Chapter 2

ON THE CONVERGENCE OF THE WALSH TYPE WAVELET PACKET SERIES

2.1 Introduction

Wavelet analysis was originally introduced in order to improve seismic signal processing by switching from short time Fourier analysis to new algorithms better suited to detect and analyze abrupt changes in signals. It corresponds to a decomposition of phase space in which the tradeoff between time and frequency localization has been chosen to provide better and better time localization at high frequencies in return for poor frequency localization. This makes analysis more adapted to the study of transient phenomena and has proven a very successful approach to many problems in signal processing, numerical analysis and quantum mechanics. Wavelet packets is an important generalization of wavelet analysis, pioneered by R. Coifman, Y. Meyer,

M. V. Wickerhauser [50] and other researchers. Wavelet packet functions comprise a rich family of building blocks functions. Wavelet packet functions are still localized in time, but offer more flexibility than wavelets in representing different types of signals. In particular, wavelet packets are better at representing signals that exhibit oscillatory or periodic behaviour.

The Walsh type wavelet packets can be considered as the smooth generalizations of the Walsh functions and they have the same convergence properties for expansion of L^p functions, $1 < p < \infty$ as the Walsh- Fourier series. The Walsh type wavelet packet expansions fails for L^1-functions (refer [55]).

The aim of this chapter is to show the uniform convergence for periodic Walsh type wavelet packet expansion for L^p functions $1 < p < \infty$.

Recently, Morten Nielsen ([54], [55]) has proved the pointwise convergence a.e. of Walsh type wavelet packet series using the concept of Schauder basis and strong type (p,p) and the pointwise convergence a.e. of expansion of function from the Block space B_q, $1 < q < \infty$.Dealing with the convergence of the Walsh type wavelet packet expansions he proved the following theorem :

THEOREM 2.1.1 *The Carleson operator for any Walsh type wavelet packet system with $w_1 \in C^1(R)$ is of strong type (p,p) for $1 < p < \infty$.* ∎

In this chapter we are generalizing the above result by proving the uniform convergence of periodized Walsh type wavelet packet series using the properties of Walsh functions.

Also, most of the work on wavelet packets has been done in one dimension or using separable wavelet packets in higher dimensions. But, separable wavelets and wavelet packet bases both have several drawbacks for the applications to field like image analysis.

In [56], Nielsen has constructed nonseparable wavelet packet bases for $L^p(R^d)$ with nice convergence properties. He also proved results on a special wavelet packets construction that can be considered the multidimensional generalization of Walsh system on $[0,1)$. He proved that this multidimensional generalization share the two most important convergence properties of the classical Walsh system : The new system is a schauder bases for $L^p(R^d)$, $1 < p < \infty$ and the expansion of every L^p function in the system converges pointwise almost everywhere.

Further in this chapter, we are generalizing the following result proved by Nielsen by proving the uniform convergence of the periodic Walsh type wavelet packet series for $L^p(R^2)$, $1 < p < \infty$.

THEOREM 2.1.2 *Let L be the Carleson operator for a basic Walsh-type wavelet packet system $\{W_n^S\}_n$ associated with an almost isotrophic dilation matrix. Suppose $W_0 \in C^1(R^d)$. Then L is of strong type (p,p), $1 < p < \infty$.* ■

2.2 Preliminaries

In this chapter, we require the following definitions for proving the theorems:

DEFINITION 2.2.1 (Multiresolution Analysis :)

A multiresolution analysis is a sequence of closed subspaces $V_j, j \in Z$, of $L^2(R)$ satisfying

$$V_j \subset V_{j+1}, \qquad j \in Z,$$

$$f \in V_j \Longleftrightarrow f(2.) \in V_{j+1}, \qquad j \in Z,$$

$$\overline{\bigcup_{j \in Z} V_j} = L^2(R),$$

$$\bigcap_{j \in Z} V_j = 0,$$

There exists a $\phi \in V_0$ *such that* $\phi(. - k)_{k \in Z}$ *is an orthonormal basis for* V_0.

Given a multiresolution analysis we can construct an associated wavelet.

DEFINITION 2.2.2 (Conjugate Quadrature Filters :)

Let $h_n \in l^1(Z)$ *be a real valued sequences, and let* $g_k = (-1)^k h_{1-k}$ *for* $k \in Z$. *Define the operators* $H, G : l^2(Z) \to l^2(Z)$ *by*

$$(Ha)_k = \sum_{n \in Z} a_n h_{n-2k}$$

$$(Ga)_k = \sum_{n \in Z} a_n g_{n-2k}$$

The filters H *and* G *are called a pair of CQFs if*

$$\begin{aligned} 2HH^* = 2GG^* &= I \\ H1 &= 1, where \quad 1 = (....1, 1, 1, ...) \\ H^*G + G^*H &= I \\ HG^* = GH^* &= 0 \end{aligned}$$

DEFINITION 2.2.3 (Non-stationary wavelet packets :)

Let (ϕ, ψ) be the scaling function and wavelet associated with a multi-resolution analysis and let $(F_0^{(p)}, F_1^{(p)})$, $p \in N$ be a family of bounded operators on $l^2(Z)$ of the form

$$(F_\epsilon^p a)_k = \sum_{n \in Z} a_n h_\epsilon^{(p)}(n - 2k)$$

$$\textit{where,} \qquad \epsilon = 0, 1$$

with $h_1^{(p)}(n) = (-1)^n h_0^{(p)}(1-n)$, a real valued sequence in $l^1(Z)$ such that each $(F_0^{(p)}, F_1^{(p)})$ is a pair of conjugate quadrature filters.

We define a family of non-stationary wavelet packets $\{w_n\}_{n=0}^{\infty}$ recursively by letting $w_0 = \phi$, $w_1 = \psi$ and then for $n \in N$,

$$\begin{aligned} w_{2n}(x) &= \sqrt{2} \sum_{q \in Z} h_0^p(q) w_n(2x - q) \\ w_{2n+1}(x) &= \sqrt{2} \sum_{q \in Z} h_1^p(q) w_n(2x - q) \end{aligned}$$

where, $2^p \leq n < 2^{p+1}$.

The trigonometric polynomials given by

$$\begin{aligned} m_0^{(p)}(\xi) &= \frac{1}{2} \sum_k h_0^{(p)}(k).e^{-ik\xi} \\ m_1^{(p)}(\xi) &= \frac{1}{2} \sum_k h_1^{(p)}(k).e^{-ik\xi} \end{aligned}$$

are called the symbols of the filters.

DEFINITION 2.2.4 (Walsh system :)

The Walsh system $\{W_n\}_{n=0}^{\infty}$ *is defined recursively on* $[0,1)$ *by letting* $W_0 = \chi_{[0,1)}$ *and*

$$\begin{aligned} W_{2n}(x) &= W_n(2x) + W_n(2x-1) \\ W_{2n+1}(x) &= W_n(2x) - W_n(2x-1) \end{aligned}$$

The Walsh system is a family of wavelet packets obtained by letting $\phi = \chi_{[0,1)}$ and $\psi = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}$ and using the Haar filters in the definition of non-stationary wavelet packets (refer [54]).

The Walsh system is closed under pointwise multiplication.

Define the binary operator $\oplus : N_0 \times N_0 \longrightarrow N_0$. Let $x, y \in [0,1)$. Then x and y have unique representations $x = \sum_{j=0}^{\infty} x_j.2^{-j-1}$ and $y = \sum_{j=0}^{\infty} y_j.2^{-j-1}$ respectively. Suppose we choose them to be finite. By $x \oplus y$ we denote the dyadic sum of x and y defined as :

$$x \oplus y = \sum_{j=0}^{\infty} |x_j - y_j| 2^{-j-1}$$

Then $\int_0^1 f(x)dx = \int_0^1 f(x \oplus y)dx$ for all $f \in L^1$ and $y \in [0,1)$. Moreover ,

$$W_n(x \oplus y) = W_n(x).W_n(y)$$

2.2.1 Walsh wavelet packets

Wavelet packets were originally introduced in Coifman, Meyer and Wickerhauser [50] to improve the frequency resolution of signals achieved by a wavelet analysis.

A wavelet decomposition or transform simply reexpresses a function in terms of the wavelet bases $\{\psi_{j,k}(t)\}$. This amounts to decomposing the function space L^2 into a direct sum of orthogonal subspaces W_j and choosing the combination of the orthonormal bases for $W's$ as the orthonormal basis of L^2. In the case of finite data with information up to a resolution level J, a wavelet transform performs a decomposition of the space V_J in to a direct sum of orthogonal subspaces

$$V_J = W_{J-1} \oplus V_{J-1} = W_{J-1} \oplus W_{J-2} \oplus V_{J-2} = \dots\dots\dots = \oplus_{j=0}^{J-1} W_j \oplus V_0$$

and the union of bases of these subspaces forms a basis for the wavelet decomposition. This, of course, is by no means the only way to decompose the space L^2 or V_J.

From multi-resolution analysis, we know that the given basis function $\{\phi_{1,k}(t)\}$ of V_1,$\{\phi(t-k)\}$ and $\{\psi(t-k)\}$ constitute an orthonormal basis for V_0 and W_0 respectively and $V_1 = V_0 \oplus W_0$ where,

$$\phi(t) = \sqrt{2} \sum_k h_k \phi(2t-k)$$

and

$$\psi(t) = \sqrt{2} \sum_k g_k \phi(2t-k)$$

where h_k and g_k denote the low pass filter and high pass filter respectively.Thus the space V can be decomposed into a direct sum of the two orthogonal subspaces defined by their basis functions given by the above two equations. This "splitting trick" or splitting algorithm can be used to decompose W which leads to the so-called wavelet packet analysis.

For example, if we analogously define

$$W_2(t) = \sqrt{2} \sum_k h_k \psi(2t-k)$$

$$W_3(t) = \sqrt{2}\sum_k g_k \psi(2t-k)$$

then $\{W_2(t-k)\}$ and $\{W_3(t-k)\}$ are orthonormal basis functions for the two subspaces whose direct sum is W_1.

DEFINITION 2.2.5 (Wavelet Packets :)

For $n = 0, 1, 2, 3.....$ we define a sequence of functions as follows:

$$W_{2n}(t) = \sqrt{2}\sum_k h_k W_n(2t-k)$$

$$W_{2n+1}(t) = \sqrt{2}\sum_k g_k W_n(2t-k)$$

when $n = 0$, $W_0(t) = \phi(t)$, *the scaling function*

and $n = 1$, $W_1(t) = \psi(t)$, *the mother wavelet.*

Various combinations of functions and their translations and dilations can give rise to various bases for the function space. So we have a whole collection of orthonormal bases generated from $\{W_n(t)\}$. We call this collection " a library of wavelet packet bases" and the function of the form $W_{n,j,k} = 2^{\frac{j}{2}} W_n(2^j t - k)$ is called a wavelet packet.

DEFINITION 2.2.6 (Haar Filter:)

The Haar low pass quadrature mirror filter $\{h_0^{(k)}\}_k$ is given by $h_0(0) = h_0(1) = \frac{1}{\sqrt{2}}, h_0(k) = 0$ otherwise and the associated high pass filter $\{h_1(k)\}_k$ is given by $h_1(k) = (-1)^k h_0(1-k)$.

DEFINITION 2.2.7 (Walsh Type Wavelet Packets :)

Let $\{w_n\}_{n\geq 0, k\in Z}$ be a family of non-stationary wavelet packets constructed by using a family $\{h_0^{(p)}(n)\}_{p=1}^{\infty}$ of finite filters for which there is a constant $k \in N$ such that

$h_0^{(p)}(n)$ is the Haar filter for every $p \geq k$. If $w_1 \in C^1(R)$ is compactly supported then we call $\{w_n\}_{n\geq 0}$ a family of Walsh type wavelet packet series.

DEFINITION 2.2.8 (Periodic Walsh Type Wavelet Packets :)

Let $\{w_n\}_{n=0}^{\infty}$ be a family of Walsh type wavelet packets. For $n \in N_0$, we define the corresponding periodic Walsh type wavelet packets $\widetilde{w_n}$ by

$$\widetilde{w_n}(x) = \sum_{k \in Z} w_n(x-k)$$

It follows from the Fubini's theorem that $\{\widetilde{w_n}\}_{n=0}^{\infty}$ is an orthonormal basis for $L^2(0,1)$ (refer [54]).

Let A be a $d \times d$-matrix such that $A : Z^d \times Z^d$. If all eigenvalues of A have absolute value strictly greater than 1 then we call A a dilation matrix.

e.g The 2×2 matrices

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

are examples of dilations matrices with determinant ± 2. The first matrix is known as quincunx dilation matrix.

DEFINITION 2.2.9 (Multi-resolution Analysis for $L^2(R^d)$:)

Multi-resolution analysis associated with a dilation matrix A is a sequence of closed subspaces $(V_j)_{j\in Z}$ of $L^2(R^d)$ satisfying

- $V_j \subset V_{j+1} \quad \forall \ j \in Z$
- $\overline{\cup_{j\in Z} V_j} = L^2(R^d) \quad$ *and* $\quad \cap_{j\in Z} V_j = 0$
- $f \in V_j \iff f(Ax) \in V_{j+1} \quad \forall \ j \in Z$
- *there exists a function $\phi \in V_0$ called a scaling function such that the system $\{\phi(. - \gamma)\}_{\gamma \in Z^d}$ is an orthonormal basis for V_0.*

DEFINITION 2.2.10 (Basic Non-Stationary Wavelet Packets(refer [56]):) *Let* $\{(m_0^{(p)}, m_1^{(p)})\}_{p=1}^{\infty}$ *be a sequence of orthogonal quadrature filters associated with* (A, Γ), $\Gamma \in Z^d$. *We define the basic non-stationary wavelet packets* $\{w_n\}_{n=0}^{\infty}$ *by* $w_0 = \phi, w_1 = \psi$ *and for* $2^k \leq n < 2^{k+1}$ *with binary expansion* $n = \sum_{j=1}^{k+1} \varepsilon_j 2^{j-1}$, *we let*

$$\widehat{w_n(\xi)} = \left[\prod_{j=1}^{k+1} m_{\varepsilon_j}^{(k-j+2)}(D_j\xi)\right]\widehat{\phi}(D^{k+1}\xi)$$

$$where \qquad D = (A^*)^{-1}$$

Generalized Haar Functions (refer [56]) :

Let A be a $d \times d$ dilation matrix with $|detA| = 2$. We are interested in the case where there is an associated multi-resolution analysis generated by a scaling function given by the characteristic function of a set $Q \subset R^d$ called a tile.

As mentioned by Lagarias and Wang ([42],[43]) for general A and $d > 3$ there is no guarantee that such a set Q exists. So we have to restrict our construction to dilation matrices A which admit such a tile. The situation is better for $1 \leq d \leq 3$ since it can be proved that a tile always exists (refer [[42],[43]]). Here we will assume that A is such that an associated tile Q exists.

The set Q has many such properties under the action of A. One such property is $AQ = Q \cup (Q + \Gamma_Q)$ for some $\Gamma_Q \in Z^d$ and we always have $|Q| = 1$ (refer [56]). Hence,

$$Q = A^{-1}Q \cup A^{-1}(Q + \Gamma_Q)$$

and $\widehat{\chi_Q(\xi)} = m_0(D\xi)\widehat{\chi_Q(D\xi)}$ where $m_0(\xi) = \frac{1}{2} + \frac{1}{2}e^{-i(\Gamma_Q\xi)}$. Also note that $|A^{-1}Q| = \frac{1}{2}$. So A^{-1} splits Q into two subtiles of equal measure.

Let

$$D_0 = \{\Omega : \Omega = A^{-j}(Q+\gamma), \gamma \in Z^d, j \geq 0 \ \ and \ \ \Omega \subset Q\}$$

denote the collection of Q dyadic sets.

DEFINITION 2.2.11 (Generalization of Haar function of $[0,1]$:) *With Q and Γ_Q as above, we define the generalized Haar function by*

$$H(x) = \chi_{A^{-1}Q}(x) - \chi_{A^{-1}(Q+\Gamma_Q)}(x)$$

The Haar system on Q is given by

$$\{\chi_Q\} \cup \{2^{\frac{j}{2}} H(A^j x - k) : \quad j \geq 0, k \in Z^d \quad and \quad supp(H(A^j x - k)) \subset Q\}$$

Generalized Walsh Functions (refer [56]) :

The Walsh system on $[0,1)$ is the system of basic wavelet packets associated with Haar multi-resolution analysis.

As mentioned in [56] generalized Walsh function can be defined through

1. Haar low pass and high pass filters starting from the Haar scaling function and wavelet.

2. By letting $w_0(x) = \chi_Q(x)$ and $w_{2n+\epsilon}(x) = w_n(Ax) + (-1)^{\epsilon} w_n(Ax - \Gamma_Q)$ $\epsilon = 0, 1$.

3. As the product system on the probability space (Q, dx) defined by using generalized Rademacher functions.

In the present chapter we concentrate on (3). The generalized Rademacher functions are obtained by letting

$$\mathbf{r}_0(x) = \sum_{k \in Z^d} H(x-k) \quad \in L^\infty(R^d)$$

where H is the Haar function and we define $\mathbf{r}_n(x) = \mathbf{r}_0(A^n x)$. Then for $n \in N_0$ with binary expansion $n = \sum_{j=0}^{\infty} \epsilon_j 2^j$ we have,

$$W_n(x) = \chi_Q(x) \prod_{j=0}^{\infty} \Big(\mathbf{r}_j(x)\Big)^{\epsilon_j}$$

which can be proved easily by induction.

DEFINITION 2.2.12 (Periodic Generalized Walsh type Wavelet packets :)

For the wavelet packet

$$w^{per}_{n,j,k}(x) = \chi_\Sigma(x).2^{\frac{j}{2}} \sum_{\gamma \in Z^d} w_n(A^j(x-\gamma)-k)$$

where Σ is any tile of R^d such as Q itself of the fundamental domain $[0,1)^d$.

LEMMA 2.2.13 (refer [56] :)

The basic wavelet packets

$$\{w_n(x-k) : 0 \le n < 2^j,\ k \in Z^d\}$$

form basis for V_j. Furthermore, $\{w_n(x-k) : n \in N_0,\ k \in Z^d\}$ form an orthonormal basis for $L^2(R^d)$.

DEFINITION 2.2.14 (Modulus of continuity refer([25]) :)

The total modulus of continuity of a function $f \in L^p$ in L^p-norm, $1 \le p \le \infty$ is defined by

$$\omega_1(f;\delta_1,\delta_2)_p := \sup\{||f(x \oplus u, y \oplus v) - f(x,y)||_p : 0 \le u \le \delta_1 \ \&\ 0 \le v \le \delta_2\}$$

While the partial modulus of continuity are defined by

$$w_{1,x}(f,\delta_1)_p = w_1(f:\delta_1,0)$$

$$w_{1,y}(f,\delta_2)_p = w_1(f:0,\delta_2)$$

Banach Steinhaus Theorem :

For $\delta_1, \delta_2 \geq 0$ and $f \in L^p$

$$\lim_{\delta_1,\delta_2 \to 0} w_1(f;\delta_1,\delta_2)_p = 0$$

DEFINITION 2.2.15 (Generalized Minkowski's Inequality :)

(i) For $f \in L^p([a,b]\times[c,d])$ for some $1 \leq p < \infty$,

$$\left\{\int_a^b \left|\int_c^d f(x,y)dy\right|^p dx\right\}^{\frac{1}{p}} \leq \int_c^d \left\{\int_a^b |f(x,y)|^p\, dx\right\}^{\frac{1}{p}} dy$$

We will also use the multivariate version i.e. when the single integrals $\int_a^b$ and $\int_c^d$ are replaced by the double ones $\int_{a_1}^{b_1}\int_{a_2}^{b_2}$and $\int_{c_1}^{d_1}\int_{c_2}^{d_2}$ respectively.

(ii) Let $1 \leq p < \infty$. If $\alpha_i, \beta_i \in K(i = 1,2,3.....)$, then

$$\left(\sum_{i=1}^{n} |\alpha_i + \beta_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |\alpha_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |\beta_i|^p\right)^{\frac{1}{p}}$$

DEFINITION 2.2.16 (Dirichlet Kernel :)

We define the Dirichlet Kernel for $f \in L^2(R^2)$ corresponding to generalized Walsh type wavelet packets as

$$K_M(x,y) = \sum_{n=0}^{M-1} \widetilde{w_n(x,y)}$$

2.3 On the Uniform Convergence of Periodized Walsh type Wavelet Packet Series

THEOREM 2.3.1 *Let $f \in L^p[0,1)$ for $1 < p < \infty$ be a function of period 1. Then,*

$$\lim_{k\to\infty} S_k f(x) = f(x)$$

uniformly in x, where $S_k f(x)$ is the k^{th} partial sum of periodic Walsh type wavelet packet series.

Proof: Morten Nielsen [54] has proved that $\widetilde{w_n(x)}$ is forming an orthonomal basis for $L^2[0,1)$.Hence every Lebesgue integrable function $f(x)$ of period 1 can be written as

$$f(x) = \sum_{n=0}^{\infty} c_n \widetilde{w_n(x)}$$

$$\text{where,} \quad c_n = \langle f, \widetilde{w_n} \rangle = \int_0^1 f(y)\widetilde{w_n(y)}dy$$

We shall find a simple expression for the partial sum for the periodic Walsh type wavelet packet series.

$$\begin{aligned} S_k f(x) &= \sum_{n=0}^{k-1} c_n \widetilde{w_n(x)} \\ &= \sum_{n=0}^{k-1} \left\{ \int_0^1 f(y)\widetilde{w_n(y)}dy \right\} \widetilde{w_n(x)} \\ &= \sum_{n=0}^{k-1} \left\{ \int_0^1 f(y)\widetilde{w_n(x)}\widetilde{w_n(y)}dy \right\} \\ &= \sum_{n=0}^{k-1} \left\{ \int_0^1 f(x \oplus y)\widetilde{w_n(y)}dy \right\} \end{aligned}$$

$$= \left\{ \int_0^1 f(x \oplus y) \sum_{n=0}^{k-1} \widetilde{w_n(y)} dy \right\}$$

$$= \left\{ \int_0^1 f(x \oplus y) K_k(y) dy \right\}$$

where $\quad K_k(y) = \sum_{n=0}^{k-1} \widetilde{w_n(y)} \quad$ is the Dirichlet kernel.

Due to the orthonormality of periodic Walsh type wavelet packet we have

$$\begin{aligned} \int_0^1 K_k(y) dy &= \int_0^1 \sum_{n=0}^{k-1} \widetilde{w_n(y)} dy \\ &= \sum_{n=0}^{k-1} \int_0^1 \widetilde{w_n(y)} \ \widetilde{w_0(y)} dy \\ &= 1 \end{aligned} \tag{2.3.1}$$

Hence,

$$\begin{aligned} S_k f(x) - f(x) &= \sum_{n=0}^{k-1} \int_0^1 f(x \oplus y) \widetilde{w_n(y)} dy - f(x).1 \\ &= \sum_{n=0}^{k-1} \int_0^1 f(x \oplus y) \widetilde{w_n(y)} dy - \sum_{n=0}^{k-1} \int_0^1 f(x) \widetilde{w_n(y)} dy \\ &= \int_0^1 [f(x \oplus y) - f(x)] \sum_{n=0}^{k-1} \widetilde{w_n(y)} dy \\ &= \int_0^1 [f(x \oplus y) - f(x)] \, K_k(y) dy \end{aligned} \tag{2.3.2}$$

Now for each natural number $k = 2^m + k', \quad 0 < k' < 2^m$ (refer[57]),

$$K_k(y) = \sum_{n=0}^{k-1} \widetilde{w_n(y)}$$

$$K_{2^m+k'}(y) = \sum_{n=0}^{2^m+k'-1} \widetilde{w_n(y)}$$

$$= \sum_{n=0}^{2^m-1} \widetilde{w_n(y)} + \sum_{n=2^m}^{2^m+k'-1} \widetilde{w_n(y)}$$

where,

$$\sum_{n=2^m}^{2^m+k'-1} \widetilde{w_n(y)} = \widetilde{w_{2^m}(y)} + \widetilde{w_{2^m+1}(y)} + \widetilde{w_{2^m+2}(y)} + \ldots\ldots. + \widetilde{w_{2^m+k'-1}(y)}$$

$$= \widetilde{w_{2^m}(y)}.\widetilde{w_0(y)} + \widetilde{w_{2^m}(y)}.\widetilde{w_1(y)} + \widetilde{w_{2^m}(y)}.\widetilde{w_2(y)} + \ldots.. + \widetilde{w_{2^m}(y)}.\widetilde{w_{k'-1}(y)}$$

$$= \widetilde{w_{2^m}(y)} \left[\widetilde{w_0(y)} + \widetilde{w_1(y)} + \widetilde{w_2(y)} + \ldots\ldots + \widetilde{w_{k'-1}(y)}\right]$$

$$= \widetilde{w_{2^m}(y)} \left[\sum_{n=0}^{k'-1} \widetilde{w_n(y)}\right]$$

Hence,

$$K_{2^m+k'}(y) = \sum_{n=0}^{2^m-1} \widetilde{w_n(y)} + \widetilde{w_{2^m}(y)} \sum_{n=0}^{k'-1} \widetilde{w_n(y)}$$

$$= K_{2^m}(y) + \widetilde{w_{2^m}(y)} K_{k'}(y)$$

Now,

$$\left\| S_k f(x) - f(x) \right\|_p = \left[\int_0^1 \left| \int_0^1 [f(x \oplus y) - f(x)] \left\{ K_{2^m}(y) + \widetilde{w_{2^m}(y)} K_{k'}(y) \right\} dy \right|^p dx \right]^{\frac{1}{p}}$$

$$
\begin{aligned}
&\leq O(1).\left\{ \left[\int_0^1 \left|\int_0^1 [f(x\oplus y)-f(x)]\, K_{2^m}(y)dy\right|^p dx\right]^{\frac{1}{p}} \right.\\
&+ \left. \left[\int_0^1 \left|\int_0^1 [f(x\oplus y)-f(x)]\, \widetilde{w_{2^m}(y)} K_{k'}(y)dy\right|^p dx\right]^{\frac{1}{p}} \right\}\\
&= O(1)\,[A+B]
\end{aligned}
$$

where,

$$
A = \left[\int_0^1 \left|\int_0^1 [f(x\oplus y)-f(x)]\, K_{2^m}(y)dy\right|^p dx\right]^{\frac{1}{p}}
$$

(Using generalized Minkowski's inequality)

$$
A \leq \int_0^1 \left[\int_0^1 \left|f(x\oplus y)-f(x)\right|^p . \left|K_{2^m(y)}\right|^p dx\right]^{\frac{1}{p}} dy \tag{2.3.3}
$$

By the property of periodic Walsh wavelet packets

$$
K_{2^m}(y) = \begin{cases} 2^m & y\in[0,2^{-m}) \\ 0 & y\in[2^{-m},1] \end{cases}
$$

From (2.3.4)

$$
\begin{aligned}
A &\leq \int_0^1 \left\|f(x\oplus y)-f(x)\right\|_p .K_{2^m(y)}dy\\
&= \int_0^{2^{-m}} \left\|f(x\oplus y)-f(x)\right\|_p .2^m dy\\
&\leq w_1(f,2^{-m}).2^m.\frac{1}{2^m}\\
&\to 0 \text{ as } \quad m\to\infty
\end{aligned} \tag{2.3.4}
$$

follows from Banach Steinhaus Theorem.

Now,

$$B = \left[\int_0^1 \left|\int_0^1 [f(x\oplus y) - f(x)]\,\widetilde{w_{2^m}(y)} K_{k'}(y)dy\right|^p dx\right]^{\frac{1}{p}}$$

Using generalized Minkowski's inequality

$$\leq \int_0^1 \left[\int_0^1 \left|f(x\oplus y) - f(x)\right|^p \left|\widetilde{w_{2^m}(y)}\right|^p \left|K_{k'}(y)\right|^p dx\right]^{\frac{1}{p}} dy$$

$$= \int_0^1 \left\|f(x\oplus y) - f(x)\right\|_p \left|\widetilde{w_{2^m}(y)}\right| \left|K_{k'}(y)\right| dy \tag{2.3.5}$$

By the definition and property of periodic Walsh wavelet packets, we have

$$\left|\widetilde{w_{2^m}(y)}\right| \leq 1 \quad and \quad K_{k'}(y) = k' \quad \text{for} \quad k' \in [0, 2^{-m}]$$

Hence from (2.3.5)

$$B \leq \int_0^{\frac{1}{2^m}} \left\|f(x\oplus y) - f(x)\right\|_p .k'dx + \int_{\frac{1}{2^m}}^{\frac{2}{2^m}} \left\|f(x\oplus y) - f(x)\right\|_p .k'dx$$

$$+ \int_{\frac{2}{2^m}}^{\frac{3}{2^m}} \left\|f(x\oplus y) - f(x)\right\|_p .k'dx + \ldots\ldots\ldots\ldots + \int_{\frac{2^m-1}{2^m}}^{1} \left\|f(x\oplus y) - f(x)\right\|_p .k'dx$$

$$= k'.w(f, 2^{-m}) + k'.w(f, 2^{-m}) + \ldots\ldots. + k'.w(f, 2^{-m})$$

$$\to 0 \qquad as \;\; m \to \infty \tag{2.3.6}$$

Thus using (2.3.3) and (2.3.6) we have

$$\left\|S_k f(x) - f(x)\right\|_p \to 0 \;\; as \qquad m \to \infty$$

which proves that periodic Walsh type wavelet packets are uniformly convergent. ∎

2.4 On the Uniform Convergence of Generalized Walsh type Wavelet Packet Series

THEOREM 2.4.1 *Let $f \in L^p[0,1)^2$ for $1 < p < \infty$ be a function of period 1. Then,*

$$\lim_{k\to\infty} S_k f(x,y) = f(x,y)$$

uniformly in x, where $S_k f(x,y)$ is the k^{th} partial sum of periodic generalized Walsh type wavelet packet series.

Proof: Morten Nielsen [56] has proved that $\widetilde{w_n(x,y)}$ is forming an orthonormal basis for $L^2[0,1]^2$. Hence Lebesgue integrable function with period 1 can be written as :

$$\begin{aligned} f(x,y) &= \textstyle\sum_{n=0}^{\infty} c_n \widetilde{w_n(x,y)} \\ \text{with,} \qquad c_n &= \langle f, \widetilde{w_n} \rangle = \textstyle\int_0^1 \int_0^1 f(r,s)\widetilde{w_n(r,s)}drds \end{aligned}$$

Now, we shall find a simple expression for the partial sum for the periodic generalized Walsh type wavelet packet series.

We have,

$$\begin{aligned} S_k f(x,y) &= \sum_{n=0}^{k-1} c_n \widetilde{w_n(x,y)} \qquad (2.4.1) \\ &= \sum_{n=0}^{k-1} \left\{ \int_0^1 \int_0^1 f(r,s)\widetilde{w_n(r,s)}drds \right\} \widetilde{w_n(x,y)} \\ &= \sum_{n=0}^{k-1} \left\{ \int_0^1 \int_0^1 f(x\oplus r, y\oplus s)\widetilde{w_n(r,s)}drds \right\} \end{aligned}$$

Using the orthonormality of Walsh type wavelet packet series, we have

$$\begin{aligned}\int_0^1\int_0^1 K_k(r,s)drds &= \int_0^1\int_0^1\sum_{n=0}^{k-1}\widetilde{w_n(r,s)}drds \\ &= \sum_{n=0}^{k-1}\int_0^1\int_0^1 \widetilde{w_n(r,s)}\ \ \widetilde{w_0(r,s)}drds \\ &= 1 \qquad (2.4.2)\end{aligned}$$

Thus using (2.4.1) and (2.4.2), we have

$$\begin{aligned}S_kf(x,y)-f(x,y) &= \sum_{n=0}^{k-1}\int_0^1\int_0^1 [f(x\oplus r,y\oplus s)]\,\widetilde{w_n(r,s)}drds - f(x,y).1 \\ &= \sum_{n=0}^{k-1}\int_0^1\int_0^1 \Big[f(x\oplus r,y\oplus s)\Big]\widetilde{w_n(r,s)}drds \\ &- f(x,y)\int_0^1\int_0^1\sum_{n=0}^{k-1}\widetilde{w_n(r,s)}drds \\ &= \sum_{n=0}^{k-1}\int_0^1\int_0^1 \Big[f(x\oplus r,y\oplus s)-f(x,y)\Big]\widetilde{w_n(r,s)}drds \\ &= \int_0^1\int_0^1 \Big[f(x\oplus r,y\oplus s)-f(x,y)\Big]K_k(r,s)drds \qquad (2.4.3)\end{aligned}$$

Now, for each natural number $k = 2^m + k'$, $0 < k' < 2^m$ according to C.W. Onneweer [57]

$$K_k(r,s) = \sum_{n=0}^{k-1}\widetilde{w_n(r,s)}$$

$$
\begin{aligned}
K_{2^m+k'}(r,s) &= \sum_{n=0}^{2^m+k'-1} \widetilde{w_n(r,s)} \\
&= \sum_{n=0}^{2^m-1} \widetilde{w_n(r,s)} + \sum_{n=2^m}^{2^m+k'-1} \widetilde{w_n(r,s)} \qquad (2.4.4)
\end{aligned}
$$

where,

$$
\begin{aligned}
\sum_{n=2^m}^{2^m+k'-1} \widetilde{w_n(r,s)} &= \widetilde{w_{2^m}(r,s)} + \widetilde{w_{2^m+1}(r,s)} + \ldots\ldots + \widetilde{w_{2^m+k'-1}(r,s)} \\
&= \widetilde{w_{2^m}(r,s)}\widetilde{w_0(r,s)} + \widetilde{w_{2^m}(r,s)}\widetilde{w_1(r,s)} + \widetilde{w_{2^m}(r,s)}\widetilde{w_2(r,s)} \\
&+ \ldots\ldots + \widetilde{w_{2^m}(r,s)}\widetilde{w_{k'-1}(r,s)} \\
&= \widetilde{w_{2^m}(r,s)}\left[\widetilde{w_0(r,s)} + \widetilde{w_1(r,s)} + \widetilde{w_2(r,s)} + \ldots.. + \widetilde{w_{k'-1}(r,s)}\right] \\
&= \widetilde{w_{2^m}(r,s)}\left[\sum_{n=0}^{k'-1} \widetilde{w_n(r,s)}\right] \qquad (2.4.5)
\end{aligned}
$$

Thus using (2.4.4) and (2.4.5),

$$
K_{2^m+k'}(r,s) = \sum_{n=0}^{2^m-1} \widetilde{w_n(r,s)} + \widetilde{w_{2^m}(r,s)} \sum_{n=0}^{k'-1} \widetilde{w_n(r,s)} \qquad (2.4.6)
$$

Hence,

$$
\begin{aligned}
K_{2^m+k'}(r,s) &= \sum_{n=0}^{2^m-1} \widetilde{w_n(r,s)} + \widetilde{w_{2^m}(r,s)} \sum_{n=0}^{k'-1} \widetilde{w_n(r,s)} \\
&= K_{2^m}(r,s) + \widetilde{w_{2^m}(r,s)} K_{k'}(r,s)
\end{aligned}
$$

and using (2.4.3)

$$\begin{aligned}\Big\|S_k f(x,y)-f(x,y)\Big\|_p &= \left[\int_0^1\int_0^1\left|\int_0^1\int_0^1\Big[f(x\oplus r,y\oplus s)-f(x,y)\Big]\right.\right.\\ &\left.\left.\Big\{K_{2^m}(r,s)+\widetilde{w_{2^m}(r,s)}K_{k'}(r,s)\Big\}drds\right|dxdy\right]^{\frac{1}{p}}\\ &= O(1).\left[\left\{\int_0^1\int_0^1\left|\int_0^1\int_0^1\Big[f(x\oplus r,y\oplus s)-f(x,y)\Big]\right.\right.\right.\\ &\left.\left.K_{2^m}(r,s)drds\right|^p dxdy\right\}^{\frac{1}{p}}\\ &+\left\{\int_0^1\int_0^1\left|\int_0^1\int_0^1\Big[f(x\oplus r,y\oplus s)-f(x,y)\Big]\right.\right.\\ &\left.\left.\left.\widetilde{w_{2^m}(r,s)}K_{k'}(r,s)drds\right|^p dxdy\right\}^{\frac{1}{p}}\right]\\ &= O(1).\,[A+B]\end{aligned}$$

where,

$$A = \left[\int_0^1\int_0^1\left\|\int_0^1\int_0^1\Big[f(x\oplus r,y\oplus s)\Big]K_{2^m(r,s)}drds\right\|^p dxdy\right]^{\frac{1}{p}}$$

Using Generalized Minkowski's inequality

$$\leq \int_0^1\int_0^1\left[\int_0^1\int_0^1\Big\|f(x\oplus r,y\oplus s)\Big\|^p\Big|K^p_{2^m(r,s)}\Big|dxdy\right]^{\frac{1}{p}}drds \qquad (2.4.7)$$

Since,

$$K_{2^m}(r,s) = \begin{cases}2^m & (r,s)\in[0,2^{-m})^2\\ 0 & (r,s)\in[2^{-m},1]^2\end{cases}$$

Hence, from (2.4.7) we have,

$$\begin{aligned} A &\leq \int_0^1\int_0^1 \left\| f\left(x\oplus r, y\oplus s\right) - f(x,y)\right\|_p dxdy \\ &= \int_0^{2^{-m}}\int_0^{2^{-m}} \left\| f\left(x\oplus r, y\oplus s\right) - f(x,y)\right\|_p .2^m dxdy \\ &\leq w_1\left(f, 2^{-m}, 2^{-m}\right).2^{-m}.\frac{1}{2^{-m}}.\frac{1}{2^{-m}} \\ &\longrightarrow 0 \quad as \quad m\to\infty \end{aligned} \tag{2.4.8}$$

which follows from Banach Steinhaus theorem.

Now,

$$B = \left[\int_0^1\int_0^1 \left|\int_0^1\int_0^1 [f(x\oplus r, y\oplus s]\, \widetilde{w_{2^m}(r,s)} K_{k'}(r,s).drds\right|^p dxdy\right]^{\frac{1}{p}}$$

Using Generalized Minkowski's inequality

$$\begin{aligned} B &= \int_0^1\int_0^1 \left[\int_0^1\int_0^1 \left|f\left(x\oplus r, y\oplus s\right) - f(x,y)\right|^p \left|\widetilde{w_{2^m}(r,s)}\right|^p \right. \\ &\quad \left. \left|K_{k'}(r,s)\right|^p dxdy\right]^{\frac{1}{p}} drds \\ &= \int_0^1\int_0^1 \left\| f\left(x\oplus r, y\oplus s\right) - f(x,y)\right\|_p \left|\widetilde{w_{2^m}(r,s)}\right| \left|K_{k'}(r,s)\right| drds \end{aligned} \tag{2.4.9}$$

Since, $\left|\widetilde{w_{2^m}(r,s)}\right| \leq 1$ and

$$\begin{aligned} K_{k'}(r,s) &= k' \qquad k'\in[0,2^{-m})^2 \\ &\leq k' \qquad otherwise \end{aligned}$$

We have from (2.4.9)

$$
\begin{aligned}
B &\leq \int_0^{2^{-m}} \int_0^{2^{-m}} \left\| f(x \oplus r, y \oplus s) - f(x,y) \right\|_p .k' dr ds \\
&+ \int_{\frac{1}{2^m}}^{\frac{2}{2^m}} \int_{\frac{1}{2^m}}^{\frac{2}{2^m}} \left\| f(x \oplus r, y \oplus s) - f(x,y) \right\|_p .k' dr ds \\
&+ \int_{\frac{2}{2^m}}^{\frac{3}{2^m}} \int_{\frac{2}{2^m}}^{\frac{3}{2^m}} \left\| f(x \oplus r, y \oplus s) - f(x,y) \right\|_p .k' dr ds \\
&+ \ldots\ldots + \int_{\frac{2^m-1}{2^m}}^{1} \int_{\frac{2^m-1}{2^m}}^{1} \left\| f(x \oplus r, y \oplus s) - f(x,y) \right\|_p .k' dr ds \\
&= k' w_1(f, 2^{-m}, 2^{-m}) + k' w_1(f, 2^{-m}, 2^{-m}) + \ldots\ldots + k' w_1(f, 2^{-m}, 2^{-m}) \\
&\longrightarrow 0 \quad as \quad m \to \infty \qquad (2.4.10)
\end{aligned}
$$

Using (2.4.8) and (2.4.10) we have,

$$\left\| S_k f(x,y) - f(x,y) \right\| \longrightarrow 0 \quad as \quad m \to \infty$$

which proves the result.