{\sin t/2}$$

$$= \sum_{k=1}^{\infty} a_k \cos k x \int_{\pi/n^{\delta}}^{\pi} \frac{\sin(k+n+1/2)t - \sin(k-n-1/2)\, t}{2 \sin t/2} dt.$$

Applying the second mean value theorem to the factor $\frac{1}{2}$ cosec $\frac{1}{2} t$ we see that the coefficient of $a_k \cos k x$, $k \neq n$, does not exceed $\dfrac{2 n^{\delta}}{\pi^2 \left| k-n-\frac{1}{2} \right|}$ in absolute value, and so

$$|\vartheta_n| \leqslant \frac{2}{\pi^2} \sum_{k=1}^{\infty}{}' |a_k| \, |\cos k x| \frac{n^{\delta}}{\left| k-n-\frac{1}{2} \right|} + 0\left(\frac{1}{n^{\alpha}}\right)$$ (where $'$ denotes that the term $k=n$ is omitted)

$$\leqslant A \sum_{k=1}^{\infty}{}' k^{-\alpha} \frac{n^{\delta}}{|k-n-y_2|} + 0\left(\frac{1}{n^{\alpha}}\right)$$

$$\leqslant A \sum_{k=1}^{n-1} + A \sum_{k=n+1}^{\infty} + 0\left(\frac{1}{n^{\alpha}}\right)$$

$$= R_1 + R_2 + R_3 .$$

$$\frac{\pi}{4}R_1 < \frac{n^\delta}{\frac{1}{2}n}\sum_{k=1}^{\left[\frac{1}{2}n\right]} k^{-\alpha} + n^\delta\left(\frac{1}{2}n\right)^{-\alpha}\sum_{k=\left[\frac{n}{2}\right]+1}^{n-1}\frac{1}{n-k-1/2}$$

$$= O(n^{\delta-\alpha}) + O(n^{\delta-\alpha}\log n).$$

Similarly

$$\frac{\pi}{4}R_2 < n^{\delta-\alpha}\sum_{k=n+1}^{2n}\frac{1}{k-n-1/2} + n^\delta\sum_{k=2n+1}^{\infty}\frac{k^{-\alpha}}{\frac{1}{2}k}$$

$$= O(n^{\delta-\alpha}\log n) + O(n^{\delta-\alpha}).$$

Therefore

$$J = \sum_{n=1}^{\infty}\frac{1}{n}\left|S_n\right|$$

$$< A\sum_{n=1}^{\infty}\frac{\log n}{n^{1+\alpha-\delta}} + A'\sum_{n=1}^{\infty}\frac{1}{n^{1+\alpha-\delta}} + A''\sum_{n=1}^{\infty}\frac{1}{n^{1+\alpha}}$$

$$< \infty,$$

for $\alpha > \delta$.

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МАТЕМАТИЧКИ ВЕСНИК
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ON THE ABSOLUTE CONVERGENCE OF FOURIER SERIES

O. P. Goyal

(Presented 26. February 1965)

1. Let $f(x)$ be L-integrable in $(0,2\pi)$ and periodic out side with period 2π, and let

(1) $$f(x) \sim \frac{a_0}{2} + \sum_{\nu=1}^{\infty}(a_\nu\cos\nu x + b_\nu\sin\nu x).$$

G. H. Hardy [2] proved that if $f(x) \in lip\,\alpha$, $0 < \alpha \leqslant 1$, then

(2) $$\sum_{n=1}^{\infty} n^{\beta-1/2}(|a_n|+|b_n|) < \infty \text{ . for } \beta < \alpha.$$

If $f(x)$ is, in addition of bounded variation, then

(3) $$\sum_{n=1}^{\infty} n^{\beta/2}(|a_n|+|u_n|) < \infty, \text{ for } \beta < \alpha.$$

A. C. Zaanen [5] has proved the following addition:

If $$|f(x+h)-f(x)| \leqslant \frac{C h^{\alpha}}{l_1(h)\, l_2(h) \cdots l_k^{1+\varepsilon}(h)}, \quad \varepsilon > 0,$$

where $l_1(h) = \log(e+h^{-1})$, $l_2(h) = \log\log(e^e+h^{-1}), \ldots$, then (2) holds for $\beta = \alpha$

B. S. Yadav and O. P. Goyal [4] have proved the following:

If $f(x)$ is of bounded variation and satisfies the condition

$$|f(x+h)-f(x)| \leqslant \frac{C h^{\alpha}}{[l_1(h)\, l_2(h) \cdots l_k^{1+\varepsilon}(h)]^2},$$

hen (3) holds for $\beta = \alpha$.

Recently the author [1] has proved the following:

Let $$f(x) \sim \frac{a_0}{2} + \sum_{\nu=1}^{\infty} a_\nu \cos \nu x$$

and ω denotes the modulus of continuity of the function $f(x)$.

Theorem A. If (I) $f(x) \in lip\,\alpha$, $0 < \alpha \leqslant \varepsilon$,

$$\text{(II) } a_n \geqslant \frac{-\omega\left(\frac{1}{n}\right)\log n}{n}, \quad n \geqslant 1,$$

and (III) $\omega\left(\frac{1}{x}\right)$ *is increasing slowly,*

then $\sum_{n=1}^{\infty} n^{\beta-1/2}(|a_n|) < \infty$, *for* $\beta < \alpha + 1/2$.

Theorem B. If (I) $f(x) \in lip\,\alpha$, $0 < \alpha \leqslant 1$,

$$\text{(II) } b_n \geqslant \frac{-\omega\left(\frac{1}{n}\right)}{n}, \quad n \geqslant 1/2,$$

and (III) $\omega\left(\frac{1}{x}\right)$ *is increasing slowly,*

then $\sum_{n=1}^{\infty} n^{\beta-1/2}(|u_n|) < \infty$, *for* $\beta < \alpha + \frac{1}{2}$.

The aim of this paper is to generalize the theorems A and B in view of Tomić's [3] paper. More precisely we intend to prove the following:

Theorem *1. Let $f(x)$ be periodic and continious in $(0,2\pi)$ and has (1) as its Fourier series. If (I) $f(x) \in lip\, \alpha$, $0 < \alpha \leqslant 1$*

and (II) a_n and b_n are postitive,

then $\sum_{n=1}^{\infty} n^{\gamma}(|a_n|+|b_n|) < \infty$, for $\gamma < \alpha$.

From Theorem 1 follows:

Theorem *2. Let $g(x)$ be a periodic and continious function in $(0, 2\pi)$ and let*

$$g(x) \sim \frac{a_0'}{2} + \sum_{\nu=1}^{\infty} (a_\nu' \cos \nu x + b_\nu' \sin \nu x)$$

and ω_1, denotes the moduls of continuity of the function $g(x)$.

If $f(x)$ satisfies the conditions of the Theorem 1 and

$$\omega_1(\delta) \leqslant \omega(\delta),\ a_2' \geqslant -a_\nu,\ b_\nu' \geqslant -b_\nu,$$

then
$$\sum_{n=1}^{\infty} n^{\gamma}(|a_n'|+|b_n'|) < \infty,\ for\ \gamma < \alpha.$$

We state here a lemma which will be needed in the proof of our theorem.

Lemma (Tomić [3]). Under the 2^{nd} hypothesis of the Theorem 1,

$$\text{if } m = \left[\frac{n}{2}\right] \text{ and } B_m = \frac{\pi}{u\, m} \sum_{\nu=1}^{m} \nu u_\nu, \text{ then}$$

(4)
$$B_m \leqslant |S_n\left(\frac{\pi}{2n}\right) < M\, \omega\left(\frac{\pi}{2n}, f\right) \text{ and}$$

(5)
$$b_m \leqslant \frac{4}{\pi}\left(B_m - B_{m-1} + \frac{B_{m-1}}{m-1}\right).$$

Proof of the Theorem 1. We have in virtue of (5),

$$\nu^{\gamma} b_\nu \leqslant \frac{4}{\pi}\left(\nu^{\gamma} B_2 - \nu^{\gamma} B_{\nu-1} + \nu^{\gamma}\frac{B_{\nu-1}}{\nu-1}\right)$$
$$\leqslant \frac{4}{\pi}\left(\nu^{\gamma} B_\nu - (\nu-1)^{\gamma} B_{\nu-1} + \nu^{\gamma}\frac{B_{\nu-1}}{\nu-1}\right), \text{ since } B_{\nu-1} \geqslant 0.$$

Therefore,

$$\sum_{\nu=1}^{m} \nu^{\gamma} b_\nu \leqslant \frac{4}{\pi}\sum_{\nu=1}^{m} \nu^{\gamma} B_\nu - \frac{4}{\pi}\sum_{\nu=1}^{m} (\nu-1)^{\gamma} B_{\nu-1} + \frac{4}{\pi}\sum_{\nu=1}^{m} \nu^{\gamma}\frac{B_{\nu-1}}{\nu-1}$$
$$\leqslant \frac{4}{\pi}\, m^{\gamma} B_m + \frac{4}{\pi}\sum_{\nu=1}^{m} \nu^{\gamma}\frac{B_{\nu-1}}{\nu-1}.$$

Hence in virtue of (4) we have

$$\sum_{\nu=1}^{m} \nu^{\gamma} b_{\nu} \leqslant A\, m^{\gamma}\, \omega\left(\frac{\pi}{m}, f\right) + A' \sum_{\nu=1}^{m} \nu^{\gamma} \frac{\omega\left(\frac{1}{\nu}, f\right)}{\nu}$$

$$\leqslant \frac{A''}{m^{\alpha-\gamma}} + A''' \sum_{\nu=1}^{m} \frac{1}{\nu^{1+\alpha-\gamma}}, \quad \text{since} \quad \omega\left(\frac{1}{\nu}\right) < \frac{c}{\nu^{\alpha}}.$$

$$\lim_{m \to \infty} \sum_{\nu=1}^{m} \nu^{\gamma} u_{\nu} \leqslant \lim_{m \to \infty} \frac{A''}{m^{\alpha-\gamma}} + A'' \lim_{m \to \infty} \sum_{\nu=1}^{m} \frac{1}{\nu^{1+\alpha-\gamma}}.$$

Therefore,

$$\sum_{\nu=1}^{\infty} \nu^{\gamma} u_{\nu} \leqslant A''' \sum_{\nu=1}^{\infty} \frac{1}{\nu^{1+\alpha-\gamma}}, \qquad \left(\text{since} \lim_{m \to \infty} \frac{1}{m^{\alpha-\gamma}} = 0, \quad \text{for} \quad \gamma < \alpha\right)$$

$$< \infty, \text{ for } \gamma < \alpha.$$

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МАТЕМАТИЧКЕ НОТЕ

У овој рубрици објављују се оригинални прилози чији обим не прелази три штампане стране.

NOTES MATHÉMATIQUES

Dans cette section on insère des notes contenant des contributions nouvelles ne dépassant pas trois pages imprimées.

МАТЕМАТИЧКИ ВЕСНИК
2 (17). 1965. стр 251–253

LACUNARY FOURIER SERIES

O. P. Goyal

Presented March 5, 1965

1. Masako Sato [1] has proved the following:

Theorem A. Let $\frac{1}{2}<a<\alpha<1, \qquad 0<\beta<(2-\alpha)/3$

and
$$\beta/2<\alpha-a\leqslant(2-\alpha-\beta)/4.$$

If
$$k^{1/(2\alpha-2a-\beta)}<n_k<e^{2k/(2+\alpha+\beta)},$$
$$|n_{k\pm1}-n_k|>4\,ekn_k^{\beta},$$

and
$$(i)\ \frac{1}{h^{\beta}}\int_0^{h^{\beta}}|f(t)-f(t\pm h)|^2dt=0\,(h^{2a})\text{ as }h\to0,$$

$$(ii)\ \frac{1}{\tau}\int_0^{\tau}|f(t)-f(t\pm h)|^2dt=0\,(1)\text{ unif. in }\tau>h^{\beta},$$

then
$$\sum(|a_{n_k}|+|b_{n_k}|)<\infty,$$

where a_{n_k} and b_{n_k} are the non-vanishing Fourier coefficients of $f(t)$.

Theorem B. Let $\frac{1}{2}<a<\alpha<1, \qquad 0<\beta<(1-\alpha)/2,$

$$\gamma>\frac{1}{2\alpha-2a-\beta},\ \text{and}\ \beta/2<\alpha-a<(1+\beta)/4.$$

If $n_k=[k^{\gamma}]$ $(k=1,2,3,\ldots)$, and the conditions (i) and (ii) of Theorem A are satisfied, then $\sum(|a_{n_k}|+|b_{n_k}|)<\infty$.

In this note we generalize the Theorems A and B and prove the following:

Theorem 1. ~~Let $0<\alpha\leqslant 1$~~. Under the hypothesis of the Theorem A,

$$\sum (|a_{n_k}|^p+|b_{n_k}|^p)<\infty, \text{ for } p>\frac{2}{2\alpha+1}. \tag{1}$$

Theorem 1′. ~~Let $0<\alpha\leqslant 1$~~. Under the hypothesis of the Theorem B, (1) holds for $p>\dfrac{2}{2\alpha+1}$.

Theorem 2. Under the hypothesis of the Theorem 1,

$$\sum n_k^{p-1/2}(|a_{n_k}|+|b_{n_k}|)<\infty, \text{ for } p<\alpha. \tag{2}$$

Theorem 2′. Under the hypothesis of the Theorem 1′, (2) holds, for $p<\alpha$.

2. Proof of the Theorem 1. In the proof we shall use the following lemma:

Lemma [1]: Under the hypothesis of the Theorem 1,

$$\sum_{n_k}^{2n_k} \rho_n^2=0\,(n_k^{-2\alpha}) \text{ where } \rho_n^2=a_n^2+b_n^2. \tag{3}$$

Putting $n_k=2^{\nu-1}$, from (3) we have,

$$\sum_{2^{\nu-1}}^{2^\nu} \rho_n^2\leqslant\frac{A}{2^{2\nu\alpha}}. \tag{4}$$

Applying Holder's inequality, we get

$$\sum_{2^{\nu-1}}^{2^\nu} \rho_n^p\leqslant\left(\sum_{2^{\nu-1}}^{2^\nu} \rho_n^2\right)^{p/2}\left(\sum_{2^{\nu-1}}^{2^\nu} 1\right)^{1-p/2}$$

$$\leqslant\frac{A'}{2^{\nu\alpha p}}\cdot 2^{\nu(1-p/2)}$$

$$\leqslant\frac{A'}{2^{\nu(\alpha p+p/2-1)}}.$$

Therefore,

$$\sum_1^\infty \rho_n^p=\sum_{\nu=1}^\infty \sum_{2^{\nu-1}}^{2_\nu}\rho_n^p$$

$$\leqslant A'\sum_{\nu=1}^\infty \frac{1}{2^{\nu(\alpha p+p/2-1)}}$$

$$<\infty, \text{ for } p>\frac{2}{2\alpha+1}.$$

Since $|a_{n_k}|^p$ and $|b_{n_k}|^p$, does not exceed $\varrho_{n_k}^p$, we have

$$\sum(|a_{nk}|^p+|b_{nk}|^p)<\infty, \text{ for } p>\frac{2}{2\alpha+1}.$$

Proof of the Theorem 2. From (4) we have, by applying Schwarz's inequality,

$$\sum_{2^{\nu-1}}^{2^\nu} \rho_n \leqslant \left(\sum_{2^{\nu-1}}^{2^\nu} \rho_n^2\right)^{1/2} \left(\sum_{2^{\nu-1}}^{2^\nu} 1^2\right)^{1/2}$$

$$\leqslant \frac{A}{2^{\nu\alpha}} \cdot 2^{\nu/2}$$

$$\leqslant \frac{A}{2^{\nu(\alpha-1/2)}}.$$

Therefore,

$$\sum_{2^{\nu-1}}^{2^\nu} n_k^{p-1/2} \rho_n \leqslant \frac{A' (2^\nu)^{p-1/2}}{2^{\nu(\alpha-1/2)}}$$

$$\leqslant A' \frac{1}{2^{\nu(\alpha-p)}}.$$

Hence

$$\sum_1^\infty n_k^{p-1/2} \rho_n = \sum_{\nu=1}^\infty \sum_{2^{\nu-1}}^{2^\nu} n_k^{p-1/2} \rho_n$$

$$\leqslant A' \sum_{\nu=1}^\infty \frac{1}{2^{\nu(\alpha-p)}}$$

$$< \infty, \quad \text{for } p<\alpha.$$

Therefore $\sum_1^\infty n_k^{p-1/2}(|a_{nk}|+|b_{nk}|) < \infty$, for $p<\alpha$.

Theorem 2′ can be proved as similar to Theorem 2.

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МАТЕМАТИЧКИ ВЕСНИК
2 (17) 1965. стр 254—256

INTEGRATION OF DIFFERENTIAL EQUATIONS OF GEODESICS ADMITTING CONFORMAL EXTENDED INFINITESIMAL TRANSFORMATIONS

Blažo Okiljević

The differential equation of geodesics in the Gaussian form is

(1) $$y'' = y'\omega_x - y'^2\omega_y$$

The aim of this paper is to show the form which the function ω has to assume in order that equation (1) admits conformal extended infinitesimal transformation and furthermore, to point out the method by which equation (1) can in this case be integrated.

Equation (1) is equivalent to a system of two differential equations of the form

(2) $$\frac{dx}{1} = \frac{dy}{y'} = \frac{dy'}{y'\omega_x - y'^2\omega_y}.$$

Let the function $\omega(x, y)$ be determined in such a way that the system (2) admits a conformal extended infinitesimal transformation

(3) $$V(f) \equiv ax f_x + b_y f_y + (b-a)\, y' f_{y'}.$$

To system (2) there corresponds a linear partial equation

(4) $$Y(f) \equiv f_x + y' f_y + (y'\omega_x - y'^2\omega_y) f_{y'} = 0.$$

In order that equation (4) admits an infinitesimal transformation (3), it is necessary that certain conditions [1] should be fulfilled, which are in this case

(5) $$Y(by) - y'\, Y(ax) = V(y')$$
$$Y[(b-a)\, y'] - y'(y'\omega_x - y'^2\omega_y)\, Y(ax) = V(y'\omega_x - y'^2\omega_y),$$

where the operators V und Y have the meaning given by the relations (3) and (4). In expanded form the conditions (5) become

$$(b-a)\, y' \equiv (b-a)\, y'$$

and

$$(b-a)(y'\omega_x - y'^2\omega_y) - (y'\omega_x - y'^2\omega_y) = ax\,[y'\omega_{x^2} - y'^2\omega_{xy}] + by\,(y'\omega_{xy} - y'^2\omega_{y^2}) \times$$
$$\times (\omega_x - 2y'\omega_y)\, y'.$$

The first condition is obviously identically satisfied, while the second takes the form

(6) $$a\omega_x - by'\omega_y + ax\omega_{x^2} - (axy' - by)\,\omega_{xy} - byy'\omega_{y^2} = 0.$$

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