Chapter 5

FRACTIONAL CALCULUS USING WAVELET PACKETS

5.1 Introduction

Fractional Calculus is the generalization of operations of differentiation and integration to non-integer order. The numerical solution of differential equations with integer order has been developed for a long time and has been a standard topic in numerical and computational Mathematics.

Differentiation and Integration are two of the most commonly used operations in calculus. Symbolically, $\frac{d^n f(x)}{dx^n}$ and $\frac{d^{-m} f(x)}{dx^{-m}}$ respectively where m and n are non-negative integers. The basic concept of fractional derivatives and fractional integration can be obtained if m and n are non-integer real values or even imaginary values. Most of the methods used for solving fractional differential equations are based on the approximation of the definitions. i.e. (i) Grunwald-Letnikov and (ii) Riemann Liouville (refer [76]) definitions. But they are complicated and time consuming.

For many engineering systems described by partial differential equations of time-varying differential equations, when evaluated in the Laplace domain, fractional functions or transcendental functions of s result. For e.g. in hole diffusion of transistors, problems in thermal processes, in electromagnetic devices, in transmission lines and in percolation processes often have mathematical models involving $\sqrt{s}$,$\sqrt{s^2+1}$,$e^{-\sqrt{s}}$ etc. To find their inverse Laplace Transforms is not a trivial matter.

The use of fractional calculus of modelling physical system has been widely considered in (refer [[47],[64]]). The differential equations involving derivatives of non-integer order have shown to be adequate models for various physical phenomena in areas like damping laws, diffusion processes etc. Recently, Atanackovic and Stankovic [2] have analyzed lateral motion of an elastic column fixed at one end and loaded at the other in terms of a system of fractional differential equations. Gejji V. and Bahakhani [27] have presented analysis of a system of fractional differential equations. They have studied existence, uniqueness and stability of solutions of a system of fractional differential equations. Other applications are in areas like electro-magnetics, electro-chemistry, arterial science, the theory of ultra slow processes and special functions.

In 1975,C. F. Chen and C. H. Hsiao [13] developed the Walsh Operational matrix for solving variational problems, state equations etc. In 1977, C.F. Chen and Y. T. Tsay [12] used these Walsh operational matrices for fractional calculus and gave their applications to distributed systems. Juinn-Lin Wu and Chin-Hsing Chen [75] have proposed a new numerical method based on the operational matrices of the orthogonal functions and solving fractional calculus and partial differential equations.

Using the operational matrix of an orthogonal function to perform integration for solving, identifying and optimizing a linear dynamic system has several advantages.

- The method is computer oriented.Thus solving higher order differential equation becomes a matter of dimension increasing.
- The solution is a multi-resolution type.
- The answer is convergent, even the size of increment is very large.

In this chapter we are using operational matrices for Walsh wavelet packets with Walsh bases as defined by Glabisz [29] in 2004, to find inverse Laplace transform for functions having fractional power like $\frac{s}{\sqrt{s^2+1}}$. Here we compare the results with the exact solutions and also with Walsh functions.

5.2 The Numerical Solution

The operational matrix $P_{m\times m}$ as defined in chapter 3, corresponds to $\frac{1}{s}$ in Laplace domain (refer [12]). The operational matrix for $\sqrt{s}$ can be obtained by by inverting the operational matrix resulting into a matrix corresponding to differentiation. and then taking the square root of the differentiator matrix thus obtained for finding a new matrix equivalent to $\sqrt{s}$.

The set of Walsh functions are defined as

$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

$$\phi_m = W_{m\times m}\psi_m$$

where, $\psi_i's$ are the block pulse functions with unit height and $\frac{1}{m}$ width.

The square matrix $W_{m\times m}$ is called the Walsh matrix. One of the properties of Walsh matrix is

$$W^2_{m\times m} = mI_m \tag{5.2.1}$$

$$W^{-1}_{m\times m} = \frac{1}{m}W_{m\times m} \tag{5.2.2}$$

We define using (5.2.1)

$$W^{-1}PW \cong H_{m\times m}$$

$$H_{m\times m} = \frac{1}{m}WPW$$

where P is the operational matrix of Walsh wavelet packets. The parameter $r = 0$, the interval's representative point, is located at it's beginning, whereas if $r = 1$, the point lies at the interval's end.

The matrix H in block pulse function domain can be expressed as

$$H = \frac{1}{m}\begin{pmatrix} r & 1 & 1 & - & - & - & 1 \\ 0 & r & 1 & 1 & - & - & 1 \\ 0 & 0 & r & - & - & - & 1 \\ | & & & & & & \\ | & & & & & & \\ 0 & 0 & 0 & - & - & - & r \end{pmatrix}$$

Matrix H can also be expressed as

$$H_{m\times m} = \frac{1}{m}\left[rI_m + Q_{m\times m} + Q^2_{m\times m} + - - - - + Q^{m-1}_{m\times m}\right] \qquad (5.2.3)$$

where

$$Q_{m\times m} = \begin{pmatrix} 0 & 1 & - & - & 0 \\ 0 & 0 & 1 & - & 0 \\ | & & & & \\ 0 & 0 & - & - & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & & & \\ | & & I_{m-1} & \\ - & - & - & - \\ 0 & - & - & 0 \end{pmatrix}$$

and also

$$Q^i_{m\times m} = \begin{pmatrix} 0 & | & I_{m-i} \\ - & | & - \\ 0 & | & 0 \end{pmatrix} \qquad for \;\; i < m$$

$$= 0 \qquad for \;\; i \geq m$$

From (5.2.3)

$$\begin{aligned}
H_{m\times m} &= \frac{1}{m}\left[rI_m + Q + Q^2 + Q^3 + ------\right] \\
&= \frac{r}{m}\left[I_m + \frac{Q}{r} + \frac{Q^2}{r} + \frac{Q^3}{r} + ...\right] \\
&= \frac{r}{m}\left[(I_m + Q) + \left(\frac{1}{r}-1\right)Q + \left(\frac{1}{r}-1\right)Q^2 + Q^2 + Q^3 + \left(\frac{1}{r}-1\right)Q^3 + ...\right] \\
&= \frac{r}{m}\left[(I_m + Q)\right]\left[1 + \left(\frac{1}{r}-1\right)Q + Q^2 + \left(\frac{1}{r}-1\right)Q^3 + Q^4 + ...\right] \\
&= \frac{r}{m}\left[(I_m + Q)\right]\left[(1 - Q + Q^2 - Q^3 + Q^4 - Q^5 + ...) + \frac{1}{r}(Q + Q^3 + Q^5 + ...)\right] \\
&= \frac{r}{m}\left[(I_m + Q)\right]\left[(I_m + Q)^{-1} + \frac{Q}{r}\left[(I_m + Q)(I_m - Q)\right]^{-1}\right] \\
&= \frac{r}{m}\left[I_m - Q\right]^{-1}\ \left[I_m + \left(\frac{1}{r}-1\right)Q\right] \qquad (5.2.4)
\end{aligned}$$

i.e. when $r = 1$ and $m = 2$, we have

$$H = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which coincide with H given in (ref [12]) corresponding to Walsh functions.

Now expressing the matrix $H_{m\times m}$ as a polynomial of $Q_{m\times m}$, we have

$$H_{m\times m} = h_m(Q_{m\times m})$$

where

$$h_m(x) = \frac{1}{m}\left[r + x + x^2 + ------ + x^{m-1}\right]$$

Hence, it follows that if q is the eigen value of $Q_{m\times m}$ then the corresponding eigen value of $H_{m\times m}$ is

$$h = h_m(q) = \frac{r}{m}\left[\frac{1+\left(\frac{1}{r}-1\right)q}{1-q}\right]$$

Since eigen values of $Q_{m\times m}$ are all zero

$$h = h_m(0) = \frac{r}{m}$$

Thus the eigen value of $H_{m\times m}$ is $\frac{r}{m}$ with multiplicity m.

5.3 Operational Matrix for Differentiation

Since the operational matrix for integration is H, we can find the operational matrix for differentiation by taking its inverse. Let the operational matrix for differentiation be $B_{m\times m}$

$$\begin{aligned}
B &= H_{m\times m}^{-1} \\
&= \frac{m}{r}\left[I_m + \left(\frac{1}{r}-1\right)Q\right]^{-1}[I_m - Q] \\
&= \frac{m}{r}[I_m + PQ]^{-1}[I_m - Q] \\
&= \frac{m}{r}\left[I_m - PQ + P^2Q^2 - P^3Q^3 + P^4Q^4 + ...\right][I_m - Q] \\
&= \frac{m}{r}\left(1+\frac{1}{p}\right)\left[\frac{1}{(1+\frac{1}{p})}I_m + \sum_{i=1}^{m-1}(-1)^i p^i Q^i\right] \\
B &= \frac{m}{r(1-r)}\left[(1-r)I_m + \sum_{i=1}^{m-1}(-1)^i\left(\frac{1}{r}-1\right)^i Q^i\right] \qquad (5.3.1)
\end{aligned}$$

which is the required value of B.

when $m = 1$, $B = \frac{1}{r}$ (using (5.3.1)) when $m = 2$,

$$B = \frac{2}{r^2}\begin{pmatrix} r & -1 \\ 0 & r \end{pmatrix}$$

when $m = 4$,

$$B = \frac{4}{r^2}\begin{pmatrix} r & -1 & (\frac{1}{r}-1) & -(\frac{1}{r}-1) \\ 0 & r & -1 & (\frac{1}{r}-1) \\ 0 & 0 & r & -1 \\ 0 & 0 & 0 & r \end{pmatrix}$$

$$with \;\; r = \frac{1}{2}$$

$$B = 16\begin{pmatrix} \frac{1}{2} & -1 & 1 & -1 \\ 0 & \frac{1}{2} & -1 & 1 \\ 0 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

which coincide with that given in (ref [12]) corresponding to Walsh functions.

Transforming back to Walsh domain yields the operational matrix for differentiation denoted by $D_{m \times m}$.

$$D_{m \times m} = \frac{1}{m} W H^{-1} W$$

Using (5.3.1) the eigen value h^{-1} of matrix H^{-1} can be expressed as eigen value q of

$Q_{m\times m}$: i.e.

$$
\begin{aligned}
b &= \frac{m}{r(1-r)}\left[(1-r)+\Sigma_{i=1}^{m-1}(-1)^i\left(\frac{1}{r}\right)^i q^i\right] \\
&= \frac{m}{r}\left[\frac{1-q}{1+\left(\frac{1}{r}-1\right)q}\right]
\end{aligned}
$$

where b is the eigen value of B.

5.4 Operational Matrix for Fractional Calculus

Now we try to find the operational matrix for fractional differentiation. In general, we have

$$
b^{\frac{1}{l}} = \left(\frac{m}{r}.\frac{1-q}{1+\left(\frac{1}{r}-1\right)q}\right)^{\frac{1}{l}}
$$

where l is an integer. The above equation can be expanded as a polynomial of q. i.e.

$$
b^{\frac{1}{l}} = (m/r)^{\frac{1}{l}}\,.\rho_{l,m}(q)
$$

where $\rho_{l,m}$ is a polynomial of order m-1. Hence the operational matrix for $1/l$ differentiation is given by

$$
B^{1/l} = (m/r)^{1/l}\,.\rho_{l,m}(Q_{m\times m})
$$

The corresponding $1/l$ differentiation operational matrix is

$$
D^{1/l}_{m\times m} = (m/r)^{1/l}\,.W^{-1}_{m\times m}\rho_{l,m}(Q_{m\times m})W_{m\times m}
$$

Let us find the operational matrix by choosing $m = 4, l = 2$

$$\begin{aligned}
\rho_{l,m}(q) &= \left(\frac{1-q}{1+\left(\frac{1}{r}-1\right)q}\right)^{1/l} \\
&= \left[(1-q)^{1/l}\left(1+\left(\frac{1}{r}-1\right)q\right)^{-1/l}\right] \\
\rho_{2,4}(q) &= \left[\frac{1-q}{1+\left(\frac{1}{r}-1\right)q}\right]^{1/2} \\
&= 1-\frac{1}{2r}q+\frac{1}{2r}\left[\frac{3}{4r}-1\right]q^2-\frac{1}{2r}\left[\frac{5}{8r^2}-\frac{3}{2r}+1\right]q^3+------
\end{aligned}$$

which is the required Maclaurin's expansion.

Hence,

$$B_{4\times 4}^{1/2} = \left(\frac{4}{r}\right)^{1/2}\left[I_{4\times 4}-\frac{1}{2r}Q_{4\times 4}+\frac{1}{2r}\left[\frac{3}{4r}-1\right]Q_{4\times 4}^2-\frac{1}{2r}\left[\frac{5}{8r^2}-\frac{3}{2r}+1\right]Q_{4\times 4}^3\right]$$

$$\begin{aligned}
\textit{when} \quad r &= 1, \\
B^{1/2} &= 2\left[I_{4\times 4}-\frac{1}{2}Q+\frac{1}{8}Q^2-\frac{1}{16}Q^3\right] \\
B^{1/2} &= \sqrt{4}\begin{pmatrix} 1 & -0.5 & -0.1250 & -0.0625 \\ 0 & 1 & -0.5 & -0.1250 \\ 0 & 0 & 1 & -0.5 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

$$when\ r = 0.8$$

$$B^{1/2} = \sqrt{\frac{4}{0.8}}\begin{pmatrix} 1 & -0.6250 & -0.0391 & -0.0635 \\ 0 & 1 & -0.6250 & -0.0391 \\ 0 & 0 & 1 & -0.6250 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{Hence,} \quad D^{1/2}_{4\times 4} = \left(\frac{m}{r}\right)^{1/2} W^{-1}_{4\times 4}\ \rho_{l,m}(Q_{m\times m})\ W_{m\times m}$$

$$= W^{-1}B^{1/2}W$$

$$D^{1/2}_{4\times 4} = \left(\frac{m}{r}\right)^{1/2} .\frac{1}{m}WB^{1/2}W$$

$$= \left(\frac{4}{r}\right)^{1/2} .\frac{1}{4}WB^{1/2}W$$

$$= \frac{1}{4r^{1/2}}\left[2WB^{1/2}W\right]$$

$$when\ r = 1$$

$$D^{1/2} = \frac{1}{4}\left[2WB^{1/2}W\right]$$

$$= \frac{1}{4}\begin{pmatrix} 4.3750 & 1.6250 & 1.1250 & 0.8750 \\ -1.6250 & 7.6250 & -0.8750 & 2.8750 \\ -1.1250 & -0.8750 & 10.6250 & -0.6250 \\ 0.8750 & -2.8750 & 0.6250 & 9.3750 \end{pmatrix}$$

$$when\ r = 0.4$$

$$D^{1/2} = \frac{1}{4(0.4)^{1/2}} \begin{pmatrix} 1.9844 & 1.0156 & 5.3906 & -0.3906 \\ -1.0156 & 4.0156 & 0.3906 & 4.6094 \\ -5.3906 & 0.3906 & 22.7656 & -9.7656 \\ -0.3906 & -4.6094 & 9.7656 & 3.2344 \end{pmatrix}$$

5.5 Example

We apply this method to find inverse Laplace transform of Bessel function i.e.

$$\frac{1}{\sqrt{s^2+1}}$$

.

Example: The Bessel Function

$$F(s) = \frac{1}{\sqrt{s^2+1}}$$

can be constructed as the Laplace Transform of the solution of time varying system. Rewriting,

$$F(s) = \frac{1}{\sqrt{s^2+1}} = \frac{s}{\sqrt{s^2+1}}.\frac{1}{s} = F_1(s).\frac{1}{s} \tag{5.5.1}$$

Suppose we are interested in the solution for $0 \leq t < 8$, the general rule of scaling is based on

$$F(s) = \frac{1}{\alpha}.F\left(\frac{s}{\alpha}\right) \quad for\ \ t \rightarrow \alpha t$$

where, α is a scaling factor.

If $\alpha = 8$ then,

$$F(s) \to \frac{1}{8}F(s/8) = \frac{1}{8}F_1(s/8)\cdot\frac{1}{(s/8)} \tag{5.5.2}$$

where

$$F_1(s/8) = \frac{s/8}{\sqrt{(s/8)^2+1}}$$

For $m = 8$ and $r = 1$, we have

$$F_1(q) = \frac{1-q}{\sqrt{2-2q+q^2}}$$

Taylor's series expansion at $q = 0$ gives

$$F_1(q) = 0.7071 - 0.3535q - 0.2650q^2 - 0.1326q^3 - 0.0276q^4$$
$$+0.0249q^5 + 0.0476q^6 + 0.0102q^7 - 0.0109q^8$$

Replacing q by Q gives the operational matrix of this problem. In (5.5.1) $1/s$, should be expressed as ψ functions.

$$1/s \to [1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1]\,\psi_{(8)}$$

Hence (5.5.2) becomes, i.e.

$$L^{-1}\left(\frac{1}{\sqrt{s^2+1}}\right) = L^{-1}\left(\frac{1}{s}F_1(s/8)\right)$$

$$= \begin{pmatrix} 0.7071 & -0.3535 & -0.2652 & -0.1326 & -0.0276 & 0.0249 & 0.0476 & 0.0102 \\ 0 & 0.7071 & -0.3535 & -0.2652 & -0.1326 & -0.0276 & 0.0249 & 0.0476 \\ 0 & 0 & 0.7071 & -0.3535 & -0.2652 & -0.1326 & -0.0276 & 0.0249 \\ 0 & 0 & 0 & 0.7071 & -0.3535 & -0.2652 & -0.1326 & -0.0276 \\ 0 & 0 & 0 & 0 & 0.7071 & -0.3535 & -0.2652 & -0.1326 \\ 0 & 0 & 0 & 0 & 0 & 0.7071 & -0.3535 & -0.2652 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7071 & -0.3535 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7071 \end{pmatrix}$$

$$= 0.7071\psi_0 + 0.3536\psi_1 + 0.0884\psi_2 - 0.0442\psi_3 - 0.0718\psi_4$$

$$- \; 0.0469\psi_5 + 0.0007\psi_6 + 0.0109\psi_7$$

Similarly we can calculate B for other values of r.

Table showing the calculations using walsh wavelet packet operational matrices for different values of r and for exact solution of Bessel function :

$t \setminus \nu$	0.4	0.5	0.8	1	Bessel function
0	0.0715	0.1056	0.2191	0.2929	1.0000
1	0.1569	0.2285	0.3652	0.4116	0.6992
2	0.1703	0.1881	0.1787	0.1355	0.0633
3	-0.0687	0.0619	-0.1419	-0.2159	-0.3692
4	0.4572	0.0321	-0.2658	-0.3253	-0.3034
5	0.4624	0.0411	-0.1137	-0.1307	0.0645
6	0.3720	0.0215	0.2674	0.1499	0.2963
7	0.2233	0.0066	0.3827	0.2892	0.1717

Table for root mean square error for different values of r :

ν	0.4	0.5	0.8	1
Root mean square Error	0.8213	0.3253	0.5334	0.5392

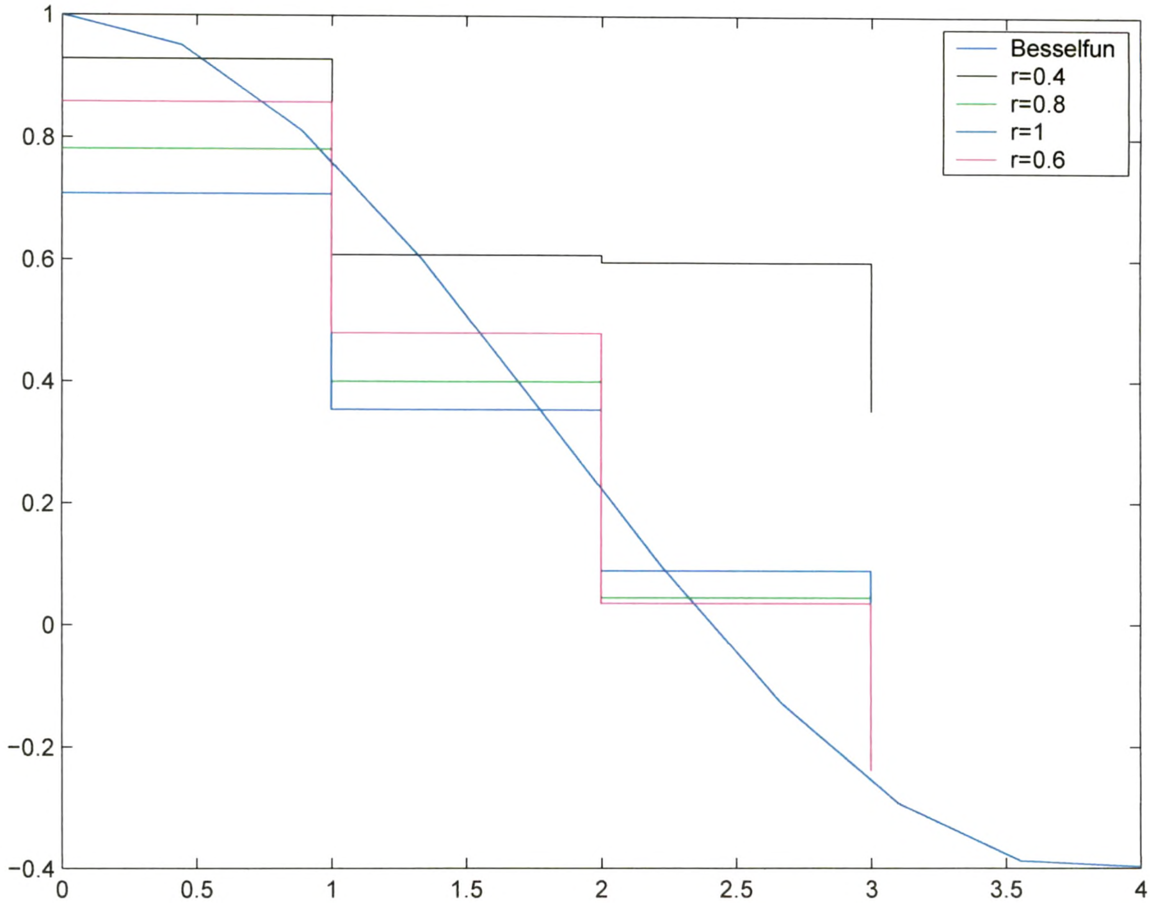

Figure 5.1: Solution Using Wavelet Packets for m=4

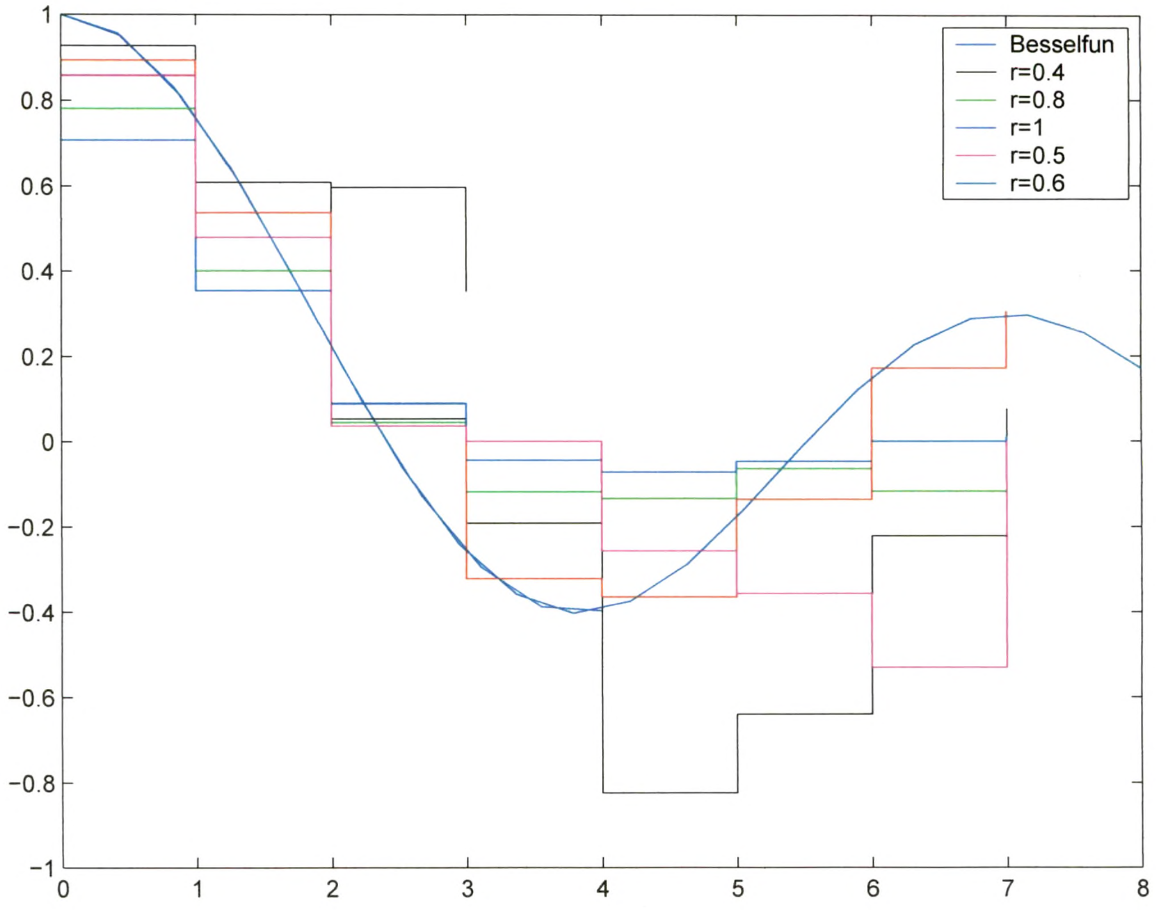

Figure 5.2: Solution Using Wavelet Packets for m=8

5.6 Conclusion

- The exact solution and the solution obtained using Walsh wavelet packet operational matrices are compared.

- In the present chapter we have approximated Bessel function which is solution to time varying system using Walsh wavelet packet and compared the approximated solution with the exact solution with m=8 and different values of r.

- From the table it is clear that the RMS error is decreasing when we take the value from $r = 1$ to $r = 0.5$ and is minimum at $r = 0.5$. For values less than 0.5 error goes on increasing.

- Also from the graph we have the same observation. i.e. we get better solution for $r = 0.5$ as compared to the values higher than $r = 0.5$.