Chapter 1

INTRODUCTION

"**Wavelets**" is a versatile tool with very rich mathematical content and great potential for applications. It has been very popular topic of conversations in many scientific and engineering researchers. It can be viewed as one of the important techniques for time frequency analysis. The subject of "**Wavelet Analysis**" has drawn much attention from both mathematicians and engineers alike. Analogous, to Fourier analysis, there are also two important mathematical entities in Wavelet Analysis : "**The Integral Wavelet Transform**" and "**Wavelet Series**". Our attention in the present study is "**Wavelet Series and its Applications**" to different engineering disciplines.

A great deal of mathematical and engineering analysis depends on methods for representing a complex phenomena in terms of elementary well understood phenomena. Examples include use of Fourier expansions in studying partial differential equations, Strum- Liouville expansions in ordinary differential equations and Riemann's decomposition of the number theoretic psi function in terms of elementary waveforms

[4], used to investigate zero's of the Riemann zeta function.

Recently wavelet theory has provided a new method for decomposing signal. For a signal say sound or an image Fourier analysis easily calculates the frequencies and the amplitudes of those frequencies which make up a signal. This provides a broad overview of the characteristics of a signal, which is important for theoretical considerations. Although Fourier inversions is possible under certain circumstances, Fourier methods are not always a good tool to recapture the signal, particularly if it is highly non-smooth, too much Fourier information is needed to reconstruct the signal locally. In these cases Wavelet analysis is often very effective because it provides a simple approach for dealing with local aspects of a signal. Wavelet analysis also provides us new methods for removing noise from signals that complement the classical method of Fourier analysis.

Wavelets are discovered time to time by scientists who wanted to solve problems in their various disciplines.

- Wavelet analysis was originally introduced in order to improve seismic signal processing by switching from short time Fourier analysis to new algorithms better suited to detect and analyze abrupt changes in signals.

- It corresponds to a decomposition of phase in which tradeoff between time and frequency localization at high frequencies has been chosen to provide better and better time localization at high frequencies in return for poor frequency localization. This makes the study well adapted to study a transient phenomena and proven a very successful approach to many problems in signal and image processing, numerical analysis, quantum mechanics, data compression, astronomy, oceanography etc.

- Wavelets separate a signal into multi-resolution components.The fine and coarse resolution components capture respectively, the fine and coarse scale features in the signal.

- The wavelet approximation can compact the energy of a signal into relatively small number of wavelet functions. This data compression feature of wavelets is valuable for applications such as non-parametric statistical estimation and classification.

- It is useful in signal processors to transmit clear messages over telephone wires.

- Different types of wavelets have been used as tools to solve problems in signal analysis, image analysis, medical diagnostics, boundary value problems, geophysical signal processing, pattern recognition and many others.

While wavelets have gained popularity in these areas, new applications are continuously being investigated.

The concept of wavelet analysis has been in place in one form or another since the beginning of twentieth century.

In pre 1930's, Joseph Fourier developed Fourier synthesis, which is the main branch leading to wavelets, using frequency analysis theories. In 1930's, the notion of frequency analysis to scale analysis was developed. A physicist, Paul Levy found that Haar basis function is superior to the Fourier basis function for studying small complicated details in Brownian matrix.

In 1960-1980, the mathematicians Guido Wiess and Ronald R. Coifman [53] created a simplest elements of a function space called Atoms, with a goal of finding the atoms for a common function.

Wavelet theory attracted attention in the 1980's through the work of several researchers from various disciplines. In 1980, Grossman and Morlet broadly defined wavelets in the context of Quantum Physics.

A new type of basis function called **WAVELETS** was introduced in 1982 by J. Morlet in view of applications for the analysis of Seismic data. The proper history of orthonormal wavelets starting in the mid 1980's with the first construction of smooth orthonormal wavelet basis on R by Meyer [50] and on R^n by Lemarie and Meyer [46].

Mallat and Meyer [[48],[51]] placed this singular calculation with general framework by formulating the notion of multi-resolution analysis. Daubecies [23] used this approach to construct families of compactly supported wavelets with certain degrees of smoothness.

Beylkin, Coifman and Rokhlin [3] pioneered the use of wavelets in numerical analysis problems related to partial differential equations.

DEFINITION 1.0.1 (Wavelets [16] :)
The family

$$\psi_{a,b}(x) = \mid a \mid^{\frac{-1}{2}} \psi\left(\frac{x-b}{a}\right), \qquad a, b \in R$$

of translated and dilated versions of function $\psi \in L^2(R)$, where ψ satisfies the "admissibility condition"

$$C_\psi := \int_{-\infty}^{\infty} \frac{\mid \widehat{\psi}(\omega) \mid^2}{\mid \omega \mid} d\omega < \infty$$

is called a Wavelet.

The introduction of wavelets to signal and image processing has provided a very flexible tool for engineers to use to create innovative techniques for solving various engineering problems.

A problem with every wavelet basis is that all the high frequency wavelets have poor frequency localization. **Wavelet packets** were introduced by Coifman, Meyer and Wickerhauser [50] to improve the poor frequency localizations of wavelet bases to high frequency and thereby provide a more efficient decomposition of signals containing with both transient and stationary components. If the signal properties change over time it is preferable to isolate different time intervals with translated windows. Wavelet packets segment the frequency axis and are uniformly translated in time.

Wavelet packet analysis is an important generalization of wavelets, pioneered by R. Coifman, Y. Meyer, M. V. Wickerhauser and other researchers (refer[[17], [18], [19], [20], [72]]).
Wavelet packets functions comprise a rich family of building blocks functions. Wavelet packet functions are still localized in time, but offer more flexibility than wavelet in representing different types of signals. In particular, wavelet packets are better at representing signals that exhibit oscillatory or periodic behavior.

Discrete wavelet packets have been thoroughly studied by Wickerhauser [72] who has also developed computer programmes and implemented them. Well known Daubechies orthogonal wavelets are a special case of wavelet packets. Wavelet packets are organized naturally into collections, and each collection is an orthogonal basis for $L^2(R)$. It is a simple but powerful extension of wavelets and multi-resolution analysis. Wavelet packets allow more flexibility in adapting the basis to the frequency contents of a signal and it is easy to develop a fast wavelet packet transform. The power of wavelet packets lies in the fact that we have much more freedom in deciding which basis function is to be used to represent a given function. The best basis selection criteria and applications to image processing can be found in (refer[[20],[71],[74]]).

Wavelet packet functions are generated by scaling and translating a family of basic function shapes, which include father wavelet ϕ and mother wavelet ψ. In addition to ϕ and ψ there is a whole range of wavelet packet functions W_n.These functions are parameterized by an oscillation or frequency index n. A father wavelet corresponds to $n = 0$, so $\phi = W_0$. A mother wavelet corresponds to $n = 1$, so $\psi = W_1$. Larger values of n correspond to wavelet packets with more oscillations and higher frequency.

DEFINITION 1.0.2 (Wavelet Packets [54] :)

For $n = 0, 1, 2, 3.....$ we define a sequence of functions as follows:

$$W_{2n}(t) = \sqrt{2}\Sigma_k h_k W_n(2t - k)$$
$$W_{2n+1}(t) = \sqrt{2}\Sigma_k g_k W_n(2t - k)$$

when $n = 0, \quad W_0(t) = \phi(t),$ *the scaling function*

and $n = 1, \quad W_1(t) = \psi(t),$ *the mother wavelet.*

Various combinations of functions and their translations and dilations can give rise to various bases for the function space. So we have a whole collection of orthonormal bases generated from $\{W_n(t)\}$. We call this collection " **a library of wavelet packet bases** " and the function of the form $W_{n,j,k} = 2^{\frac{j}{2}} W_n(2^j t - k)$ is called a wavelet packet.

The problem of convergence of wavelet series has been studied by Meyer [51], Walter [[69],[70]], Tao [67] and Kelly et al.[[39],[40]]. Meyer was the first to study convergence results for wavelet expansions. He has shown that regular wavelet expansions converge in L^p, $1 < p < \infty$ and also in L^∞ for expansions of uniformly continuous functions, the expansions of continuous functions converge everywhere. The results in [51] were based on assumptions of regularity for basic wavelets and their derivatives. In addition Walter [[69], [70]] established point-wise convergence results for regular wavelets expansions of continuous functions. Kelly et. al [[39], [40]] have extended and obtained results analogous to those obtained by Carleson [5] and Hunt [36] for the Fourier series. Tao [67] extended the results of Meyer [52] and Kelly et al. [[39],[40]] and has shown that the wavelet expansion of any L^p function converges point-wise almost everywhere under the wavelet projection, hard sampling and soft sampling summation methods, for $1 < p < \infty$. Kelly et al.[39] have given necessary and sufficient conditions for given point-wise (sup-norm) rates of convergence of wavelet or multi-resolution analysis, in terms of Sobolev conditions on the basic wavelet or scaling function. It has already been shown by Mallat [48] and Meyer [51] that Sobolev class of functions is determined by the L^2 rates of convergence of its wavelet expansions. Necessary and sufficient conditions for L^2 rates convergence which are analogous to sup-norm conditions (refer [67]) have been obtained by de Boor et al. who have also studied sup-norm convergence.

Recently, many mathematicians are doing their research work in studying convergence properties of wavelet packet series. Siddiqi has studied the convergence of the wavelet packet series generalizing few results of Walter [69] and Kelly [39]. Khalil Ahmed and Rakesh Kumar [1] have studied pointwise convergence of wavelet packet series using the results of Kelly [[39],[40]] and Hernandez Weiss [32]. Morten Nielsen [54] have studied the point-wise convergence of Walsh type wavelet packet series using Carleson Hunt theorem.

The main aim of this thesis is to study the convergence properties of the periodized and generalized Walsh type wavelet packet series by generalizing the results obtained by C. W. Onneweer [57] and Ferenc Moricz [25]. Here, the idea of the operational matrix of Walsh wavelet packet bases and Haar wavelet packet bases as defined by Glabisz [29] has been widely applied. The present study also includes the application of wavelet packets in many fields such as solving the state space analysis, variational problems and fractional differential equations. The main characteristic of the operational matrix is to convert a differential equation to algebraic one. It does not only simplify the problem but also speed up the computation.

In this thesis, we give a wavelet packet approach for investigating different problems. This thesis consists of five chapters.

- Chapter 1, Introduction
- Chapter 2,
  1. Uniform Convergence of periodized Walsh type wavelet packet series
  2. Uniform convergence of Generalized Walsh type wavelet packet series
- Chapter 3,
  1. State Space Analysis using Wavelet Packets
  2. Wavelet Packet series approach to State Space analysis using Bilinear systems
- Chapter 4, Variational Problems and Wavelet Packets
- Chapter 5, Fractional Calculus using Wavelet Packets

The outline of the thesis with brief contents is as follows :

Chapter 1 is introductory. Chapter 2 deals with the uniform convergence of periodized Walsh type wavelet packet series.

Morten Nielsen [54, 55] has proved the point-wise convergence a.e. of the periodic Walsh type wavelet packet expansions $1 < p < \infty$. Here, we have generalized the results obtained by C. W. Onneweer [57] and Ferenc Moricz [25]. The Walsh type wavelet packets can be considered as smooth generalizations of the Walsh functions and they have the same convergence properties for expansion of L^p functions $1 < p < \infty$.

THEOREM 1.0.3 *Let $f \in L^p[0,1)$, $1 < p < \infty$ be a function of period 1. Then,*

$$\lim_{k\to\infty} S_k f(x) = f(x)$$

uniformly in x, where $S_k f(x)$ is the k^{th} partial sum of periodic Walsh type wavelet packet series. ∎

Most of the work on wavelet packets has been done in one dimension or using separable wavelet packets in higher dimensions. Morten Nielsen has studied wavelet packets associated with multi-resolution analysis with a scaling function given by the characteristic function of some sets (called a tile) in R^d. The functions in this case of wavelet packets are called generalized Walsh functions. He proved that the functions constitute a Schauder bases for $L^p(R^d)$, $1 < p < \infty$ and the expansion of L^p functions converge point wise a.e. Here we have generalized the result to the uniform convergence of the periodic Walsh type wavelet packet series for $L^p(R^d), 1 < p < \infty$ (refer [6]).

THEOREM 1.0.4 *Let $f \in L^p[0,1)^2$ for $1 < p < \infty$ be a function of period 1. Then,*

$$\lim_{k\to\infty} S_k f(x,y) = f(x,y)$$

uniformly in (x, y), where $S_k f(x, y)$ is the k^{th} partial sum of periodic Walsh type wavelet packet series.

Functions can be represented in orthogonal function bases. This way of representing functions can be widely used to solve differential equations describing dynamics of mechanical systems, control and identification problems. Orthogonal Walsh functions were applied by Corrington [21] and, Chen and Hsiao [13] were first to define notion of operational matrix for this type of functions. Chen, Tsay and Wu [12] defined operational matrices for block pulse functions. Hwang and Shih [37] and King and Paraskevopoulos [41] gave orthogonal matrices for Laguerre polynomials, Chang and Wang [11] for Legendre polynomials.

Now a days, wavelet bases have been increasingly used in description of different classes of functions (refer [[48],[65]]). Chen and Hsiao [15] were first to present an operational integration matrix based on Haar wavelets and they applied it to the analysis of lumped and distributed parameters of dynamic systems. State analysis of time delayed systems via Haar wavelets was proposed by Hsiao [33]. Hsiao and Wang [[34],[35]] applied Haar wavelets to the state analysis and parameter estimation of bilinear systems and to the analysis of nonlinear stiff systems.

Glabisz [29] defined the operational matrices of Walsh wavelet packet basis and Haar wavelet packet basis. The main characteristic of the operational matrix is to convert a differential equation in to an algebraic equation. It not only simplifies the problem but also speed up the computation.

Chapter 3, consists of state space analysis by using wavelet packet series.

The classical design methods for control system analysis suffer from certain limitations due to the fact that the transfer function model is applicable only to linear time invariant systems and is restricted to Single Input Single Output (SISO) systems, as it becomes highly cumbersome for use in Multi Input Multi Output (MIMO) systems.

Another limitation of the transfer function is that it reveals only the system output for a given input and provides no information about the internal behavior of the system. The limitations of the classical methods based on transfer function models have led to the development of state variable approach of analysis and design. It is a powerful technique for analysis and design of the linear, non-linear, time invariant or time varying MIMO systems. It is easily amendable to digital computers.

Here, we used the operational matrices for Walsh wavelet packets using Haar basis and Walsh basis respectively as shown below (refer [8]):

Consider a linear time invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ x(0) &= x_0 \end{aligned}$$

where, $x(t) \in R^n$, $u(t) \in R^m$ and $y(t) \in R^l$ and A, B and C are constant matrices of appropriate dimensions.

Using the operational matrices for given values of m and r, where $m = 2^J$ is referred as degree of approximation and r represents the position of the representative point for each of m intervals of x, we are obtaining the state space matrices using the procedure given by P. N. Paraskevopoulos [58].

This work also includes the comparative study of the results obtained using Haar bases and Walsh bases with different values of m and r.

Also in this chapter we are using the wavelet packet series approach to state space analysis using bilinear systems. Bilinear systems may be considered as a specialization of non-linear systems under the assumption of linearity in control or respectively in state but not in both jointly.

A bilinear time invariant system is described as

$$\dot{x}(t) = Ax(t) + Nx(t)u(t) + Bu(t) \tag{1.0.1}$$
$$x(t=0) = x(0)$$

where $x(t) \in R^n$, $u(t) \in R^m$ and A and B are constant matrices of appropriate dimensions.

An orthogonal series approach to state space analysis of bilinear time invariant systems using Chebychev orthogonal polynomials in the time interval $[0, 1]$ was given by P. N. Paraskepoulos [59]. It involved only multiplication of matrices of small dimensions.

Using the above approach we are obtaining the solution of bilinear time invariant systems using Walsh wavelet packet operational matrices . Also comparative study of the results obtained using Walsh wavelet packets and exact solution have been made.

In Chapter 4, we are using the direct method for solving variational problems using Walsh wavelet packets.

A numerical algorithm for solving variational problem using Walsh bases and Haar bases has been presented. The variational problem is solved by the means of direct method.

Let,

$$y'' = -x^2, \qquad y(0) = y(1) = 0$$

which is Poisson's problem in one variable which in fact requires only two integration to discover the exact solution

$$y(x) = \frac{x(1-x^3)}{12}$$

which can be converted to variational problem

$$J(y) = \int_0^1 \left[\frac{1}{2}(y')^2 - x^2 y\right] dx$$

The boundary conditions are

$$y(0) = 0, \quad y(1) = 0$$

The idea of the direct method for solving a variational problem is to convert the problem of extremization of a function in to one which involves a finite number of variables. Ritz's method is well known in this area.

Here, we first introduce a direct method for solving variational problems using the operational matrices for Walsh bases and Haar bases and later on comparative study has made using the results obtained for the exact solutions, Walsh functions, Walsh bases and Haar bases.

In chapter 5, we use the wavelet packet operational matrix approach for solving fractional differential equations.

Fractional calculus is the generalization of operators of differential and integration to non-integer order. A differential equation involving the fractional calculus operators such as $\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}}$ and $\frac{d^{\frac{-1}{2}}}{dt^{\frac{-1}{2}}}$ is called the fractional differential equation. They have many applications in science and engineering. The point to be noted with this type of differential equations is that the analytical solution exists not only for limited number of cases but it is also difficult to solve with numerical methods.

In this thesis, we use the operational matrix for Walsh bases to solve these type of problems. Here, we have discussed the applications of fractional calculus, to find inverse Laplace transform for functions having fractional power like $\frac{s}{\sqrt{s^2+1}}$ using the example of Bessel function.

The Bessel function

$$F(s) = \frac{1}{\sqrt{s^2+1}}$$

can be constructed as the Laplace transform of time varying system.

This work extends the work of C. F. Chen and Y. T. Tsay [12]. An example based on fractional differential equation has been demonstrated and comparative study of the exact solution and Walsh wavelet packet solutions has been done.