CHAPTER III

In chapter I we have described a fully efficient least squares method for estimating α, δ, β and ρ, and the errors of these estimates. The method is one of successive approximation and requires a reasonably accurate initial estimate of ρ. In chapter II we have pointed out that the least squares estimate of ρ, $\hat{r}$ say, are of the form

$$\hat{r} = \sum_{1}^{n-1} w_x(\hat{r}) y_x / \sum_{1}^{n-1} w_x(\hat{r}) y_{x-1} \quad ,$$

where $w_x(\hat{r})$ are complicated functions of $\hat{r}$. There we have described, for each of the cases n = 5,6,7 and 8, a estimate which is a function of the ratio of linear functions of y's, which leads to very simple estimates of reasonably high efficiency over the useful range of ρ.

In this chapter, the estimate of ρ is considered under two alternative methods: Shah and Khatri [32]:

(i) Patterson's [26] method of estimating ρ by considering a ratio of two quadratic functions of y's, which he calls a "Quadratic Estimate".

(ii) Modified Hartley's method suggested by Khatri and Shah [16] (chapter VI).

The relative efficiencies and biases of these estimates are considered in details for n = 5,6,7 and 8 equally spaced ordinates, the observed y's being independently and normally distributed about their expectation with constant variance.

1. Quadratic Estimate of ρ.

(A) Patterson's quadratic estimator:

The curve $y = \alpha + \delta x + \beta \rho^{x}$ can be written as

$$y_{x+1} = \alpha(1-\rho) + \delta + \delta(1-\rho)x + \rho y_x \quad \ldots \quad (3.1)$$

which is one of the family of estimates obtained by a procedure which Hartley [10] has termed internal regression. The estimate of ρ given by the multiple regression of y_{x+1} on x and y_x in equation (3.1) can be expressed as

$$r = A/B = \sum_{1}^{n-1} w'_x y_x \Big/ \sum_{1}^{n-1} w'_x y_{x-1} \;, \quad \ldots \quad (3.2)$$

where $$w'_{x+1} = y_x - \Big(\sum_{1}^{n-1} y_x\Big)/(n-1) - (x-\bar{x}) \frac{\sum_{1}^{n-1} x y_x - \bar{x}\sum_{1}^{n-1} y_x}{x^2 - (n-1)\bar{x}^2}, \quad (3.3)$$

and $\bar{x} = (\sum_{1}^{n-1} x)/(n-1)$, $(x=0,1,2,\ldots, n-2)$.

Estimates of this type, as described by Patterson, given by ratios of quadratic functions of the y's, are known as quadratic ~~fxxx~~ estimators. We can write the difference equation (3.1) in a more general form as

$$y_{x+1} = (k+1\rho)^{-1}\Big[\{k y_x + 1 y_{x+1}\} + \delta k(1-\rho)x + k(\alpha + \delta - \rho\alpha)\Big] \;, \ldots (3.4)$$

where $k \neq 0$. The estimate of the regression coefficient is

$$\underline{y}_1' D (k\underline{y}_o + l\underline{y}_1) / (k\underline{y}_o' + l\underline{y}_1') D (k\underline{y}_o + l\underline{y}_1), \quad \ldots (3.5)$$

where $\underline{y}_1' = (y_1, y_2, \ldots y_{n-1})$, $\underline{y}_o' = (y_o, y_1, \ldots, y_{n-2})$ and the matrix D, of order $(n-1)x(n-1)$, is such that

$$D\, \underline{1}_{n-1} = 0 \quad \text{and} \quad D\, \underline{t} = 0 \qquad \ldots (3.6)$$

where $\underline{1}_{n-1}' = (1,1,\ldots,1)$: $(n-1)x1$ and $\underline{t}' = (0,1,2,\ldots n-2)$. And equation (3.2) can be conveniently expressed in matrix notation as

$$A = \underline{y}_1' \underline{w}_o = \underline{y}_1' D_o \underline{y}_o \qquad \ldots (3.7)$$

$$B = \underline{y}_o' \underline{w}_o = \underline{y}_o' D_o \underline{y}_o \qquad \ldots (3.8)$$

where $\underline{w}_o' = (w_1, w_2, \ldots w_{n-1})$ and D_o will be described later.

$$r\,(0,k,l) = \underline{y}_1' D (k\underline{y}_o + l\underline{y}_1) / \underline{y}_o' D (k\underline{y}_o + l\underline{y}_1) \ldots (3.9)$$

The significance of 0 in r(0,k,l) will be explained later and k and l are self explanatory. The matrix D may not be symmetrical in general.

As a particular case of D, the matrix $D_o = (d_{ij})$ found from (3.3) can be written as

$d_{ij} = -2\left\{n(2n-1-3i-3j)+6ij\right\}/n(n-1)(n-2), \ i \neq j$ ...(3.10)

and

$d_{ii} = 1-2\left\{n(2n-1-6i)+6i^2\right\}/n(n-1)(n-2), \ i \neq j$ ... (3.11)

$i, j = 1,2,3\ldots n-1.$

Here, the matrix D_0 is symmetric and idempotant. When n=5,6,7 and 8, we have the following matrices. Please note that each column (or row) sum is equal to zero and the sum of all multiples of each element of column (or row) by 1,2,3,... respectively is also equal to zero.

When n = 5,

$$D = \frac{1}{10}\begin{bmatrix} 3 & -4 & -1 & 2 \\ -4 & 7 & -2 & -1 \\ -1 & -2 & 7 & -4 \\ 2 & -1 & -4 & 3 \end{bmatrix}$$

When n = 6,

$$D = \frac{1}{10}\begin{bmatrix} 4 & -4 & -2 & 0 & 2 \\ -4 & 7 & -2 & -1 & 0 \\ -2 & -2 & 8 & -2 & -2 \\ 0 & -1 & -2 & 7 & -4 \\ 2 & 0 & -2 & -4 & 4 \end{bmatrix}$$

When $n = 7$

$$D = \frac{1}{105}\begin{bmatrix} 50 & -40 & -25 & -10 & 5 & 20 \\ -40 & 74 & -22 & -13 & -4 & 5 \\ -25 & -22 & 86 & -16 & -13 & -10 \\ -10 & -13 & -16 & 86 & -22 & -25 \\ 5 & -4 & -13 & -22 & 74 & -40 \\ 20 & 5 & -10 & -25 & -40 & 50 \end{bmatrix}.$$

When $n = 8$

$$D = \frac{1}{168}\begin{bmatrix} 90 & -60 & -42 & -24 & -6 & 12 & 30 \\ -60 & 120 & -36 & -24 & -12 & 0 & 12 \\ -42 & -36 & 138 & -24 & -18 & -12 & -6 \\ -24 & -24 & -24 & 144 & -24 & -24 & -24 \\ -6 & -12 & -18 & -24 & 138 & -36 & -42 \\ 12 & 0 & -12 & -24 & -36 & 120 & -60 \\ 30 & 12 & -6 & -24 & -42 & -60 & 90 \end{bmatrix}.$$

It is easy to show that $r(0,k,l)$ is the estimate of obtain-ed from the regression of $k'y_x + l'y_{x+1}$ on $ky_x + ly_{x+1}$, where k' and l' take any values such that $kl' \neq k'l$.

(B) Modified Hartley's Method:

The difference equation (3.1) can be written as

$$y_{x+1}-y_x=(k+l\rho)^{-1}\Big[\{(\alpha-\alpha\rho+\delta)+\delta(1-\rho)x\}+(\rho-1)(ly_{x+1}+ky_x)\Big], \quad \ldots \quad (3.12)$$

for $k+l\neq 0$. Summing over the values of x, we have

$$y_{x+1}=(k+l\rho)^{-1}\Big[\{(\alpha-\alpha\rho+\delta)(x+1)+\delta(1-\rho)x(x+1)/2+\alpha+\beta\}$$

$$+(\rho-1)(lS_{x+1}+kS_x)\Big], \quad \ldots \quad (3.13)$$

where $S_x=\sum_{j=0}^{x} y_j$ with $S_{-1}=0$.

The equation (3.13) can be written as

$$y_i = a_0+a_1\xi_{1,i}+a_2\xi_{2,i}+a_3(lS_i+kS_{i-1}), \quad \ldots (3.14)$$

$$(i=0,1,2,\ldots,n-1)$$

where $a_3=(\rho-1)(k+l\rho)^{-1}$, $k+l\neq 0$, and ξ_1 and ξ_2 are the first and second order Fisher's [9] orthogonal polynomials obtained for $x=0,1,2,\ldots,n-1$, i.e.

$\xi_{1t}=x-\bar{x}$, $\xi_{2t}=x^2-\overline{x^2}-\frac{\mu_3}{\mu_2}(x-\bar{x})$, where

$\mu_3=\sum(x^2-\overline{x^2})(x-\bar{x})$, $\mu_2=\sum(x-\bar{x})^2$, $\overline{x^2}=\sum x^2/n$ and

$\bar{x}=\sum x/n$.

The regression obtained in (3.14) is called an internal regression as defined by Hartley [10]. Solving the internal least square equations, the estimate of a_3 is given as

$$\underline{y}' E(l\underline{S}_0 + k\underline{S}_{-1}) / (l\underline{S}_0 + k\underline{S}_{-1})' E (l\underline{S}_0 + k\underline{S}_{-1}) \quad , \ldots (3.15)$$

where $\underline{y}' = (y_0, y_1, y_2, \ldots, y_{n-1})$, $\underline{S}_0' = (S_0, S_1, \ldots, S_{n-1})$,

$\underline{S}_{-1}' = (0, S_1, S_2, \ldots, S_{n-2})$, $E = I - \frac{1}{n}\underline{1}_n \underline{1}_n' - \frac{1}{\eta_1}\underline{\xi}_1\underline{\xi}_1' - \frac{1}{\eta_2}\underline{\xi}_2\underline{\xi}_2'$,

$\underline{1}_n' = (1, 1, \ldots, 1)$: nx1,

We may write $\underline{S}_0 = T_n\underline{y}$ and $\underline{S}_{-1} = U_n\underline{y}$ where

$$T_n = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 \\ 1 & 1 & 1 & \ldots & 0 \\ \\ 1 & 1 & 1 & \ldots & 1 \end{bmatrix} : n \times n \quad \text{and} \quad W = \left[\begin{array}{c|c} 0 & 0 \\ \hline T_n & 0 \end{array}\right]$$

with $T_n - I = W_n$. Then the estimate of ρ can be shown as

$$\underline{y}' P \underline{y} / \underline{y}' N \underline{y} \quad \ldots \quad (3.16)$$

where $P = T_n E (lT_n + kW_n)$ and $N = W_n E (lT_n + kW_n)$.

Noting $E \underline{1}_n = 0$, this can be easily verified as

$$r(l,k,l) = \underline{y}_1' M (k\underline{y}_0 + l\underline{y}_1) / \underline{y}_0' M (k\underline{y}_0 + l\underline{y}_1), \ldots (3.17)$$

where $M = (m_{ij})$ and $m_{ij} = m_{ji}$ and

$$m_{ij} = i - ij/n - \frac{(\sum_1^i \xi_{1t})(\sum_1^j \xi_{1t})}{\eta_1} - \frac{(\sum_1^i \xi_{2t})(\sum_1^j \xi_{2t})}{\eta_2} . \quad \ldots(3.18)$$

The notation 1 in r(1,k,1) will be explained later. It may be noted that the estimate r(1,1,0) was given by Khatri and Shah [16] and r(1,1,1) by Trivedi [14].

In practice, it is very easy to construct the matrix M by using the orthogonal vectors from Fisher and Yates table [9] for different values of n.

When n = 5,

$$M = \frac{1}{35} \begin{bmatrix} 4 & -5 & -2 & 3 \\ -5 & 8 & -1 & -2 \\ -2 & -1 & 8 & -5 \\ 3 & -2 & -5 & 4 \end{bmatrix}$$

When n = 6,

$$M = \frac{1}{140} \begin{bmatrix} 25 & -20 & -20 & 0 & 15 \\ -20 & 32 & -4 & -8 & 0 \\ -20 & -4 & 48 & -4 & -20 \\ 0 & -8 & -4 & 32 & -20 \\ 15 & 0 & -20 & -20 & 25 \end{bmatrix}$$

When n = 7,

$$M = \frac{1}{42}\begin{bmatrix} 10 & -5 & -8 & -4 & 2 & 5 \\ -5 & 10 & -2 & -4 & -1 & 2 \\ -8 & -2 & 16 & 2 & -4 & -4 \\ 4 & -4 & 2 & 16 & -2 & -8 \\ 2 & -1 & -4 & -2 & 10 & -5 \\ 5 & 2 & -4 & -8 & -5 & 10 \end{bmatrix}$$

and When n = 8,

$$M = \frac{1}{168}\begin{bmatrix} 49 & -14 & -35 & -28 & -7 & 14 & 21 \\ -14 & 44 & -10 & -24 & -14 & 4 & 14 \\ -35 & -10 & 65 & 12 & -11 & -14 & -7 \\ -28 & -24 & 12 & 80 & 12 & -24 & -28 \\ -7 & -14 & -11 & 12 & 65 & -10 & -35 \\ 14 & 4 & -14 & -24 & -10 & 44 & -14 \\ 21 & 14 & -7 & -28 & -35 & -14 & 45 \end{bmatrix}$$

2. Expectation and variance of the General quadratic estimate.

The method adopted to determine the expectations and variances of quadratic estimate of ρ is in principle exactly the same as described by Finney [7] and Patterson [26] but they are given here for completeness of the text. This results given in this section apply to any quadratic estimate of the type considered above. It is assumed that y_x are supposed to be independently and normally distributed with variance σ^2.

Finney's formula [7] can be written for the asymptotic variance and expectation of $r = A/B$ (as described in chapter II) as

$$\text{Var } r = \left\{ \text{Var } A + \rho^2 \text{Var } B - 2\rho \text{ Cov }(A,B) \right\} / \left\{ E^*(B) \right\}^2, \quad \ldots (3.19)$$

and $$E(r) = \rho + (V_A - \rho V_B)/ E^*(B) + \left\{ \rho \text{Var } B - \text{Cov}(A,B) \right\} / \left\{ E^*(B) \right\}^2, \quad \ldots (3.20)$$

where V_A and V_B are the terms in σ^2 in $E(A)$ and $E(B)$, respectively and $E^*(B) = E(B) - V_B$. The variances and covariances of A and B can be obtained by repeated use of the formula

$$\begin{aligned} \text{Cov}(\underline{s}'D\underline{t}, \underline{u}'E\underline{v}) &= E(\underline{s}')\, DC_{tv}\, E'\, E(\underline{u}) \\ &+ E(\underline{s}')\, DC_{tv}\, E(\underline{v}) + E(\underline{t}')D'C_{sv}E'E(\underline{u}) \\ &+ E(\underline{t}')\, D'C_{su}\, E\, E(\underline{v}) \quad \ldots \quad (3.21) \end{aligned}$$

Here $\underline{s}$, $\underline{t}$, $\underline{u}$, $\underline{v}$ are jointly normally distributed variables with covariance matrices

$$C_{tv} = \begin{bmatrix} \text{cov}(t_1,v_1) & \text{cov}(t_1,v_2) & \cdots \\ \text{cov}(t_2,v_1) & \text{cov}(t_2,v_2) & \cdots \\ \text{cov}(t_3,v_1) & \text{cov}(t_3,v_2) & \cdots \\ \vdots & \vdots & \end{bmatrix} \qquad \ldots(3.22)$$

In addition the expression

$$E(\underset{\sim}{s}'D\underset{\sim}{t}) = E(\underset{\sim}{s}')\, DE(\underset{\sim}{t}) + \text{trace}\,(DC'_{st}) \qquad \ldots(3.23)$$

is also required.

Let us define two matrices

$$\underline{R}' = (1, \rho, \ldots \rho^{n-2}) \qquad \ldots(3.24)$$

and

$$U = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & & 1 & 0 \end{bmatrix} \qquad \ldots(3.25)$$

And further let us define

$$F_0 = \underline{R}'D\underline{R}, \quad F_1 = \underline{R}'D'D\underline{R}, \quad F_2 = \underline{R}'D'UD\underline{R},$$
$$F_3 = \underline{R}'DD'\underline{R}, \quad F_4 = \underline{R}'DUD'\underline{R}, \quad F_5 = \underline{R}'DD\underline{R}, \qquad \ldots(3.26)$$
$$F_6 = \underline{R}'DUD\underline{R}, \text{ and } F_7 = \underline{R}'DU'D\underline{R}.$$

The variances and covariances of A and B are then

$$\text{Var } A = \beta^2\sigma^2\Big\{(k+l\rho)^2F_1 + \rho^2(k^2+l^2)F_3 + 2kl\rho^2 F_4$$

$$+ 2\rho l(k+l\rho)F_5 + 2k\rho(k+l\rho)F_6\Big\} \quad \ldots \quad (3.27)$$

$$\text{Var } B = \beta^2\sigma^2\Big\{(k+l\rho)^2F_1 + (k^2+l^2)F_3 + 2kl F_4 + 2k(k+l\rho)F_5$$

$$+ 2l(k+l\rho)F_6\Big\}, \quad \ldots \quad (3.28)$$

$$\text{Cov}(A,B) = \beta^2\sigma^2\Big\{(k+l\rho)^2F_2 + \rho(k^2+l^2)F_3 + 2kl\rho F_4$$

$$+ (k+l\rho)(\rho k+l)F_3 + (k+l\rho)(kF_6+l\rho F_2)\Big\} \quad \ldots (3.29)$$

Also $\quad E^*(B) = (k + l\rho)\, F_0 \, . \quad \ldots (3.30)$

Substituting (3.27) to (3.29) in (3.19) we have the required variance:

$$\text{Var } r = \frac{\sigma^2}{\beta^2}\Big\{(1+\rho^2)F_1 - 2\rho F_2) \;/\; F_0^2\Big\}, \quad \ldots (3.31)$$

which is independent of k and l. It should be noted that the asymptotic variance of the linear estimate

$$\underline{y}_1'\, D\, \underline{R} \,/\, \underline{y}_0'\, D\, \underline{R} \quad \ldots (3.32)$$

is also given by equation (3.31). The first two terms for the bias in r, depending on k and l are

$$\frac{\sigma^2}{\beta^2}\left\{(1-\rho k)\ \mathrm{tr}\ D + k\ \mathrm{tr}\ DU - 1\rho\ \mathrm{tr}\ Du' \quad /(k+1\rho)F_0\right\} \quad \ldots \quad (3.33)$$

and

$$\frac{\sigma^2}{\beta^2}\left\{(k+\rho 1)(\rho F_1 - F_2) + (\rho k-1)F_5 - kF_6 + 1\rho F_1 \quad /(k+1\rho)F_0^2\right\} \ldots (3.34)$$

3. Quadratic Estimates with minimum variance when $\rho = \rho_0$.

We shall consider here how to construct the matrix D such that the quadratic estimate of ρ has minimum asymptotic variance when ρ takes some particular value, ρ_0 say.

The asymptotic variance of (3.32) when $\rho = \rho_0$ can be written as

$$\phi = \frac{\sigma^2}{\beta^2}\left\{(1+\rho_0^2)\sum_1^{n-1} w_x^2 - 2\rho_0 \sum_1^{n-2} w_x\ w_{x+1}\right\} \Big/ \left\{\left(\sum_1^{n-1} w_x\ \rho_0^{x-1}\right)^2\right\} \ldots (3.35)$$

where w_x are proportional to the elements of D_1R, with the restriction $\sum w_x = 0$, $\sum x w_x = 0$ and $\sum w_x \rho_0^{x-1} = \lambda$.

Minimization of ϕ with respect to w_x subject to the above conditions gives

$$(1+\rho_0^2)w_x - \rho_0(w_{x+1} + w_{x-1}) = \lambda_2' + \lambda_3'\ x + \lambda_4'\ \rho_0^{x-1}\ , \quad \ldots (3.36)$$

where $x = 1, 2, \ldots n-1$, $w_0 = 0$ and $w_n = 0$ and λ_2', λ_3', λ_4' are constants for all $\underline{x}$. We can write the above (n-1) equations in the matrix form as

$$V\ \underline{w} = \lambda_2'\ \underline{1}_{n-1} + \lambda_3'\ \underline{x} + \lambda_4'\ \underline{R}_0 \quad \ldots (3.37)$$

Where

$$V=\begin{bmatrix} 1+\rho_0^2 & -\rho_0 & 0 & 0 & \cdots \\ -\rho_0 & 1+\rho_0^2 & -\rho_0 & 0 & \cdots \\ 0 & -\rho_0 & 1+\rho_0^2 & -\rho_0 & \cdots \\ 0 & 0 & -\rho_0 & 1+\rho_0^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad \ldots(3.38)$$

$\underline{R}_0'=(1, \rho_0, \rho_0^2, \ldots \rho_0^{n-2})$, $\underline{x}' = (1,2,\ldots n-1)$ and

$\underline{w}'=(w_1, w_2, \ldots w_{n-1})$.

Hence,

$$\underline{w} = \lambda_2' V^{-1} \underline{1}_{n-1} + \lambda_3' V^{-1} \underline{x} + \lambda_4' V^{-1} \underline{R}_0 \quad \ldots(3.39)$$

Since $\underline{1}'\underline{w} = 0$ and $\underline{x}'\underline{w} = 0$, we have two equations to determine λ_2' and λ_3' by taking $\lambda_4' = 1$. We have

$$\begin{bmatrix} \lambda_2' \\ \lambda_3' \end{bmatrix} = \begin{bmatrix} \underline{1}'_{n-1} V^{-1} \underline{1}_{n-1} & \underline{1}'_{n-1} V^{-1} \underline{x} \\ \underline{x}' V^{-1} \underline{1}_{n-1} & \underline{x}' V^{-1} \underline{x} \end{bmatrix} \begin{bmatrix} \underline{1}'_{n-1} V^{-1} \underline{R}_0 \\ \underline{x}' V^{-1} \underline{R}_0 \end{bmatrix} \quad \ldots(3.40)$$

Substituting the values of λ_2' and λ_3' of (3.40) in (3.39), we will get the required weights

$$\underline{w} = \left[V^{-1} - V^{-1} X(X'V^{-1}X)^{-1}X'V^{-1} \right] \underline{R}_0 \quad \ldots(3.41)$$

Where $$X' = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & n-1 \end{bmatrix} : 2x(n-1)$$

Thus the required matrix D which minimizes the variance of the quadratic estimate when $\rho = \rho_0$ is

$$D_{\rho_0} = V^{-1} - V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1} , \qquad \ldots \qquad (3.42)$$

where $V^{-1} = (c_{ij})$ is given by $c_{ij} = c_{ji}$,

$$c_{ij} = \rho_0^{j-1}(1-\rho_0^{2i})(1-\rho_0^{2(n-j)})/(1-\rho_0^{2n})(1-\rho_0^2), (i \leq j) \qquad \ldots \qquad (3.43)$$

when $\rho_0 < 1$, and

$$c_{ij} = i(n-j)/n \qquad (i \leq j), \qquad \ldots \qquad (3.44)$$

~~When~~ when $\rho_0 = 1$.

The quadratic estimates with D can be written as $r(\rho_0, k, 1)$, the estimate is obtained by putting $\rho = \rho_0$. When $\rho = \rho_0$, this is the same estimate as we discussed in section (A) with matrix given by (3.10) and (3.11) and therefore is more efficient for low values of ρ.

When $\rho_0 = 1$, this is the same estimate $r(1,k,1)$ as discussed in section (B) and therefore it is more efficient for high values of ρ. In this case the matrix D_1 of (3.42) is same as the matrix M of (3.18). We have verified this for n = 5,6,7 and 8 and in general the above statement may be true.

4. Expectation and Variances of r(0,kl) and r(1,k,1).

Using the formulae (3.24), (3.25) and (3.26), we can write down the values of F_0, F_1 and F_2 for the estimates r(0,k,1) and r(1,k,1) considering the matrices D_o and M respectively.

(A) For the case of r(0,k,1) when D_o is given by (3.10) and (3.11):

$$F_o = F_1 : \; n=5: \; (1-\rho)^4(3+4\rho+3\rho^2)/10,$$

$$n=6; \; (1-\rho)^4(4+8\rho+11\rho^2+8\rho^3+4\rho^4)/10,$$

$$n=7; \; (1-\rho)^4(50+120\rho+204\rho^2+232\rho^3+204\rho^4 +120\rho^5+50\rho^6)/105,$$

$$n=8; \; (1-\rho)^4(15+40\rho+76\rho^2+104\rho^3+118\rho^4 +104\rho^5+76\rho^6+40\rho^7+15\rho^8)/28$$

$$F_2 : \; n=5: \; (1-\rho)^4(-10-5\rho-10\rho^2)/100,$$

$$n=6: \; (1-\rho)^4(-8+6\rho+4\rho^2+6\rho^3-8\rho^4)/100,$$

$$n=7: \; (1-\rho)^4(-20+57\rho+120\rho^2+176\rho^3+120\rho^4 +57\rho^5-20\rho^6)/315,$$

$$n=8: \; (1-\rho)^4(-5+28\rho+72\rho^2+128\rho^3+142\rho^4 +128\rho^5+72\rho^6+28\rho^7-5\rho^8)/98 .$$

(B) For the case of r(1,k,1) when M is given by (3.18).

F_0: n=5: $(1-\rho)^4(4+6\rho+4\rho^2)/35$,

n=6: $(1-\rho)^4(25+60\rho+82\rho^2+60\rho^3+25\rho^4)/140$,

n=7: $(1-\rho)^4(10+30\rho+54\rho^2+64\rho^3+54\rho^4+30\rho^5+10\rho^6)/42$,

n=8: $(1-\rho)^4(49+168\rho+352\rho^2+520\rho^3+594\rho^4$

$+520\rho^5+352\rho^6+168\rho^7+49\rho^8)/168$.

F_1: n=5: $(1-\rho)^4(54+88\rho+54\rho^2)/1225$,

n=6: $(1-\rho)^4(1650+4480\rho+6164\rho^2+4480\rho^3+1650\rho^4)/1960$,

n=7: $(1-\rho)^4(234+816\rho+1542\rho^2+1872\rho^3+1542\rho^4$

$+816\rho^5+234\rho^6)/1764$,

n=8: $(1-\rho)^4(5292+21504\rho+48576\rho^2+75264\rho^3+86808\rho^4$

$+75264\rho^5+48576\rho^6+21504\rho^7+5292\rho^8)/28224$.

F_2: n=5: $(1-\rho)^4(-16-17\rho-16\rho^2)/1225$,

n=6: $(1-\rho)^4(-100+280\rho+424\rho^2+280\rho^3-100\rho^4)/1960$,

n=7: $(1-\rho)^4(24+186\rho+408\rho^2+528\rho^3+408\rho^4+186\rho^5+24\rho^6)/1764$,

n=8: $(1-\rho)^4(1176+7392\rho+19008\rho^2+31524\rho^3+36892\rho^4$

$+31584\rho^5+19008\rho^6+7392\rho^7+1176\rho^8)/28224$.

5. Efficiencies of the two estimates:

The efficiencies of r(0,k,1) and r(1,k,1) for n=5,6,7 and 8 are set out in Table 3.1. The efficiencies are obtained by taking the ratio of the least squares estimate obtained by Shah and Patel [34] to the variance given by (3.31).

It can be seen that the efficiencies of the quadratic estimates r(0,k,1) are better than those of the linear estimates as described in Chapter II (Shah [31]). It is interesting to note that the efficiencies are almost 99.9% for $r > 0.5$ in the modified Hartley's method and for quadratic estimates the efficiencies are better and almost 97.0% for $r < 0.4$. This suggests that the further work for this curve in the direction given by Patterson [28] is under consideration.

6. Biases in r(0,1,0) and r(1,1,0).

The total bias given by (3.33) and (3.34) can be written as :

$$\text{bias} = \theta \text{ Var } r.$$

Values of θ for the estimates r(0,1,0) and r(1,1,0) are set out for comparision in Table 3.2. It is easy to see that the bias in the linear estimates considerably low as compared to that of in r(0,1,0) and r(1,1,0). It is of interest to note that the modified Hartley's estimates have less bias than quadratic estimates.

Table 3.1

Efficiencies of simple estimates of ρ.

ρ	n=5		n=6		n=7		n=8	
	r(0,k,1)	r(1,k,1)	r(0,k,1)	r(1,k,1)	r(0,k,1)	r(1,k,1)	r(0,k,1)	r(1,k,1)
0.0	100.00	98.76	100.00	94.70	100.00	89.74	100.00	84.69
0.1	99.92	99.52	99.64	97.36	99.94	94.16	99.31	90.42
0.2	99.80	99.83	98.92	98.81	98.14	97.01	97.57	94.62
0.3	99.75	99.94	98.22	99.51	96.64	98.63	95.35	97.32
0.4	99.77	99.98	97.81	99.91	95.35	99.45	93.19	98.97
0.5	99.82	100.00	97.43	99.94	94.43	99.81	91.47	99.56
0.6	99.88	100.00	97.27	99.98	93.87	99.96	90.31	99.87
0.7	99.93	100.00	97.07	99.99	93.54	99.98	89.29	99.96
0.8	99.97	100.00	96.59	100.00	93.12	99.81	89.08	99.99
0.9	99.99	100.00	97.20	100.00	93.04	99.99	89.12	99.99
1.0	100.00	100.00	97.22	100.00	93.34	100.00	89.10	100.00

Table 3.2

Comparision of Biases in r(0,1,0) and r(1,1,0)

	n=5		n=6		n=7		n=8	
ρ	r(0,1,0)	r(1,1,0)	r(0,1,0)	r(1,1,0)	r(0,1,0)	r(1,1,0)	r(0,1,0)	r(1,1,0)
0.0	-.333	-.222	-.800	-.606	-1.067	-0.718	-1.238	-0.667
0.1	-.356	-.204	-1.059	-.672	-1.954	-0.884	-1.667	-0.930
0.2	-.362	-.178	-1.278	-.710	-2.475	-1.015	-2.088	-1.157
0.3	-.354	-.149	-1.453	-.723	-2.735	-1.110	-2.495	-1.348
0.4	-.338	-.120	-1.584	-.716	-2.810	-1.169	-2.875	-1.499
0.5	-.317	-.092	-1.667	-.694	-2.776	-1.195	-3.202	-1.599
0.6	-.292	-.068	-1.711	-.661	-2.668	-1.190	-3.451	-1.653
0.7	-.268	-.046	-1.718	-.622	-2.532	-1.162	-3.604	-1.661
0.8	-.244	-.028	-1.697	-.581	-2.311	-1.117	-3.654	-1.632
0.9	-.221	-.013	-1.655	-.540	-2.262	-1.061	-3.612	-1.575
1.0	-.200	.000	-1.600	-.500	-2.151	-1.000	-3.500	-1.500

7. Estimates $r(0,k,l)$ and $r(1,k,l)$ with low biases.

As suggested by Patterson [26] it is possible to choose k and l in the families of $r(0,k,l)$ and $r(1,k,l)$ such that the biases are zero. The value

$$k/l = \left\{F_0(\text{tr } D - \rho \,\text{tr } DU') + (\rho^2-1) F_1\right\}/F_0(\rho \,\text{tr}D - \text{tr}DU) - 2\rho F_1 + 2F_2 , \qquad \ldots \quad (3.45)$$

is such that the bias is zero in $r(0,k,l)$ when the matrix D_0 is used instead of D, and in $r(1,k,l)$ when the matrix M is used instead of D. Values of k/l are given in Table 3.3.

Now by choosing some value of k/l from the table 3.3 it may be possible to reduce the biases near to zero for at least some values of ρ. This is pointed out in Tables (3.4) and (3.5) for $r(0,k,l)$ and $r(1,k,l)$ respectively.

8. Remarks: By choosing the proper values of k and l, it can be seen that the biases in $r(0,k,l)$ and $r(1,k,l)$ are sufficiently low over the whole range of ρ. And particularly for high values of ρ where the estimate $r(1,k,l)$ is almost efficient, the bias in $r(1,k,l)$ is less than that of the linear estimates also.

Thus we can conclude that the estimate obtained by the modified Hartley's method are much improved than that of linear estimates and quadratic estimates on the whole.

Table 3.3

Values of k/l for zero bias in r(0,k,l) & r(1,k,l)

	n=5		n=6		n=7		n=8	
ρ	r(0,k,l)	r(1,k,l)	r(0,k,l)	r(1,k,l)	r(0,k,l)	r(1,k,l)	r(0,k,l)	r(1,k,l)
0.0	3.00	3.50	2.50	2.40	2.81	2.89	3.23	4.00
0.1	2.91	3.90	2.13	2.15	2.25	2.31	2.46	2.80
0.2	2.96	4.61	1.93	2.04	1.93	2.00	2.03	2.22
0.3	3.12	5.71	1.83	2.03	1.73	1.84	1.77	1.91
0.4	3.36	7.34	1.79	2.07	1.61	1.76	1.60	1.73
0.5	3.66	9.83	1.80	2.16	1.53	1.73	1.49	1.64
0.6	4.02	13.78	1.84	2.28	1.48	1.75	1.41	1.60
0.7	4.43	20.66	1.90	2.43	1.46	1.79	1.36	1.59
0.8	4.90	34.83	1.99	2.61	1.45	1.85	1.32	1.60
0.9	5.43	81.67	2.09	2.80	1.45	1.92	1.30	1.63
1.0	5.00	∞	2.20	3.00	1.45	2.01	1.29	1.67

Table 3.4

Bias in r(0,k,1)

	n=5	n=6	n=7	n=8
ρ	r(0,4,1)	r(0,1.8,1)	r(0,1.6,1)	r(0,1.6,1)
0.0	-.083	.867	.183	1.262
0.1	-.095	.581	.168	.840
0.2	-.089	.364	-.063	.403
0.3	-.072	.202	-.238	.224
0.4	-.049	.083	-.155	.001
0.5	-.026	.001	-.157	-.174
0.6	.001	-.053	-.138	-.301
0.7	.025	-.084	-.109	-.384
0.8	.046	-.100	-.069	-.426
0.9	.070	-.104	-.024	-.436
1.0	.080	-.100	.015	-.423

Table 3.5

Bias in r(1,k,1)

	n=5	n=6	n=7	n=8
ρ	r(1,8,1)	r(1,2.2,1)	r(1,1.8,1)	r(1,1.6,1)
0.0	-.125	.055	.436	1.000
0.1	-.103	-.016	.235	.654
0.2	-.074	-.047	.101	.400
0.3	-.041	-.050	.019	.220
0.4	-.009	-.036	-.023	.101
0.5	.020	-.010	-.034	.030
0.6	.047	.020	-.026	-.003
0.7	.067	.050	-.006	-.011
0.8	.085	.079	.020	-.002
0.9	.099	.104	.046	.017
1.0	.111	.125	.071	.039