CHAPTER V

DEGREE OF APPROXIMATION OF PARTIAL SUMS OF CERTAIN FOURIER-BESSEL SERIES.

5.1 The Fourier- Bessel series of first type (FB-I), corresponding to a function $f \in L^1[0,1]$, is given by

$$(5.1.1) \qquad f(x) \sim \sum_{m=1}^{\infty} a_m \, J_\nu(x j_m), \qquad 0 \leq x \leq 1,$$

where $j_1 < j_2 < j_3 < \cdots$ denote the successive positive zeros of the Bessel function $J_\nu(t)$ of the first kind, of order $\nu \geqslant -1/2$, and

$$a_m = \frac{2}{J_{\nu+1}^{2}(j_m)} \int_0^1 t \, f(t) \, J_\nu(t j_m) \, dt.$$

Cooke[1] defined the modified Fourier-Bessel series

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[1] Cooke [25].

for $f \in L^1[0,1]$, as

$$(5.1.2) \quad f(x) \sim x^{\alpha} \sum_{m=1}^{\infty} \frac{2\, J_{\nu}(x j_m)}{J_{\nu+1}^{2}(j_m)} \int_0^1 t^{1-\alpha} f(t)\, J_{\nu}(t j_m)\, dt,$$

$$0 \leq x \leq 1.$$

Evidently, the series (5.1.2) reduces to the series (5.1.1), if $\alpha = 0$, and to a Fourier-trigonometric series, if $\alpha = \nu = 1/2$ or $\alpha = \frac{1}{2} = -\nu$.

For any function $f \in L^1[0,1]$, the series

$$(5.1.3) \qquad f(x) \sim \sum_{m=1}^{\infty} b_m\, J_{\nu}(x\lambda_m), \qquad 0 \leq x \leq 1,$$

where

$$b_m = \frac{\int_0^1 t\, f(t)\, J_{\nu}(t\lambda_m)\, dt}{\int_0^1 t\, J_{\nu}^{2}(t\lambda_m)\, dt},$$

and $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ are the successive positive zeros of $\ell\, t\, J_{\nu}'(t) + h\, J_{\nu}(t)$, with either $\ell = 0$ or $\nu + h/\ell > 0$, is called the Fourier-Bessel series of second type (FB-II) or Dini Series.

For $0 < a < b$ and $f \in L^1[a,b]$, the Fourier-Bessel series of third type (FB-III) is

$$(5.1.4) \qquad f(x) \sim \sum_{m=1}^{\infty} d_m\, c_{\nu}(x\gamma_m, b\gamma_m), \qquad a \leq x \leq b,$$

where $c_\nu(\alpha,\beta) = J_\nu(\alpha) Y_\nu(\beta) - J_\nu(\beta) Y_\nu(\alpha)$, $\gamma_1 < \gamma_2 < \gamma_3 < \cdots$ are the successive positive zeros of $c_\nu(at,bt)$ and

$$d_m = \frac{\pi^2 \gamma_m^2 J_\nu^2(a\gamma_m) \int_a^b t\, f(t)\, c_\nu(t\gamma_m, b\gamma_m)\, dt}{2\left\{J_\nu^2(a\gamma_m) - J_\nu^2(b\gamma_m)\right\}}.$$

Let $S_n(\alpha,x)$ and $S_n(x)$ denote, respectively, the n-th partial sums of series (5.1.2) and (5.1.4). Then,

$$(5.1.5) \qquad S_n(\alpha,x) = x^c \int_0^1 t^{1-\alpha} f(t)\, T_n(t,x)\, dt,$$

where

$$(5.1.6) \qquad T_n(t,x) = \sum_{m=1}^{n} \frac{2 J_\nu(xj_m)\, J_\nu(tj_m)}{J_{\nu+1}^2(j_m)};$$

and

$$(5.1.7) \qquad S_n(x) = \int_a^b t\, f(t)\, R_n(t,x)\, dt,$$

where

$$(5.1.8) \quad R_n(t,x) = \frac{\pi^2}{2} \sum_{m=1}^{n} \frac{\gamma_m^2 J_\nu^2(a\gamma_m) c_\nu(x\gamma_m, b\gamma_m) c_\nu(t\gamma_m, b\gamma_m)}{J_\nu^2(a\gamma_m) - J_\nu^2(b\gamma_m)}$$

For the Fourier series of a function $f \in L^1[-\pi,\pi]$ and periodic with period 2π, the following theorem is known[1]:

THEOREM 5.A. If $\int_0^t |\varphi(t)|\, dt = o(t)$, as $t \to 0+$, where $\varphi(t) = f(x+t) + f(x-t) - 2f(x)$, then the n-th partial

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[1] Zary [11], pp. 141-142.

sum, $s_n(x)$, of the Fourier series of f has the order $o(\log n)$, almost everywhere.

The following theorem concerning the Fourier series of a real even function $f \in L^1$ has been proved by Otto-Szász[1]:

THEOREM 5.B. Let $\int_0^x |f(t)|\,dt = o\left(\frac{x}{\log 1/x}\right)$, as $x \to 0+$. Then $s_n(x)$, the n-th partial sum of the Fourier series (a cosine series) has the order given by

$$s_n(x) = o(\log \log n).$$

Wilson[2] studied a similar problem for Legendre series. Khoti[3] and Schorberg[4] investigated the orders of the partial sums of the series (5.1.1) and (5.1.3), respectively.

In this chapter, we investigate the degrees of approximation of the generating function by the partial sums of its modified Fourier-Bessels series (5.1.2) and also by the partial sums of FB-III. Khoti's theorems become a particular case of our Theorems 5.1 and 5.2 (below), by putting $\alpha = 0$ there. Also Theorem 5.B is a particular case of our theorem 5.2, when $\alpha = -\nu = 1/2$ and

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1) Otto-Szász [85].
2) Wilson [107].
3) Khoti [95].
4) Schorberg [81].

Theorem 5.A follows from Theorem 5.3, when $\nu = \pm 1/2$.

Our theorems are as follows[1]:

THEOREM 5.1. *Let* $g(x) = x^{1/2-\alpha} f(x)$ *and let* $\varphi_x(u) = g(x \pm u) - g(x)$. *If* $g \in L^1[0,1]$, g *vanishes at* 0 *and* 1 *and if*

$$(5.1.9) \qquad \phi(t) = \int_0^t |\varphi_x(u)|\,du = o(t), \text{ as } t \to 0+,$$

then for $0 < x < 1$,

$$S_n(\alpha, x) - f(x) = o(\log n), \text{ as } n \to \infty,$$

almost everywhere.

THEOREM 5.2. *Let* g, φ_x *and* ϕ *be as defined in Theorem 5.1,* $g \in L^1[0,1]$, $g(0) = g(1) = 0$, *and let*

$$(5.1.10) \qquad \phi(t) = o\left(\frac{t}{\log 1/t}\right), \text{ as } t \to 0+.$$

Then for $0 < x < 1$,

$$S_n(\alpha, x) - f(x) = o(\log \log n), \text{ as } n \to \infty.$$

THEOREM 5.3. *If* $F(x) = x^{1/2} f(x)$, $F \in L^1[a,b]$, $0 < a < b$, $F(a) = F(b) = 0$ *and if*

$$(5.1.11) \qquad \Theta(t) = \int_0^t |\theta_x(u)|\,du = o(t), \text{ as } t \to 0$$

where $\theta_x(u) = F(x \pm u) - F(x)$; *then for* $a < x < b$,

$$S_n(x) - f(x) = o(\log n),$$

almost everywhere.

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[1] Agrawal and Patel [2]; [5].

THEOREM 5.4. *If in the hypothesis of Theorem 5.3, (5.1.11) is replaced by*

$$(5.1.12) \qquad \theta(t) = o\left(\frac{t}{\log 1/t}\right), \text{ as } t \to 0+,$$

then for $a < x < b$,

$$S_n(x) - f(x) = o(\log\log n).$$

5.2. The following properties of $T_n(t,x)$ and $R_n(t,x)$ are needed to complete the proofs of our theorems (K_1, K_2, K_3, ... , are suitabale positive constants):

LEMMA 5.1. (Young[1]). For $0 \le x \le 1$ and $0 \le t \le 1$, it is true that,

$$(5.2.1) \qquad \sqrt{xt}\, |T_n(t,x)| \le K_1 n, \qquad n \geq 1;$$

and for $0 \le x \le 1$, $0 \le t \le 1$, $x \ne t$,

$$(5.2.2) \qquad \sqrt{xt}\, |T_n(t,x)| \le \frac{K_2}{|t-x|}, \qquad n \geq 1.$$

LEMMA 5.2. (Young[2]). For $\nu \geq -1/2$ and $0 < x < 1$, we have

$$(5.2.3) \qquad \int_0^x \sqrt{xt}\, T_n(t,x)\, dt \to 1/2, \text{ as } n \to \infty,$$

and we have uniformly, for $\varepsilon > 0$, $\varepsilon \le x \le 1-\varepsilon$,

$$(5.2.4) \qquad \int_0^1 \sqrt{xt}\, T_n(t,x)\, dt \to 1, \quad \text{as } n \to \infty.$$

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1) Young [111], p. 293. Young [111], p. 302.

LEMMA 5.3. For $0 < a < x < b$,

$$(5.2.5) \qquad \lim_{n \to \infty} \int_a^b t^{\nu+1} R_n(t,x)\, dt = x^{\nu};$$

and

$$(5.2.6) \qquad \lim_{n \to \infty} \int_a^b \sqrt{xt}\, R_n(t,x)\, dt = 1.$$

PROOF. (5.2.5) has been proved by Titchmarsh[1].
The integral (5.2.6) follows from (3.5.9), since $c>0$ is arbitrary there.

5.3. PROOF OF THEOREM 5.1. We have, by (5.1.5),

$$x^{1/2-\alpha} S_n(\alpha,x) - x^{1/2-\alpha} f(x) = \int_0^1 \Big\{ t^{1/2-\alpha} f(t) - x^{1/2-\alpha} f(x) \Big\} \sqrt{xt}\, T_n(t,x)\, dt + x^{1/2-\alpha} f(x) \Big\{ \int_0^1 \sqrt{xt}\, T_n(t,x)\, dt - 1 \Big\},$$

$$(5.3.1) \qquad = I + I', \text{ say.}$$

Let $\varepsilon > 0$ be given. In view of (5.1.9), let us choose $\delta > 0$ such that

$$(5.3.2) \qquad \frac{\phi(t)}{t} < \varepsilon, \quad \text{for } 0 < t \leq \delta.$$

Let n be chosen so large that $\delta > 1/n$. Then,

$$I = \left(\int_0^{x-\delta} + \int_{x-\delta}^{x-1/n} + \int_{x-1/n}^{x} + \int_x^{x+1/n} + \int_{x+1/n}^{x+\delta} + \int_{x+\delta}^{1} \right) \times$$

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1) Titchmarsh [96], p. xv.

$$\times\left\{ t^{(1/2)-\alpha} f(t) - x^{(1/2)-\alpha} f(x) \right\} \sqrt{xt}\, T_n(t,x)\, dt$$

(5.3.3) $= I_1 + I_2 + I_3 + I_4 + I_5 + I_6$, say.

By the analogue of Riemann-Lebesgue Lemma[1], we get,

(5.3.4) $I_1 = o(1), \quad I_6 = o(1), \quad$ as $n \to \infty$.

By (5.2.2), we obtain, by using (5.3.2),

$$|I_2| \leq K_2 \int_{x-\delta}^{x-1/n} \frac{|g(t)-g(x)|}{x-t}\, dt =$$

$$= K_2 \int_{1/n}^{\delta} \frac{|g(x-u)-g(x)|}{u}\, du$$

$$= K_2 \left[\frac{\phi(\delta)}{\delta} - \frac{\phi(1/n)}{1/n} + \int_{1/n}^{\delta} \frac{\phi(u)}{u^2}\, du \right]$$

$$K_2 \left[\varepsilon + \varepsilon \log(\delta n) \right] =$$

(5.3.5) $= o(\log n)$, as $n \to \infty$.

Similarly,

(5.3.6) $I_5 = o(\log n)$, as $n \to \infty$.

Further, using (5.2.1), we get,

$$|I_3| \leq K_1 n \int_{x-1/n}^{x} |g(t)-g(x)|\, dt =$$

$$= K_1 n \int_0^{1/n} |g(x-u)-g(x)|\, du$$

$$= K_1 n\, \phi(1/n) = K_1 \varepsilon, \quad \text{by (5.3.2).}$$

Therefore,

(5.3.7) $I_3 = o(1)$, as $n \to \infty$.

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1) Watson [103], p. 589.

Similarly,

(5.3.8) $\qquad I_4 = o(1), \quad \text{as } n \to \infty.$

By (5.3.3) to (5.3.8), it follows that

(5.3.9) $\qquad I = o(\log n), \text{ as } n \to \infty.$

Now, by (5.2.4), it is true that for every x, $0 < x < 1$,

(5.3.10) $\qquad I' = o(1), \quad \text{as } n \to \infty.$

Since, $x > 0$, the theorem follows from (5.3.1), (5.3.9) and (5.3.10).

PROOF OF THEOREM 5.2. For any given $\varepsilon > 0$, in view of (5.1.10), let us choose $\eta > 0$, such that

(5.3.11) $$\frac{\phi(t)}{t} < \frac{\varepsilon}{\log 1/t}, \quad \text{for } 0 < t \leq \eta.$$

Choosing n so large that $\eta > 1/n$, we now evaluate I and I' from (5.3.1). We have,

$$I = \left\{ \int_0^{x-\eta} + \int_{x-\eta}^{x-1/n} + \int_{x-1/n}^{x} + \int_x^{x+1/n} + \int_{x+1/n}^{x+\eta} + \int_{x+\eta}^{1} \right\} \times$$

$$\times \left\{ g(t) - g(x) \right\} \sqrt{\pi t}\, T_n(t,x)\, dt,$$

(5.3.12) $\qquad = I_{11} + I_{22} + I_{33} + I_{44} + I_{55} + I_{66}, \text{ say.}$

As in (5.3.4),

(5.3.13) $\qquad I_{11} = o(1), \quad I_{66} = o(1), \quad \text{as } n \to \infty.$

Again, using Lemma 5.1 and (5.3.11), as in (5.3.5), we obtain,

$$(5.3.14) \qquad I_{22} = o(\log\log n), \quad \text{as } n \to \infty,$$

and

$$(5.3.15) \qquad I_{55} = o(\log\log n), \quad \text{as } n \to \infty.$$

In the same way, using Lemma 5.1 and (5.3.11), we obtain,

$$(5.3.16) \qquad I_{33} = o(1), \quad I_{44} = o(1), \quad \text{as } n \to \infty.$$

By (5.3.1), (5.3.10) and (5.3.12) to (5.3.16), the proof of the theorem is completed.

Theorems 5.3 and 5.4 can be proved in a way similar to the proofs of Theorems 5.1 and 5.2, respectively, by using Lemma 5.3 and Lemmas 2.4, 3.4 and 3.6 from previous chapters.

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