CHAPTER VII

BOUNDEDNESS OF DIFFERENCE KERNELS OF DINI AND FOURIER SERIES.

7.1. Let

$$(7.1.1) \qquad \begin{cases} \varphi_\nu(t) = \sqrt{\dfrac{\pi t}{2}}\; J_\nu(t), & t > 0, \\[2ex] \varphi_\nu(0) = \lim\limits_{t \to 0+} \varphi_\nu(t); \end{cases}$$

where $\nu \geqslant -1/2$.

Consider, for any function $f \in L^1[0,1]$, the Fourier cosine series,

$$(7.1.2) \qquad f(x) \sim \frac{1}{2} a_0 + \sum_{m=1}^{\infty} a_m \cos m\pi x, \qquad 0 \leqslant x \leqslant 1,$$

and the Dini series (FB-II),

$$(7.1.3) \qquad f(x) \sim \sum_{m=1}^{\infty} b_m \, \varphi_\nu(x\lambda_m), \qquad 0 \leqslant x \leqslant 1,$$

where $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ are the successive positive zeros of $\ell t J_\nu'(t) + h J_\nu(t)$, with $\frac{h}{\ell} + \nu > 0$ or $\ell = 0$, and

$$(7.1.4) \quad b_m = \frac{4 \lambda_m \pi^{-1} \int_0^1 f(t)\, \varphi_\nu(t\lambda_m)\, dt}{(\lambda_m^2 - \nu^2) J_\nu^2(\lambda_m) + \lambda_m^2 J_\nu'^2(\lambda_m)} .$$

The respective partial sums of these series are given by

$$\sigma_n(x,f) = \frac{1}{2} a_0 + \sum_{m=1}^{n} a_m \cos m\pi x$$

$$(7.1.5) \qquad = \int_0^1 f(t)\, K_n(t,x)\, dt,$$

where

$$(7.1.6) \quad K_n(t,x) = 1 + \sum_{m=1}^{n} 2 \cos m\pi t \, \cos m\pi x,$$

and

$$(7.1.7) \quad S_n(x,f) = \sum_{m=1}^{n} b_m\, \varphi_\nu(x\lambda_m) = \int_0^1 f(t)\, U_n(t,x)\, dt,$$

where

$$(7.1.8) \quad U_n(t,x) = \sum_{m=1}^{n} \frac{2 \sqrt{xt}\, \lambda_m^2\, J_\nu(t\lambda_m)\, J_\nu(x\lambda_m)}{(\lambda_m^2 - \nu^2) J_\nu^2(\lambda_m) + \lambda_m^2 J_\nu'^2(\lambda_m)} .$$

If $\ell = 0$, the series (7.1.3) becomes FB-I corresponding to f. If $\nu = \pm 1/2$, it reduces to a trigonometric sine or cosine series. In fact, if $\nu = 1/2$, $\varphi_\nu(x) = \sin x$ and if $\nu = -1/2$, $\varphi_\nu(x) = \cos x$.

The functions $\Delta_n(t,x)$, defined as

(7.1.9) $$\Delta_n(t,x) = U_n(t,x) - K_n(t,x), \quad n = 1, 2, \ldots,$$

are called the difference kernels of Dini and Fourier cosine series.

S.H.Tung[1] has proved certain boundedness properties of the difference kernels of the series FB-I and Fourier cosine seriese corresponding to any function $f \in L^1[0,1]$. He has also established the equiconvergence of these two series. Gupta and Khoti[2] has extended these results to modified Fourier-Bessel series of a function $f \in L^1[0,1]$. Titchmarsh[3] has established equiconvergence of series FB-III with series FB-I in $[a,b]$, $0 < a < b$.

In the present chapter, we extend Tung's results to series FB-II and Fourier cosine series. The corresponding results for Fourier sine and Fourier sine-cosine series follow exactly in a similar manner.

We prove the following theorems:

<u>THEOREM 7.1. The difference kernel</u> $\Delta_n(t,x)$ <u>is bounded independently of</u> n <u>and</u> t, <u>at any fixed point</u> $x \in (0,1)$, <u>for</u> $t \in N_\delta[x] = [x-\delta, x+\delta]$, <u>where</u> $0 < \delta \le \min\left\{\frac{x}{2}, \frac{1-x}{2}\right\}$.

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[1] Tung [100]. [2] Gupta and Khoti [39]. [3] Titchmarsh [96].

<u>THEOREM 7.2. The difference kernel $\Delta_n(t,x)$ is bounded independently of n and $t \in [0,1]$, at any fixed point</u> $x \in (0,1)$.

<u>THEOREM 7.3. For any</u> $f \in L^1[0,1]$, <u>its Dini series is equiconvergent with its Fourier cosine series at every</u> $x \in (0,1)$.

7.2. The following lemmas will be used in the proofs of these theorems:

<u>LEMMA 7.1.</u> (Tung[1]). If α is real, $\beta \geqslant 0$, and $0 < \eta \leqslant y-k \leqslant 1-\eta < 1$ for some integer k, then

$$\sum_{m=1}^{n} m^{-\beta} \sin(2m\pi y + \alpha) \quad \text{and} \quad \sum_{m=1}^{n} m^{-\beta} \cos(2m\pi y + \alpha)$$

are bounded independently of α, y and n.

<u>LEMMA 7.2.</u> (Hobson[2]). For $c \geqslant -1/2$,

$$\sum_{m=1}^{n} m^{-1} \sin(m+c)\tau$$

is bounded independently of c and n, where $0 \leqslant \tau \leqslant 2\pi-\eta$, $\eta > 0$.

<u>LEMMA 7.3.</u> (Moore[3]). The following estimation is true

$$\lambda_n = n\pi + Q + \frac{K(\lambda_n)}{n},$$

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1) Tung [100]. 2) Hobson [47]. 3) Moore [67].

where

$$q = \begin{cases} k\pi + \pi(2\nu+1)/4, & \text{if } \ell \neq 0, \\ k\pi + \pi(2\nu+1)/4, & \text{if } \ell = 0, \end{cases}$$

k is an integer, positive, negative or zero and $K(\lambda_n)$ remains bounded for sufficiently large values of n.

7.3. PROOF OF THEOREM 7.1. From the asymptotic expansion[1],

$$(7.3.1)\quad J_\nu(z) = \sqrt{\frac{2}{\pi z}}\left\{\cos(z-\gamma) - \frac{\nu^2-1/4}{2z}\sin(z-\gamma) + O(|z|^{-2})\right\},$$

where $\gamma = (2\nu+1)\pi/4$, we have,

$$(\lambda_m^2-\nu^2)\, J_\nu^2(\lambda_m) + \lambda_m^2\, J_\nu'^2(\lambda_m) =$$

$$= \lambda_m^2\left\{J_\nu^2(\lambda_m) + J_{\nu+1}^2(\lambda_m)\right\} - 2\nu\lambda_m J_\nu(\lambda_m) J_{\nu+1}(\lambda_m)$$

$$= \frac{2\lambda_m}{\pi}\left\{1 - \frac{\nu\sin 2(\lambda_m-\gamma)}{\lambda_m} + O(\lambda_m^{-2})\right\}$$

$$(7.3.2)\quad = \frac{2\lambda_m}{\pi}\left\{1 + O(m^{-2})\right\},$$

by Lemma 7.3.

By (7.1.8), (7.3.1) and (7.3.2), we get,

$$U_n(t,x) = \sum_{m=1}^{n}\left\{1 + O(m^{-2})\right\}\Big[2\cos(t\lambda_m-\gamma)\cos(x\lambda_m-\gamma) - \frac{\nu^2-1/4}{2}\left\{\frac{2\sin(t\lambda_m-\gamma)\cos(x\lambda_m-\gamma)}{t\lambda_m} + \frac{2\cos(t\lambda_m-\gamma)\sin(x\lambda_m-\gamma)}{x\lambda_m}\right\} + O(\lambda_m^{-2})\Big]$$

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[1] Watson [103], p. 199.

$$(7.3.3) \qquad = U_n^{(1)} + U_n^{(2)} + U_n^{(3)},$$

where

$$U_n^{(1)} = \sum 2\cos(t\lambda_m - \gamma)\cos(x\lambda_m - \gamma),$$

$$U_n^{(2)} = -\frac{\nu^2 - 1/4}{2tx}\left\{\sum \frac{t+x}{\lambda_m}\sin(t\lambda_m + x\lambda_m - 2\gamma) - \right.$$

$$\left. - \sum \frac{t-x}{\lambda_m}\sin(t-x)\lambda_m \right\},$$

$$U_n^{(3)} = \sum O(m^{-2}),$$

and $\sum$ denotes the summation $\sum_{m=1}^{n}$.

$U_n^{(3)}$ is, obviously, bounded independently of n and t.

Now, if $t+x = 2\eta$, then by Lemma 7.3,

$$(7.3.4) \qquad \begin{cases} (t+x)\lambda_m - 2\gamma = 2\eta m\pi + \alpha + 2\eta\beta_m, \text{ where} \\ \alpha = (t+x)k\pi + (t+x)\dfrac{2\nu+1}{4}\pi - \dfrac{2\nu+1}{2}\pi, \text{ if } \ell \neq 0, \\ \alpha = (t+x)k\pi + (t+x)\dfrac{2\nu-1}{4}\pi - \dfrac{2\nu+1}{2}\pi, \text{ if } \ell = 0, \\ \text{and } \beta_m = K(\lambda_m)/m. \end{cases}$$

Therefore,

$$\sum \frac{\sin\{(t+x)\lambda_m - 2\gamma\}}{\lambda_m} = \sum \frac{\sin(2\eta m\pi + \alpha)\cos 2\eta\beta_m}{\lambda_m} +$$

$$+ \sum \frac{\cos(2\eta m\pi + \alpha)\sin 2\eta\beta_m}{\lambda_m}$$

$$(7.3.5) \qquad = \sum \frac{\sin(2\eta m\pi+\alpha)}{m\pi} + \sum O(m^{-2}),$$

since,

$$(7.3.6) \quad \sin 2\eta\beta_m = O(m^{-1}) \text{ and } \cos 2\eta\beta_m = 1 + O(m^{-2}).$$

For any $x \in (0,1)$, if $\delta \leq \min\{x/2,(1-x)/2\}$, then for $t \in N_\delta[x] = [x-\delta, x+\delta]$, it is true that $0 < 3\delta/2 \leq \eta \leq 1-3\delta/2 < 1$.

Hence, by Lemma 7.1, the first sum on the right of (7.3.5) is bounded independently of n, η and α, i.e., bounded independently of n and $t \in N_\delta[x]$. This shows that the first sum in $U_n^{(2)}$ is bounded independently of n and $t \in N_\delta[x]$.

Again, if $\tau = \pi(t-x)$ and $\alpha' = \alpha/\pi$, then by Lemma 7.3,

$$(7.3.7) \qquad (t-x)\lambda_m = (m+\alpha')\tau + \tau\theta_m/\pi.$$

Hence, using estimates of the type of (7.3.6), we obtain, in a similar way,

$$\sum \frac{\sin(t-x)\lambda_m}{\lambda_m} = \sum \frac{\sin(m+\alpha')\tau \cos(\tau\theta_m/\pi)}{\lambda_m} + \sum \frac{\cos(m+\alpha')\tau \sin(\tau\theta_m/\pi)}{\lambda_m}$$

$$(7.3.8) \qquad = \sum (m\pi)^{-1} \sin(m+\alpha')\tau + \sum O(m^{-2}).$$

By Lemma 7.2, the first sum on the right in (7.3.8) is bounded independently of n and $t \in N_\delta[x]$. We have, thus proved that $U_n^{(2)}$ is bounded independently of n and $t \in N_\delta[x]$.

In view of (7.1.9) and (7.3.3), it remains to consider the difference,

$$U_n^{(1)}(t,x) - K_n(t,x) = \sum \cos\left\{(t+x)\lambda_m - 2\gamma\right\} +$$

$$+ \sum \cos(t-x)\lambda_m - \Big\{1 +$$

$$+ \sum \cos(t+x)m\pi \Big\} -$$

$$- \sum \cos(t-x)m\pi$$

$$= I_1 + I_2 - I_3 - I_4, \text{ say.}$$

By (7.3.4) and (7.3.6),

$$I_1 = \sum \cos(2\eta m\pi + \alpha)\cos(2\eta\beta_m) - \sum \sin(2\eta m\pi + \alpha)\sin 2\eta\beta_m$$

$$= \sum \cos(2\eta m\pi + \alpha) - \sum \sin(2\eta m\pi + \alpha)\cdot O(m^{-1}) +$$

$$+ \sum O(m^{-2}).$$

By one argument similar to one given for (7.3.5) based on Lemma 7.1, it follows that I_1 is bounded independently of n and $t \in N_\delta[x]$.

A similar reasoning, based on Lemma 7.2, proves the boundedness of I_3, independently of n and $t \in N_\delta[x]$.

Finally, by (7.3.7),

$$I_2 - I_4 = \sum \cos(n+\alpha')\tau \cos(\tau\beta_n/\pi) -$$

$$- \sum \sin(n+\alpha')\tau \sin(\tau\beta_n/\pi) - \sum \cos n\tau.$$

By Lemma 7.2, the second sum on the right is bounded independently of n and $t \in N_\delta[x]$. Therefore, we consider,

$$\sum \cos(n+\alpha')\tau \cos(\tau\beta_n/\pi) - \sum \cos n\tau =$$

$$= -2 \sin \frac{\alpha'\tau}{2} \sum \sin(n+\alpha'/2)\tau + \sum O(n^{-2})$$

$$= U + \sum O(n^{-2}), \text{ say.}$$

Then,

$$|U| = \left| \frac{2\sin(\alpha'\tau/2)\ \sin(n+1+\alpha')\tau\ \sin n\tau}{\sin(\tau/2)} \right|$$

$$\leqslant \left| \frac{2\sin(\alpha'\tau/2)}{\sin(\tau/2)} \right| .$$

Since $0 < \tau \leqslant \delta$, it follows that this sum is bounded independently of n and $t \in N_\delta[x]$.

The proof of the theorem is, now, complete.

7.4. <u>PROOF OF THEOREM 7.2.</u> We have

$$K_n(t,x) = \left\{ 1 + \sum \cos(t+x)m\pi \right\} + \sum \cos(t-x)m\pi .$$

Choosing δ as in Theorem 7.1, for $t \in [0, x-\delta]$, we have $0 < \delta \leqslant (t+x)/2 < 1-\delta < 1$, and $0 < \delta \leqslant x-t < 1-\delta < 1$. Hence, by Lemma 7.1, $K_n(t,x)$ is bounded independently of n and $t \in [0, x-\delta]$. Similar will be the case when $t \in [x+\delta, 1]$.

Also[1],

$$(7.4.1)\quad U_n(t,x) = \frac{\sqrt{tx}}{2i} \int_{D_n - i\infty}^{D_n + i\infty} \frac{w\,\phi(w,x)\,J_\nu(tw)\,dw}{\ell\,w\,J_\nu'(w) + h\,J_\nu(w)},$$

for $0 \leq t < x \leq 1$, and for $0 \leq x < t \leq 1$,

$$(7.4.2)\quad U_n(t,x) = \frac{\sqrt{tx}}{2i} \int_{D_n - i\infty}^{D_n + i\infty} \frac{w\,\phi(w,t)\,J_\nu(xw)\,dw}{\ell\,w\,J_\nu'(w) + h\,J_\nu(w)},$$

where $\lambda_n < D_n < \lambda_{n+1}$,

$$(7.4.3)\quad \begin{cases} \phi(w,x) = (h+\ell\nu)\,\phi_1(w,x) - w\,\phi_2(w,x), \\ \phi_1(w,x) = J_\nu(w)\,Y_\nu(xw) - Y_\nu(w)\,J_\nu(xw) \text{ and} \\ \phi_2(w,x) = J_{\nu+1}(w)\,Y_\nu(xw) - Y_{\nu+1}(w)\,J_\nu(xw). \end{cases}$$

Using asymptotic expansions of $J_\nu(w)$ and $Y_\nu(w)$ [2], we obtain,

$$(7.4.4)\quad \left| w\,\phi_1(w,x) \right| \leq \frac{2C_1}{\sqrt{x}}\, e^{(1-x)|v|},$$

$$(7.4.5)\quad \left| w\,\phi_2(w,x) \right| \leq \frac{2C_2}{\sqrt{x}}\, e^{(1-x)|v|},$$

and

$$(7.4.6)\quad \left| \frac{w}{\ell\,w\,J_\nu'(w) + h\,J_\nu(w)} \right| \leq C_3 \sqrt{|w|}\, e^{-|v|},$$

where $w = D_n + iv$, $-\infty < v < \infty$, and $C_1, C_2, C_3, \ldots$, denote

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[1] Watson [103] p. 602. [2] Watson [103], p. 199.

suitable positive constants depending at most upon ν.

Now, if $0 \leq t \leq x-\delta$, by (7.4.1) and (7.4.3) to (7.4.6), it follows that

$$|U_n(t,x)| \leq C_4 \int_0^{\infty} e^{-(x-t)v}\, dv \leq C_4/\delta.$$

Similarly, if $x+\delta \leq t \leq 1$,

$$|U_n(t,x)| \leq C_4/\delta.$$

Hence, $U_n(t,x)$ is bounded independently of n and t, for any fixed $x \in (0,1)$, $t \in [0,x-\delta] \cup [x+\delta,1]$.

These conclusions, together with Theorem 7.1, prove the theorem.

7.5. <u>PROOF OF THEOREM 7.3.</u> By Theorem 7.2, for any fixed $x \in (0,1)$,

$$|\Delta_n(t,x)| \leq C,$$

where C is a positive constant independent of n and $t \in [0,1]$.

Also, for $f \in L^1[0,1]$, and any given $\varepsilon > 0$, there exists $\delta > 0$, such that[1],

$$\int_{x-\delta/2}^{x+\delta/2} |f(t)|\, dt < \varepsilon/2C.$$

Therefore,

$$(7.5.1) \qquad \left| \int_{x-\delta/2}^{x+\delta/2} f(t)\, \Delta_n(t,x)\, dt \right| < \varepsilon/2.$$

Now, let $E = [0,1] - (x-\delta/2, x+\delta/2)$. By the localization

[1] Natanson [73], p. 148.

theorem in Fourier series,

$$(7.5.2) \qquad \lim_{n\to\infty} \int_E f(t)\, K_n(t,x)\, dt = 0.$$

Moreover[1], the absolute convergence of

$$\int_E t^{1/2} f(t)\, dt$$

implies that

$$(7.5.3) \qquad \lim_{n\to\infty} \int_E f(t)\, U_n(t,x)\, dt = 0.$$

By taking $f \equiv 1$, we obtain,

$$\lim_{n\to\infty} \int_E U_n(t,x)\, dt = 0.$$

Hence, by the general convergence theorem[2], (7.5.3) is true for $f \in L^1[0,1]$.

The theorem, now, follows from (7.5.1) to (7.5.3).

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1) Watson [103], §§ 18.23, 18.32.

2) Hobson [49], pp. 422-425.