CHAPTER VI

ORDER OF COEFFICIENTS OF FOURIER-BESSEL SERIES OF A SPECIAL KIND.

6.1 Let, for $\nu \geqslant -1/2$,

(6.1.1) $$Q_\nu(\alpha,\beta) = J_\nu(\alpha)\, Y'_\nu(\beta) - J'_\nu(\beta)\, Y_\nu(\alpha).$$

Denote by $j_1 < j_2 < j_3 < \dots$ the successive positive zeros of $J_\nu(t)$ and by $k_1 < k_2 < k_3 < \dots$ those of $\delta(k)$, where, for $0 < a < b$,

(6.1.2) $$\delta(k) = J'_\nu(bk)\, Y'_\nu(ak) - J'_\nu(ak)\, Y'_\nu(bk).$$

Given $f \in L^2[0,1]$, the Fourier-Bessel series of first type (FB-I),

(6.1.3) $$f(x) \sim \sum_{n=1}^{\infty} a_n\, J_\nu(x j_n), \quad 0 \leqslant x \leqslant 1,$$

where

$$a_m = \frac{2}{J_{\nu+1}^2(j_m)} \int_0^1 t\, f(t)\, J_\nu(tj_m)\, dt,$$

is well known.

F. Kito[1] introduced the following series of special kind, known as Fourier-Bessel series of fourth type (FB-IV) in this thesis, corresponding to $f \in L^1[a,b]$, $0 < a < b$, while studying the vibrations of a cylindrical shell immersed in water:

$$(6.1.4) \qquad f(x) \sim \sum_{m=1}^{\infty} p_m\, Q_\nu(xk_m, ak_m), \quad a \leq x \leq b,$$

where ν is a positive integer and

$$p_m = \frac{\int_a^b t\, f(t)\, Q_\nu(tk_m, ak_m)\, dt}{\int_a^b t\, Q_\nu^2(tk_m, ak_m)\, dt}$$

$$(6.1.5) \qquad = \frac{\pi^2\ k_m^2 \int_a^b t\, f(t)\, Q_\nu(tk_m, ak_m)\, dt}{2\left[\left(1 - \frac{\nu^2}{b^2k_m^2}\right)\frac{J_\nu'^2(ak_m)}{J_\nu'^2(bk_m)} - \left(1 - \frac{\nu^2}{a^2k_m^2}\right)\right]}$$

Kito[2] used $G_\nu(z)$ instead of $Y_\nu(z)$, where $G_\nu(z) = -(\pi/2)Y_\nu(z)$. He has also studied the convergence of this series.

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1) Kito [56].   2) Kito [56]; [57].

G.P.Tolstov[1] has studied the series (6.1.3) for a function satisfying differentiability conditions on it, to consider the order of coefficients of its terms and certain aspects of its uniform and absolute convergence. D.P.Khoti[2] has established similar results for series FB-III. M.G.Scherberg[3] also dealt with the convergence of series FB-II. He investigated the order of coefficients of this series. G.N.Watson[4] and G.P.Tolstov[5] have also found the order estimates of the coefficients of series (6.1.3) under the bounded variation and square integrability conditions, respectively.

In the present chapter, we establish such estimates under various differentiability, bounded variation and square integrability conditions. We also infer the uniform and absolute convergence under some of these conditions.

Our theorems are as follows[6]:

<u>THEOREM 6.1. Let</u> f <u>be a bounded and twice differentiable function defined for</u> $0 < a \leq x \leq b$, <u>such that</u> $f(a) = f(b) = 0$, $f'(a) = f'(b) = 0$ <u>and</u> f'' <u>is bounded (this</u>

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1)Tolstov [99], pp. 228-233. 2)Khoti [53].
3)Scherberg [81]. 4)Watson [103], p. 595.
5)Tolstov [99], pp. 223-224. 6)Agrawal and Patel [4]; [8].

derivative may not exist at certain points). Then the coefficients p_n of the series (6.1.4) of the function f satisfy the relation

$$|p_n| = O(1/k_n), \quad \text{as } n \to \infty.$$

THEOREM 6.2. Let f be a function satisfying the conditions of theorem 6.1. Then the series (6.1.4) converges uniformly and absolutely on [a,b], for $\nu \geq -1/2$.

THEOREM 6.3. Let f be a function defined on [a,b], such that f is differentiable 2s times (s > 1) and such that $f(a) = f'(a) = f''(a) = \ldots = f^{(2s-1)}(a) = 0$;

$$f(b) = f'(b) = f''(b) = \ldots = f^{(2s-1)}(b) = 0;$$

and $f^{(2s)}(x)$ is bounded (this derivative may not exist at certain points). Then the coefficients p_n of the series (6.1.4) of the function f satisfy the relation

$$|p_n| = O(1/k_n^{2s-1}), \text{ as } n \to \infty.$$

THEOREM 6.4. Let f be a function satisfying the conditions of Theorem 6.3 for $s \geq 1$. Then, for $\nu \geq -1/2$, the series (6.1.4) converges uniformly and absolutely on [a,b].

THEOREM 6.5. Let f be a function of bounded variation on [a,b]. Then,

$$\left| p_n\, Q_\nu(xk_n, ak_n) \right| = O(1/k_n), \text{ as } n \to \infty.$$

THEOREM 6.6. *If* $f \in L^2[a,b]$, *then*,

$$\int_a^b t\, f(t)\, Q_\nu(tk_m, ak_m)\, dt = o(1/k_m), \text{ as } m \to \infty,$$

and the general term of the series (6.1.4) *converges to zero for every* $x \in [a,b]$.

6.2. The following lemmas are needed to prove the above theorems ($K_1, K_2, K_3, \ldots$, denote suitable positive constants, throughout the chapter):

LEMMA 6.1. *If* $k > 0$ *is sufficiently large and* ν *is a real number, then*

$$|Q_\nu(xk, ak)| < \frac{K_1}{k\sqrt{ax}},$$

where $0 < a \leq x \leq b$.

PROOF. We have[1],

$$(6.2.1)\quad Q_\nu(xk, ak) = \frac{H_\nu^{(2)}(xk)\, H_\nu^{(1)'}(ak) - H_\nu^{(1)}(xk)\, H_\nu^{(2)'}(ak)}{2i}.$$

Also, it is true that[2],

$$(6.2.2)\quad \left|H_\nu^{(1)}(z)\right| < \frac{K_2}{|z|^{1/2}}, \quad \left|H_\nu^{(2)}(z)\right| < \frac{K_2}{|z|^{1/2}},$$

for all sufficiently large real values of z and[3],

$$(6.2.3)\quad H_{-\nu}^{(1)}(z) = e^{\nu\pi i} H_\nu^{(1)}(z), \quad H_{-\nu}^{(2)}(z) = e^{-\nu\pi i} H_\nu^{(2)}(z).$$

The estimations (6.2.2) are true for $\nu \geq 0$, but in

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1) Watson [103], §3.61, relations (1) and (3).

2) Watson [103], p.211. 3) Watson [103], p. 74.

view of (6.2.3), (6.2.2) become true for all real values of ν and for all sufficiently large real values of z.

Using recurrence relations[1], we also have,

(6.2.4) $\left|H_\nu^{(1)\prime}(z)\right| < \frac{K_2}{\sqrt{|z|}}, \quad \left|H_\nu^{(2)\prime}(z)\right| < \frac{K_2}{\sqrt{|z|}},$

for all sufficiently large real z and ν any real number.

The lemma, now, follows from (6.2.1), (6.2.2) and (6.2.4).

<u>LEMMA 6.2. For</u> $\nu \geq -1/2$ <u>and sufficiently large real</u> $k > 0$,

$$k\sqrt{x}\, Q_\nu(xk, ak) = K_3 \cos(xk - ak) + \frac{\sigma(k)}{xk}, \quad 0 < a \leq x,$$

<u>where</u> $\sigma(k)$ <u>remains bounded as</u> $k \to \infty$.

<u>PROOF.</u> Using the asymptotic expressions[2],

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\left\{\sin\left(x - \frac{\nu\pi}{2} + \frac{\pi}{4}\right) + \frac{\delta_\nu(x)}{x}\right\},$$

$$Y_\nu(x) = \sqrt{\frac{2}{\pi x}}\left\{\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \frac{\varphi_\nu(x)}{x}\right\},$$

where $\delta_\nu(x)$ and $\varphi_\nu(x)$ remain bounded as $x \to \infty$, and the recurrence relations[3], we obtain,

$$Q_\nu(xk, ak) = \frac{1}{2} J_\nu(xk)\left\{Y_{\nu-1}(ak) - Y_{\nu+1}(ak)\right\} - \frac{1}{2}\left\{J_{\nu-1}(ak) - J_{\nu+1}(ak)\right\} Y_\nu(xk)$$

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[1] Watson [103], p. 74. [2] Tolstov [99], p. 213.

[3] Watson [103], p. 66.

$$= \frac{1}{k\sqrt{ax}} \Big[\sin(xk - \frac{\nu\pi}{2} + \frac{\pi}{4}) \Big\{ \sin(ak - \frac{\nu\pi}{2} + \frac{\pi}{4}) -$$

$$- \sin(ak - \frac{\nu\pi}{2} - \frac{3\pi}{4}) \Big\} - \sin(xk - \frac{\nu\pi}{2} - \frac{\pi}{4}) \times$$

$$\times \Big\{ \sin(ak - \frac{\nu\pi}{2} + \frac{3\pi}{4}) - \sin(ak - \frac{\nu\pi}{2} - \frac{\pi}{4}) \Big\} \Big] +$$

$$+ \frac{\sigma(k)}{k^2 x \sqrt{x}} ,$$

where $\sigma(k)$ remains bounded as $k \to \infty$.

The lemma, now, easily follows from this.

LEMMA 6.3. For $\nu \geqslant -1/2$ and sufficiently large real $k > 0$,

$$\frac{K_4}{k^2} \leqslant \int_a^b x\, Q_\nu^{\,2}(xk,ak)\, dx \leqslant \frac{K_5}{k^2} .$$

PROOF. We have, by Lemma 6.1,

$$\int_a^b x\, Q_\nu^{\,2}(xk,ak)\, dx \leqslant \frac{K_1^{\,2}}{ak^2} \int_a^b dx = \frac{K_5}{k^2} .$$

This proves the right hand part of the inequality.

Also,

$$\int_a^b x\, Q_\nu^{\,2}(xk,ak)\, dx = \frac{1}{k^2} \int_{ak}^{bk} t\, Q_\nu^{\,2}(t,ak)\, dt,$$

and from Lemma 6.2,

$$t\,k\,Q_\nu^{\,2}(t,ak) \geqslant K_3^{\,2} \cos^2(t-ak) - K_6/t.$$

Hence,

$$\int_a^b x\, Q_\nu^{\,2}(xk,ak)\, dx \geqslant \frac{K_3^{\,2}}{k^3} \int_{ak}^{bk} \cos^2(t-ak)\, dt - \frac{K_6}{k^3} \log \frac{b}{a} =$$

$$= \frac{K_4}{k^2} ,$$

K_4 is positive when k is sufficiently large.

The proof of the lemma is, now, complete.

<u>LEMMA 6.4.</u> (Naylor[1]). For sufficiently large m and any real ν,

$$k_m = \frac{m\pi}{b-a} + \frac{(4\nu^2+3)(b-a)}{8m\pi ab} + O(m^{-3}).$$

6.3. <u>PROOF OF THEOREM 6.1.</u> Let $F(x) = x^{1/2} f(x)$. Then the conditions of the theorem are also true for F, i.e.,

$$(6.3.1) \qquad \begin{cases} F(a) = F'(a) = 0,\ F(b) = F'(b) = 0 \text{ and } F'' \text{ is} \\ \text{bounded, save at certain points.} \end{cases}$$

Since[2] $K_7 J_\nu(x) + K_8 Y_\nu(x)$ is a general solution of the Bessel's equation,

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0,$$

for arbitrary K_7 and K_8, it follows that $Q_\nu(x, ak)$ is a solution of this equation. If we put $z = x^{1/2} y$, then $z(x) = x^{1/2} Q_\nu(x, ak)$ becomes a solution of

$$x^2 z'' + (x^2 - \mu) z = 0,$$

where $\mu = \nu^2 - 1/4$. Finally, by substituting $x = tk$, we observe that $z(t) = (tk)^{1/2} Q_\nu(tk, ak)$ is a solution of

$$(6.3.2) \qquad z'' + (k^2 - \mu/t^2) z = 0.$$

Let $u(t) = t^{1/2} Q_\nu(tk, ak)$. Since $u(t)$ differs from

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1) Naylor [72], p. 70. 2) Tolstov [99], p. 203 ; Bowman [18], p. 116.

$z(t)$ only by a constant, $u(t)$ is also a solution of (6.3.2). Hence, we have

$$u'' + (k^2 - \mu/t^2)\, u = 0,$$

that is,

$$u = \frac{1}{k^2}\left(\frac{\mu}{t^2} - u''\right). \tag{6.3.3}$$

Let us, now, consider

$$I(k) = \int_a^b x\, f(x)\, Q_\nu(xk, ak)\, dx$$

$$= \int_a^b F(x)\, u(x)\, dx$$

$$= \frac{1}{k^2}\int_a^b F(x)\left\{\frac{\mu}{x^2}\, u(x) - u''(x)\right\} dx$$

$$= \frac{1}{k^2}\int_a^b \left\{F(x) \cdot \frac{\mu}{x^2} - F''(x)\right\} u(x)\, dx +$$

$$+ \frac{1}{k^2}\int_a^b \left\{F''(x)\, u(x) - F(x)\, u''(x)\right\} dx$$

$$= \frac{1}{k^2}\int_a^b \left\{\frac{\mu}{x^2}\, F(x) - F''(x)\right\} u(x)\, dx +$$

$$+ \frac{1}{k^2}\Big[F'(x)\, u(x) - F(x)\, u'(x)\Big]_a^b . \tag{6.3.4}$$

Now,

$$u'(x) = \frac{1}{2} x^{-1/2}\, Q_\nu(xk, ak) + kx^{1/2}\left\{J_\nu'(xk)\, Y_\nu'(ak) - J_\nu'(ak)\, Y_\nu'(xk)\right\};$$

so that, if k is a zero of $S(k)$,

$$u'(a) = \frac{1}{2} a^{-1/2} Q_\nu(ak, ak) \quad \text{and} \quad u'(b) = \frac{1}{2} b^{-1/2} Q_\nu(bk, ak).$$

By Lemma 6.1, it, now follows that $u(a)$, $u(b)$, $u'(a)$ and $u'(b)$ are all bounded. Therefore, by (6.3.1) and (6.3.4),

$$I(k_n) = \frac{1}{k_n^2} \int_a^b \left\{ \frac{\mu}{x^2} F(x) - F''(x) \right\} u(x)\, dx,$$

for each n.

Choosing K_9, such that,

$$\left| \frac{\mu}{x^2} F(x) - F''(x) \right| \leqslant K_9, \qquad x \in E,$$

where $[a,b] - E$, a set of measure zero, is the set of points at which F'' is not bounded, we have by Schwarz inequality,

$$\left| I(k_n) \right| \leqslant \frac{K_9}{k_n^2} \sqrt{b-a} \sqrt{\int_a^b |u(x)|^2\, dx}$$

$$\text{(6.3.5)} \qquad \leqslant \frac{K_{10}}{k_n^3}, \qquad \text{by Lemma 6.3.}$$

By (6.1.5), (6.3.5) and Lemma 6.3, the proof of the theorem is completed.

6.4. <u>PROOF OF THEOREM 6.2.</u> Since f satisfies the hypothesis of Theorem 6.1, using its conclusion and Lemma 6.1, we get,

$$\text{(6.4.1)} \qquad \left| p_n\, Q_\nu(xk_n, ak_n) \right| \leqslant \frac{K_{11}}{k_n^2}.$$

Also, by Lemma 6.4, for sufficiently large m,

(6.4.2) $$k_m \geqslant m/2.$$

The uniform and absolute convergence of (6.1.4) in $[a,b]$, now, follows from (6.4.1) and (6.4.2).

6.5. PROOF OF THEOREM 6.3. Let $F(x) = x^{1/2} f(x)$ and $u(x) = x^{1/2} Q_\nu(xk_m, ek_m)$. Then as in the proof of Theorem 6.1,

$$I(k_m) = \frac{1}{k_m^2} \int_a^b \left\{ \frac{\mu}{x^2} F(x) - F''(x) \right\} u(x)\, dx$$

$$= \frac{1}{k_m^2} \int_a^b F_1(x)\, u(x)\, dx,$$

where $F_1(x) = \frac{\mu}{x^2} F(x) - F''(x)$.

Now, F_1 also satisfies the conditions of Theorem 6.1, so that

$$I(k_m) = \frac{1}{k_m^4} \int_a^b F_2(x)\, u(x)\, dx,$$

where $F_2(x) = \frac{\mu}{x^2} F_1(x) - F_1''(x)$.

If $s > 2$, F_2 satisfies the hypothesis of Theorem 6.1 and the argument can be repeated till we obtain,

$$I(k_m) = \frac{1}{k_m^{2s}} \int_a^b F_s(x)\, u(x)\, dx,$$

where $F_s(x) = \frac{\mu}{x^2} F_{s-1}(x) - F''_{s-1}(x)$.

Let G be the set on which $f^{(2s)}(x)$ is bounded. Then $F^{(2s)}(x)$ is also is bounded on G, and mes. $\left([a,b] - G\right) = 0$.

Hence, as in the proof of Theorem 6.1,

$$|I(k_m)| \leqslant \frac{K_{12}}{k_m^{2s+1}} . \tag{6.5.1}$$

The theorem, now, follows from (6.5.1) and Lemma 6.3.

6.6. PROOF OF THEOREM 6.4. By Theorem 6.3, Lemma 6.1, and (6.4.2), we have,

$$\left|p_m\, Q_\nu(xk_m, ak_m)\right| \leqslant \frac{K_{13}}{m^{2s}} , \quad m = 1, 2, 3, \ldots , \quad s \geqslant 1.$$

This proves the theorem.

6.7. PROOF OF THEOREM 6.5. By hypothesis, $t^{1/2} f(t)$ is of bounded variation, so that we may write,

$$t^{1/2} f(t) = g_1(t) - g_2(t),$$

where g_1 and g_2 are bounded monotone increasing functions of t on [a,b]. We, therefore, have

$$\int_a^b t\, f(t)\, Q_\nu(tk_m, ak_m)\, dt = \int_a^b t^{1/2}\, g_1(t)\, Q_\nu(tk_m, ak_m)\, dt - \int_a^b t^{1/2}\, g_2(t)\, Q_\nu(tk_m, ak_m)\, dt$$

$$= I_1 + I_2, \text{ say.} \tag{6.7.1}$$

Using the second mean value theorem, we get,

$$I_1 = g_1(a) \int_a^{\eta} t^{1/2} Q_\nu(tk_n, ak_n)\, dt + $$
$$+ g_1(b) \int_{\eta'}^{b} t^{1/2} Q_\nu(tk_n, ak_n)\, dt$$

for some η, η' in $[a,b]$. Hence, by Lemma 6.2,

$$I_1 = g_1(a) \int_a^{\eta} \left\{ \frac{K_3}{k_n^2} \cos(tk_n - ak_n) + \frac{\sigma(k_n)}{t\, k_n^2} \right\} dt + $$
$$+ g_1(b) \int_{\eta'}^{b} \left\{ \frac{K_3}{k_n^2} \cos(tk_n - a\, k_n) + \frac{\sigma(k_n)}{t\, k_n^2} \right\} dt$$

(6.7.2) $$= O(1/k_n^2).$$

Similarly,

(6.7.3) $$I_2 = O(1/k_n^2).$$

By (6.7.1) to (6.7.3),

(6.7.4) $$\int_a^b t\, f(t)\, Q_\nu(tk_n, ak_n)\, dt = O(1/k_n^2).$$

Using Lemmas 6.1 and 6.3 and the estimation (6.7.4), we obtain,

$$\left| D_n\, Q_\nu(xk_n, ak_n) \right| = O(1/k_n), \quad \text{as } n \to \infty.$$

This proves the theorem.

6.8. <u>PROOF OF THEOREM 6.6.</u> We set $F(x) = x^{1/2} f(x)$, $a \le x \le b$. Then $F \in L^2[a,b]$. Hence, by Bessel's inequality,

$$\sum_{n=1}^{\infty} p_n^2 \, \| \sqrt{x} \, Q_\nu(xk_n, ak_n) \|_2^2 \leqslant \| F \|_2^2 .$$

It follows that the series

$$\sum_{n=1}^{\infty} p_n^2 \int_a^b x \, Q_\nu^2(xk_n, ak_n) \, dx$$

is convergent, so that,

$$(6.8.1) \qquad \lim_{n \to \infty} p_n^2 \int_a^b x \, Q_\nu^2(xk_n, ak_n) \, dx = 0.$$

Also, by Lemma 6.3,

$$(6.8.2) \qquad \int_a^b x \, Q_\nu^2(xk_n, ak_n) \, dx \geqslant \frac{K_4}{k_n^2} .$$

Therefore, by (6.8.1) and (6.8.2),

$$(6.8.3) \qquad p_n = o(k_n), \text{ as } n \to \infty.$$

Again, by Lemma 6.3,

$$\left| \int_a^b x \, f(x) \, Q_\nu(xk_n, ak_n) \, dx \right| \leqslant |p_n| \frac{K_5}{k_n^2} = o(k_n) \cdot \frac{K_5}{k_n^2} ,$$

as $n \to \infty$.

Hence,

$$\int_a^b x \, f(x) \, Q_\nu(xk_n, ak_n) \, dx = o(1/k_n), \text{ as } n \to \infty.$$

This proves the first part of the theorem. The second part of the theorem follows from (6.8.3) and Lemma 6.1.

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