SUMMARY

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" On the Convergence of Wavelet Packet Series and Its Applications"

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Summary

"**Wavelets**" is a versatile tool with very rich mathematical content and great potential for applications. It has been very popular topic of conversations in many scientific and engineering researchers. It can be viewed as one of the important techniques for time frequency analysis. The subject of "**Wavelet Analysis**" has drawn much attention from both mathematicians and engineers alike. Analogous, to Fourier Analysis, there are also two important mathematical entities in Wavelet Analysis : "**The Integral Wavelet Transform**" and "**Wavelet Series**". Our attention in the present study is "**Wavelet Series and its Applications**" to different engineering disciplines.

A great deal of mathematical and engineering analysis depends on methods for representing a complex phenomena in terms of elementary well understood phenomena.Examples include the use of Fourier expansions in studying partial differential equations, Strum- Liouville expansions in ordinary differential equations and Riemann's decomposition of the number theoretic psi function in terms of elementary waveforms [1], used to investigate zero's of the Riemann zeta function.

Recently wavelet theory has provided a new method for decomposing signal. For a signal say sound or an image Fourier analysis easily calculates the frequencies and the amplitudes of those frequencies which make up a signal. This provides a broad overview of the characteristics of a signal, which is important for theoretical considerations. Although Fourier inversions is possible under certain circumstances, Fourier methods are not always a good tool to

recapture the signal, particularly if it is highly non-smooth, too much Fourier information is needed to reconstruct the signal locally. In these cases Wavelet analysis is often very effective because it provides a simple approach for dealing with local aspects of a signal. Wavelet analysis also provides us new methods for removing noise from signals that complement the classical method of Fourier analysis.

Wavelets are discovered time to time by scientists who wanted to solve problems in their various disciplines.

- Wavelet analysis was originally introduced in order to improve seismic signal processing by switching from short time Fourier analysis to new algorithms better suited to detect and analyze abrupt changes in signals.

- It corresponds to a decomposition of phase in which tradeoff between time and frequency localization at high frequencies has been chosen to provide better and better time localization at high frequencies in return for poor frequency localization. This makes the study well adapted to study a transient phenomena and proven a very successful approach to many problems in signal and image processing, numerical analysis, quantum mechanics, data compression, astronomy, oceanography etc.

- Wavelets separate a signal into multi-resolution components. The fine and coarse resolution components capture respectively, the fine and coarse scale features in the signal.

- The wavelet approximation can compact the energy of a signal into relatively small number of wavelet functions. This data compression feature of wavelets is valuable for applications such as non-parametric statistical estimation and classification.

- It is useful in signal processors to transmit clear messages over telephone wires.

- Different types of wavelets have been used as tools to solve problems in signal analysis, image analysis, medical diagnostics, boundary value problems, geophysical signal processing, pattern recognition and many others.

While wavelets have gained popularity in these areas, new applications are continuously being investigated.

Wavelets are discovered time to time by scientists who wanted to solve technical problems in their various disciplines. It is useful in Signal Processors to transmit clear messages over telephone wires. Oil prospectors desired a better way to interpret seismic traces. Wavelets are one of the tools in computer imaging and animation.In the future, scientists may put wavelet analysis for diagnosing breast cancer, looking for heart abnormalities or predicting the weather.

Wavelet analysis was originally introduced in order to improve seismic signal processing by switching from short-time Fourier analysis to new algorithms better suited to detect and analyze abrupt changes in signals. It corresponds to a decomposition of phase in which the trade off between time and frequency localization at high frequencies has been chosen to provide better and better time localization at high frequencies in return for poor frequency localization. This makes the analysis well-adapted to the study of transient phenomena and has proven a very successful approach to many problems in signal processing, numerical analysis and quantum mechanics.

The main aim of this thesis is to study the convergence properties of the periodized and generalized Walsh type wavelet packet series by generalizing the results obtained by C.W. Onneweer [12] and Ferenc Moricz [5]. Mortein Nielsen [10, 11] have proved the pointwise convergence of the periodized and generalized Walsh type wavelet packet series using the concept of Carleson operator and strong type (p,p). In the present study we are proving the uniform convergence of the periodized and generalized Walsh type wavelet packet series.

The present study also includes the applications of wavelet packets in many fields such as solving State space analysis, Bilinear systems,Variational Problems and Fractional differential equations. The wavelet packets are applied in these fields using the idea of operational matrices of Walsh wavelet packets using Walsh bases and Haar bases as defined by Glabisz [7]. The main characteristic of the operational matrix is to convert a differential equation into algebraic one. It does not only simplify the problem but also speed up the computation.

The thesis consists of five chapters.

Chapter 1. begins with the introduction about wavelets and wavelet packets.

In Chapter 2, we are proving the uniform convergence of periodized Walsh type wavelet packet series as well as generalized Walsh type wavelet packet series.

THEOREM 0.1 *Let $f \in L^p[0,1)$, $1 < p < \infty$ be a function of period 1. Then,*

$$\lim_{k \to \infty} S_k f(x) = f(x)$$

uniformly in x, where $S_k f(x)$ is the k^{th} partial sum of periodic Walsh type wavelet packet series.

Morten Nielsen [10, 11] has proved the point-wise convergence a.e. of the periodic Walsh type wavelet packet expansions $1 < p < \infty$. We have generalized the results obtained by C.W.Onneweer [12] and Ferenc Moricz [5] The Walsh type wavelet packets can be considered as smooth generalizations of the Walsh functions and they have the same convergence properties for expansion of L^p functions $1 < p < \infty$.

Most of the work on wavelet packets has been done in one dimension or using separable wavelet packets in higher dimensions. Morten Nielsen [11] has studied wavelet packets associated with multi-resolution analysis with a scaling function given by the characteristic function of some sets (called a tile) in R^d. The functions in this case of wavelet packets are called generalized Walsh functions. He proved that the functions constitute a Schauder bases for $L^p(R^d)$, $1 < p < \infty$ and the expansion of L^p functions converge point wise a.e. Here we have generalized the result to the uniform convergence of the periodic generalized Walsh type wavelet packet series for $L^p(R^d)$, $1 < p < \infty$ as follows :

THEOREM 0.2 *Let $f \in L^p[0,1)^2$ for $1 < p < \infty$ be a function of period 1. Then,*

$$\lim_{k \to \infty} S_k f(x,y) = f(x,y)$$

uniformly in (x,y), where $S_k f(x,y)$ is the k^{th} partial sum of periodic Walsh type wavelet packet series.

Chapter 3, deals with of state space analysis using wavelet packet series.

Here, we have used the operational matrices for Walsh wavelet packets using Haar basis and Walsh basis respectively as shown below (refer [2]).

Consider a linear time invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

$$x(0) = x_0$$

where, $x(t) \in R^n$, $u(t) \in R^m$ and $y(t) \in R^l$ and A, B and C are constant matrices of appropriate dimensions.

Using the operational matrices for given values of m and r, where $m = 2^J$ is referred as degree of approximation and r represents the position of the representative point, for each of m intervals of x we are obtaining the state space matrices using the procedure given by P. N. Paraskevopoulos [13].

This work also includes the comparative study of the results obtained using Haar bases and Walsh bases with different values of m and r.

Also in this chapter we have used the wavelet packet series approach to state space analysis for Bilinear systems. Bilinear systems may be considered as a specialization of non-linear systems under the assumption of linearity in control or respectively in state but not in both jointly.

A bilinear time invariant system is described as

$$\dot{x}(t) = Ax(t) + Nx(t)u(t) + Bu(t)$$
$$x(t = 0) = x(0)$$

where $x(t) \in R^n$, $u(t) \in R^m$ and A and B are constant matrices of appropriate dimensions.

An orthogonal series approach to state space analysis of bilinear time invariant systems using Chebychev orthogonal polynomials in the time interval $[0, 1]$ was given by P. N. Paraskepoulos [14]. It involved only multiplication of matrices of small dimensions.

Using the above approach we are obtaining the solution of bilinear time invariant systems using Walsh wavelet packet operational matrices. Also comparative study of the results obtained using Walsh wavelet packets and exact solution have been made.

In Chapter 4, we are using the Direct method for solving variational problems using Walsh wavelet packets.

A numerical algorithm for solving variational problem using Walsh bases and Haar bases has been presented. The variational problem is solved by the means of direct method (refer [3]).

Let,

$$y'' = -x^2, \qquad y(0) = y(1) = 0$$

which is Poisson's problem in one variable which in fact requires only two integration to discover the exact solution

$$y(x) = \frac{x(1-x^3)}{12}$$

which can be converted to variational problem

$$J(y) = \int_0^1 \left[\frac{1}{2}(y')^2 - x^2 y\right] dx$$

The boundary conditions are

$$y(0) = 0, \qquad y(1) = 0$$

The idea of the direct method for solving a variational problem is to convert the problem of extremization of a function in to one which involves a finite number of variables. Ritz's method is well known in this area.

Here, we first introduce a direct method for solving variational problems using the operational matrices for Walsh bases and Haar bases and later on comparative study has made using the results obtained for the exact solutions, Walsh functions, Walsh bases and Haar bases.

In chapter 5, we use the wavelet packet operational matrix approach for solving fractional differential equations.

Fractional calculus is the generalization of operators of differential and integration to non-integer order. A differential equation involving the fractional calculus operators such as $\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}}$ and $\frac{d^{\frac{-1}{2}}}{dt^{\frac{-1}{2}}}$ is called the fractional differential equation. They have many applications in science and engineering. The point to noted with this type of differential equations is that the analytical solution exists only for limited number of cases.Also it is difficult to solve with numerical methods.

In this thesis, we use the operational matrix for Walsh bases to solve these type of problems. Here, we have discussed the applications of fractional calculus, to find inverse Laplace transform for functions having fractional power like $\frac{s}{\sqrt{s^2+1}}$ using the example of Bessel function.

The Bessel function

$$F(s) = \frac{1}{\sqrt{s^2+1}}$$

can be constructed as the Laplace transform of time varying system.

This work extends the work of C. F. Chen and Y. T. Tsay [4]. An example based on fractional differential equation has been demonstrated and comparative study of the exact solution and Walsh wavelet packet solutions has been done.

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