Chapter 4

(ψ, ϕ)-Wardowski Contractions

4.1 Introduction

In order to solve the more complex nonlinear analysis problems, the concept of metric spaces has been extended in many aspects. In particular, Czerwik [15] introduced the concept of b-metric spaces and established the Banach contraction principle in this framework with the fact that b-metric need not be continuous. Recently, Aghajani et al. [3] defined the G_b -metric spaces by using the notions of b-metric spaces and G-metric spaces, and they discussed some basic properties of G_b -metric. They also pointed out that the class of G_b-metric spaces is effectively larger than that of G-metric spaces. Thereafter, several results have been extended from metric spaces to b-metric spaces, more so, a lot of results on the fixed point theory of various classes of mappings in the framework of b-metric spaces have been established by different researchers in this area.

On the other hand, Rhoades' problem on discontinuity at fixed points is one of the interesting problems of fixed point theory. Rhoades [52] brought up the issue of whether there is a contractive condition strong enough to produce a fixed point which does not require the map to be continuous at the fixed point. Following the initial answer provided by R. P. Pant [47], several further solutions to this open problem have been offered using various techniques. In this context, Wardowski [61] introduced the F-contraction and proved fixed point results for such mappings. Later, Liu et al. [42] introduced the (ψ, ϕ)-type contraction for metric spaces as follows.

Here, Φ denotes the collection of non-decreasing, continuous functions $\phi : [0, \infty) \to [0, \infty)$ such that for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \phi(t_n) = 0$ if and only if $\lim_{n\to\infty} t_n = 0$.

A function $\psi : [0,\infty) \to [0,\infty)$ is called a comparison function if it is monotone increasing and $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$, where ψ^n is n^{th} iterate of ψ. F_{com} denotes the collection of all comparison functions.

Definition 4.1.1. [42, p.4131] Let T be a self-mapping defined on the metric space (X, d). Then, T is said to be (ψ, ϕ)-type contraction, if there exists $\phi \in \Phi$ and $\psi \in F_{com}$, such that

$$d(Tx, Ty) > 0 \implies \phi(d(Tx, Ty)) \leq \psi(\phi(M(x, y))), \text{ for all } x, y \in X,$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}d(x, Ty), d(y, Tx)\}$.

In recent years, fixed points of discontinuous mappings have found applications in neural networks. Some current solutions to the Rhoades' Open Problem are used to provide some applications for neural networks with discontinuous activation functions [45, 51].

The objective of this chapter is to find out contractive condition which does not force the mapping to be continuous at their common fixed points. For this, generalized $(\psi, \phi)-G_b$-Wardowski contraction for three mappings is introduced to establish a common fixed point theorem in setting of complete G_b-metric spaces. Further, its application to neural networks is discussed.

4.2 Preliminaries

Here, we recollect some basic definitions and results that are prerequisites for this chapter.

Definition 4.2.1. [3, p.1088] A G_b-metric space (X, G_b) is said to be *symmetric if* $G_b(x, y, y) = G_b(y, x, x)$, *for all* $x, y \in X$.

Definition 4.2.2. [3, p.1089] For a sequence $\{x_n\}$ and a point x in (X, G_b), we say that:

(1) $\{x_n\}$ G_b-converges to x, if $\lim_{n,m\to\infty} G_b(x_n, x_m, x) = 0$, that is, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying $G_b(x_n, x_m, x) < \varepsilon$, for all $n, m \geq n_0$.

(2) $\{x_n\}$ is G_b-Cauchy if $\lim_{n,m,k\to\infty} G_b(x_n, x_m, x_k) = 0$, that is, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying $G_b(x_n, x_m, x_k) < \varepsilon$, for all $n, m, k \geq n_0$.

(3) (X, G_b) is G_b-complete if every G_b- Cauchy sequence in X is G_b- convergent in X.

Proposition 4.2.3. *[3, Prop.1.11, p.1090] For a sequence $\{x_n\}$ and a point x in (X, G_b), the following are equivalent:*

(a) $\{x_n\}$ *G_b-converges to x,*

(b) $\lim_{n\to\infty} G_b(x_n, x_n, x) = 0,$

(c) $\lim_{n\to\infty} G_b(x_n, x, x) = 0.$

Proposition 4.2.4. *[3, Prop.1.10, p.1089] A sequence $\{x_n\}$ in (X, G_b) is G_b-Cauchy if and only if* $\lim_{n,m\to\infty} G_b(x_n, x_m, x_m) = 0.$

Definition 4.2.5. [3, p.1090] Let (X, G) and (X, G') be two G_b-metric spaces. Then a function $f : X \to X'$ is G_b-continuous at a point $x \in X$ if and only if $\{f(x_n)\} \to f(x)$, whenever $\{x_n\} \to x$.

In a G-metric space, the metric is jointly continuous in all three of its variables. But this statement is not true in case of G_b-metric spaces.

Proposition 4.2.6. *[3, Prop.1.5, p.1089] Let (X, G_b) be a G_b-metric space. Then for $s \geq 1$ and for each $x, y, z, a \in X$, the following properties hold:*

1. $G_b(x, y, z) = 0 \implies x = y = z,$
2. $G_b(x, y, z) \leq s[G_b(x, x, y) + G_b(x, x, z)],$
3. $G_b(x, y, y) \leq 2sG_b(y, x, x),$
4. $G_b(x, y, z) \leq s[G_b(x, a, z) + G_b(a, y, z)].$

4.3 Results for (ψ, ϕ)-Wardowski contraction in G_b-metric spaces

To take this section forward, we firstly introduce the $(\psi, \phi) - G_b$-Wardowski contraction and generalized $(\psi, \phi) - G_b$-Wardowski contraction. Subsequently, common fixed point result via such contraction in G_b-metric space is demonstrated.

Definition 4.3.1. Let S be a self-mapping defined on the G_b-metric space (X, G). Suppose that there exist $\phi \in \Phi$ and $\psi \in F_{com}$, such that for $s \geq 1$

$$G(Sx, Sy, Sz) > 0 \implies \phi(2s^4 G(Sx, Sy, Sz)) \leq \psi(\phi(M_1(x, y, z))),$$

for all $x, y, z \in X$, where

$$M_1(x, y, z) = \max\Big\{G(x, y, z), G(x, Sx, Sy), G(y, Sy, Sz), G(z, Sz, Sx), \frac{1}{4s}[G(Sx, y, z) + G(x, Sy, z) + G(x, y, Sz)]\Big\}.$$

Then, S is said to be a $(\psi, \phi) - G_b$-Wardowski contraction.

Definition 4.3.2. Let S, T, R be self-mappings defined on the G_b-metric space (X, G). Suppose that there exist $\phi \in \Phi$ and $\psi \in F_{com}$, such that for $s \geq 1$

$$G(Sx, Ty, Rz) > 0 \implies \phi(2s^4 G(Sx, Ty, Rz)) \leq \psi(\phi(M_2(x, y, z))), \tag{4.1}$$

for all $x, y, z \in X$, where

$$M_2(x, y, z) = \max\Big\{G(x, y, z), G(x, Sx, Ty), G(y, Ty, Rz), G(z, Rz, Sx), \frac{1}{4s}[G(Sx, y, z) + G(x, Ty, z) + G(x, y, Rz)]\Big\}.$$

Then, we say that (S, T, R) is generalized $(\psi, \phi) - G_b$-Wardowski contraction.

Now, the main result of this chapter is furnished below.

Theorem 4.3.3. *Let $S, T, R : X \to X$ be generalized $(\psi, \phi) - G_b$-Wardowski contraction in a complete G_b-metric space. Then S, T, R have a unique common fixed point $u \in X$, also $S^n x \to u$, $T^n x \to u$ and $R^n x \to u$, for each $x \in X$. Further, at least one of S, T and R is not continuous at u if and only if*

$$\lim_{x \to u} M_2(x, u, u) \neq 0 \text{ or } \lim_{y \to u} M_2(u, y, u) \neq 0 \text{ or } \lim_{z \to u} M_2(u, u, z) \neq 0.$$

Proof. For any $x_0 \in X$, we can construct a sequence $\{x_n\}$ by setting

$$x_{3n+1} = Sx_{3n}, \; x_{3n+2} = Tx_{3n+1}, \; x_{3n+3} = Rx_{3n+2}, \; n \geq 0.$$

Suppose that $x_n = x_{n+1}$, for some $n \in \mathbb{N}$.
If $x_{3n} = x_{3n+1}$, then x_{3n} is a fixed point of S.
If $x_{3n+1} = x_{3n+2}$, then x_{3n+1} is a fixed point of T.
If $x_{3n+2} = x_{3n+3}$, then x_{3n+2} is a fixed point of R.
Thus, at least, one of the mappings S, T or R has a fixed point.
We assume that $x_n \neq x_{n+1}$, for all n. Let $d_n = G(x_n, x_{n+1}, x_{n+2}) > 0$, for all n. Hence

$$G(Sx_{3n}, Tx_{3n+1}, Rx_{3n+2}) = G(x_{3n+1}, x_{3n+2}, x_{3n+3}) = d_{3n+1} > 0,$$

implies that

$$\begin{aligned} \phi(2s^4 d_{3n+1}) &= \phi(2s^4 G(x_{3n+1}, x_{3n+2}, x_{3n+3})) \\ &\leq \psi(\phi(M_2(x_{3n}, x_{3n+1}, x_{3n+2}))), \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} & M_2(x_{3n}, x_{3n+1}, x_{3n+2}) \\ & = \max\Big\{ G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, Sx_{3n}, Tx_{3n+1}), G(x_{3n+1}, Tx_{3n+1}, Rx_{3n+2}), \\ & \qquad G(x_{3n+2}, Rx_{3n+2}, Sx_{3n}), \frac{1}{4s}[G(Sx_{3n}, x_{3n+1}, x_{3n+2}) \\ & \qquad + G(x_{3n}, Tx_{3n+1}, x_{3n+2}) + G(x_{3n}, x_{3n+1}, Rx_{3n+2})]\Big\} \\ & = \max\Big\{ G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3}), \\ & \qquad \frac{1}{4s}[G(x_{3n+1}, x_{3n+1}, x_{3n+2}) + G(x_{3n}, x_{3n+2}, x_{3n+2}) \\ & \qquad + G(x_{3n}, x_{3n+1}, x_{3n+3})]\Big\}. \end{aligned}$$

From definition of G_b-metric space, we have

$$\begin{aligned} G(x_{3n+1}, x_{3n+1}, x_{3n+2}) &\leq G(x_{3n+1}, x_{3n+2}, x_{3n+3}) = d_{3n+1}, \\ G(x_{3n}, x_{3n+2}, x_{3n+2}) &\leq G(x_{3n}, x_{3n+1}, x_{3n+2}) = d_{3n}, \\ G(x_{3n}, x_{3n+1}, x_{3n+3}) &\leq s[G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n+1}, x_{3n+2}, x_{3n+3})] \\ &= s[d_{3n} + d_{3n+1}]. \end{aligned}$$

Hence,

$$\begin{aligned} M_2(x_{3n}, x_{3n+1}, x_{3n+2}) &= \max\{d_{3n}, d_{3n+1}, \frac{s+1}{4s}(d_{3n} + d_{3n+1})\} \\ &= \max\{d_{3n}, d_{3n+1}\}. \end{aligned}$$

If $M_2(x_{3n}, x_{3n+1}, x_{3n+2}) = d_{3n+1}$, then from (4.2), we have

$$\phi(2s^4 d_{3n+1}) \leq \psi(\phi(d_{3n+1})) < \phi(d_{3n+1}),$$

which is not possible. Hence, $M_2(x_{3n}, x_{3n+1}, x_{3n+2}) = d_{3n}$.
Using (4.2), we have

$$\phi(2s^4 d_{3n+1}) \leq \psi(\phi(d_{3n})) < \phi(d_{3n}), \text{ for all } n \in \mathbb{N}. \tag{4.3}$$

Again, from (4.1), we have

$$\begin{aligned} \phi(2s^4 d_{3n+2}) &= \phi(2s^4 G(x_{3n+2}, x_{3n+3}, x_{3n+4})) \\ &= \phi(2s^4 G(Tx_{3n+1}, Rx_{3n+2}, Sx_{3n+3})) \\ &\leq \psi(\phi(M_2(x_{3n+3}, x_{3n+1}, x_{3n+2}))), \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} &M_2(x_{3n+3}, x_{3n+1}, x_{3n+2}) \\ &= \max\Big\{ G(x_{3n+3}, x_{3n+1}, x_{3n+2}), G(x_{3n+3}, Sx_{3n+3}, Tx_{3n+1}), \\ &\qquad G(x_{3n+1}, Tx_{3n+1}, Rx_{3n+2}), G(x_{3n+2}, Rx_{3n+2}, Sx_{3n+3}), \\ &\qquad \frac{1}{4s}[G(Sx_{3n+3}, x_{3n+1}, x_{3n+2}) + G(x_{3n+3}, Tx_{3n+1}, x_{3n+2}) \\ &\qquad + G(x_{3n+3}, x_{3n+1}, Rx_{3n+2})] \Big\} \\ &= \max\Big\{ G(x_{3n+3}, x_{3n+1}, x_{3n+2}), G(x_{3n+3}, x_{3n+4}, x_{3n+2}), \\ &\qquad \frac{1}{4s}[G(x_{3n+4}, x_{3n+1}, x_{3n+2}) + G(x_{3n+3}, x_{3n+2}, x_{3n+2}) \\ &\qquad + G(x_{3n+3}, x_{3n+1}, x_{3n+3})] \Big\} \end{aligned}$$

$$\leq \max\Big\{G(x_{3n+1}, x_{3n+2}, x_{3n+3}), G(x_{3n+2}, x_{3n+3}, x_{3n+4}),$$
$$\frac{s+1}{4s}[G(x_{3n+1}, x_{3n+2}, x_{3n+3}) + G(x_{3n+2}, x_{3n+3}, x_{3n+4})]\Big\}$$
$$= \max\{d_{3n+1}, d_{3n+2}\}.$$

If $M_2(x_{3n+3}, x_{3n+1}, x_{3n+2}) = d_{3n+2}$, then from (4.4), we get

$$\phi(2s^4 d_{3n+2}) \leq \psi(\phi(d_{3n+2})) < \phi(d_{3n+2}),$$

which is not possible. Hence, $M_2(x_{3n+3}, x_{3n+1}, x_{3n+2}) = d_{3n+1}$.
Using (4.4), we have

$$\phi(2s^4 d_{3n+2}) \leq \psi(\phi(d_{3n+1})) < \phi(d_{3n+1}). \tag{4.5}$$

Similarly, we can obtain

$$\phi(2s^4 d_{3n+3}) \leq \psi(\phi(d_{3n+2})) < \phi(d_{3n+2}). \tag{4.6}$$

From (4.3),(4.5) and (4.6), we have

$$\phi(d_{n+1}) \leq \phi(2s^4 d_{n+1}) \leq \psi(\phi(d_n)) \leq \psi^2(\phi(d_{n-1})) \leq ... \leq \psi^n(\phi(d_1)).$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} \psi^n(\phi(d_1)) = 0$.
Thus, $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+2}) = 0$.
Since $x_n \neq x_{n+1}$ for every n, so by property $(GB3)$, we obtain

$$G(x_n, x_{n+1}, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+2}).$$

Hence,

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0.$$

Since, $G(x_n, x_n, x_{n+1}) \leq sG(x_n, x_{n+1}, x_{n+1}),$ for all $n \geq 0$.

$$\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0.$$

Now, we prove that $\{x_n\}$ is a G_b-Cauchy sequence in X. It is sufficient to show that $\{x_{3n}\}$ is a G_b-Cauchy in X. On contrary, assume that $\{x_{3n}\}$ is not a G_b-

Cauchy sequence. There exists $\varepsilon > 0$ for which we can find subsequences $\{x_{3m_k}\}$ and $\{x_{3n_k}\}$ of $\{x_{3n}\}$ such that m_k is the smallest index for which $3m_k > 3n_k > k$ and

$$G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) < \varepsilon \leq G(x_{3n_k}, x_{3m_k}, x_{3m_k}).$$

Since

$$\begin{aligned}\varepsilon &\leq G(x_{3n_k}, x_{3m_k}, x_{3m_k}) \\ &\leq s[G(x_{3n_k}, x_{3n_k+1}, x_{3n_k+1}) + G(x_{3n_k+1}, x_{3m_k}, x_{3m_k})] \\ &\leq s[G(x_{3n_k}, x_{3n_k+1}, x_{3n_k+1}) + G(x_{3n_k+1}, x_{3m_k}, x_{3m_k-1})],\end{aligned}$$

taking upper limit as $k \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} \; G(x_{3n_k+1}, x_{3m_k}, x_{3m_k-1}), \tag{4.7}$$

which implies that, $G(x_{3n_k+1}, x_{3m_k}, x_{3m_k-1}) > 0,$ for all $k \in \mathbb{N}$.
Hence, from (4.1), we have

$$\begin{aligned}\phi(2s^4 G(x_{3n_k+1}, x_{3m_k-1}, x_{3m_k})) &= \phi(2s^4 G(Sx_{3n_k}, Tx_{3m_k-2}, Rx_{3m_k-1})) \\ &\leq \psi(\phi(M_2(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}))),\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}&M_2(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}) \\ &= \max\Big\{G(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}), G(x_{3n_k}, Sx_{3_k}, Tx_{3m_k-2}), \\ &\quad G(x_{3m_k-2}, Tx_{3m_k-2}, Rx_{3m_k-1}), G(x_{3m_k-1}, Rx_{3m_k-1}, Sx_{3n_k}), \\ &\quad \frac{1}{4s}[G(Sx_{3n_k}, x_{3m_k-2}, x_{3m_k-1}) + G(x_{3n_k}, Tx_{3m_k-2}, x_{3m_k-1}) \\ &\quad + G(x_{3n_k}, x_{3m_k-2}, Rx_{3m_k-1})]\Big\} \\ &= \max\Big\{G(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}), G(x_{3n_k}, x_{3n_k+1}, x_{3m_k-1}), \\ &\quad G(x_{3m_k-2}, x_{3m_k-1}, x_{3m_k}), G(x_{3m_k-1}, x_{3m_k}, x_{3n_k+1}), \\ &\quad \frac{1}{4s}[G(x_{3n_k+1}, x_{3m_k-2}, x_{3m_k-1}) + G(x_{3n_k}, x_{3m_k-1}, x_{3m_k-1})\end{aligned}$$

$$+ G(x_{3n_k}, x_{3m_k-2}, x_{3m_k})]\Big\}.$$

Since

$$\begin{aligned} G(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}) \leq & s[G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) \\ & + G(x_{3m_k-3}, x_{3m_k-2}, x_{3m_k-1})], \end{aligned}$$

taking upper limit as $k \to \infty$, we get

$$\limsup_{k\to\infty} \; G(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}) \leq s\varepsilon. \tag{4.9}$$

Also,

$$\begin{aligned} & G(x_{3n_k}, x_{3n_k+1}, x_{3m_k-1}) \\ & \leq s[G(x_{3m_k-1}, x_{3m_k-3}, x_{3m_k-3}) + G(x_{3m_k-3}, x_{3n_k}, x_{3n_k+1})] \\ & \leq sG(x_{3m_k-1}, x_{3m_k-3}, x_{3m_k-3}) + s^2G(x_{3m_k-3}, x_{3n_k}, x_{3n_k}) \\ & \quad + s^2G(x_{3n_k}, x_{3n_k}, x_{3n_k+1}) \\ & \leq sG(x_{3m_k-1}, x_{3m_k-3}, x_{3m_k-3}) + 2s^3G(x_{3m_k-3}, x_{3m_k-3}, x_{3n_k}) \\ & \quad + s^2G(x_{3n_k}, x_{3n_k}, x_{3n_k+1}). \end{aligned}$$

Taking upper limit as $k \to \infty$, we get

$$\limsup_{k\to\infty} \; G(x_{3n_k}, x_{3m_k+1}, x_{3m_k-1}) \leq 2s^3\varepsilon. \tag{4.10}$$

Again,

$$\begin{aligned} & G(x_{3m_k-1}, x_{3m_k}, x_{3n_k+1}) \\ & \leq sG(x_{3n_k+1}, x_{3n_k}, x_{3n_k}) + s^2G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) \\ & \quad + s^2G(x_{3m_k-3}, x_{3m_k}, x_{3m_k-1}). \end{aligned}$$

Hence,

$$\limsup_{k\to\infty} \; G(x_{3m_k-1}, x_{3m_k}, x_{3n_k+1}) \leq s^2\varepsilon. \tag{4.11}$$

Also,

$$\begin{aligned}
&G(x_{3n_k+1}, x_{3m_k-2}, x_{3m_k-1})\\
&\leq s^2 G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) + s^2 G(x_{3n_k}, x_{3n_k}, x_{3n_k+1})\\
&\quad + sG(x_{3m_k-3}, x_{3m_k-2}, x_{3m_k-1})
\end{aligned}$$

implies that

$$\limsup_{k\to\infty} G(x_{3n_k+1}, x_{3m_k-2}, x_{3m_k-1}) \leq s^2\varepsilon. \tag{4.12}$$

Also,

$$G(x_{3n_k+1}, x_{3m_k-1}, x_{3m_k-1}) \leq G(x_{3n_k+1}, x_{3m_k-2}, x_{3m_k-1})$$

implies that

$$\limsup_{k\to\infty} G(x_{3n_k+1}, x_{3m_k-1}, x_{3m_k-1}) \leq s^2\varepsilon. \tag{4.13}$$

Again,

$$G(x_{3n_k}, x_{3m_k-1}, x_{3m_k-1}) \leq G(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1})$$

implies that

$$\limsup_{k\to\infty} G(x_{3n_k}, x_{3m_k-1}, x_{3m_k-1}) \leq s\varepsilon. \tag{4.14}$$

Also,

$$\begin{aligned}
&G(x_{3n_k}, x_{3m_k-2}, x_{3m_k})\\
&\leq sG(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) + sG(x_{3m_k-3}, x_{3m_k-2}, x_{3m_k}).
\end{aligned}$$

Hence,

$$\limsup_{k\to\infty} G(x_{3n_k}, x_{3m_k-2}, x_{3m_k}) \leq s\varepsilon. \tag{4.15}$$

Using (4.9)-(4.15), we get

$$\begin{aligned}
\limsup_{k\to\infty} M_2(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}) &\leq \max\{s\varepsilon, 2s^3\varepsilon, s^2\varepsilon, \frac{1}{4s}(2s^2\varepsilon + s\varepsilon)\}\\
&= 2s^3\varepsilon.
\end{aligned}$$

Now, using (4.7) and (4.8), we get

$$\begin{aligned}\phi(2s^4\frac{\varepsilon}{s}) &\leq \phi(2s^4 \limsup_{k\to\infty} \ G(x_{3n_k+1}, x_{3m_k-1}, x_{3m_k}))\\ &= \phi(2s^4 \limsup_{k\to\infty} \ G(Sx_{3n_k}, Tx_{3m_k-2}, Rx_{3m_k-1}))\\ &\leq \psi(\phi(\limsup_{k\to\infty} \ M_2(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1})))\\ &\leq \psi(\phi(2s^3\varepsilon))\\ &< \phi(2s^3\varepsilon),\end{aligned}$$

which is a contradiction. Hence, $\{x_{3n}\}$ is a G_b-Cauchy sequence and so, $\{x_n\}$ is a G_b-Cauchy sequence. Since X is a complete G_b-metric space, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Therefore,

$$\begin{aligned}&\lim_{n\to\infty} x_{3n+1} = \lim_{n\to\infty} Sx_{3n} = \lim_{n\to\infty} x_{3n+2} =\\ &= \lim_{n\to\infty} Tx_{3n+1} = \lim_{n\to\infty} x_{3n+3} = \lim_{n\to\infty} Rx_{3n+2} = u.\end{aligned}$$

To prove that, $u = Ru$.
We have,

$$G(Sx_{3n}, Tx_{3n+1}, Rx_{3n+2}) \leq s[G(Sx_{3n}, Tx_{3n+1}, Ru) + G(Ru, Ru, Rx_{3n+2})].$$

Suppose $G(Sx_{3n}, Tx_{3n+1}, Ru) = 0$ and $G(Ru, Ru, Rx_{3n+2}) = 0$, for some $n \in \mathbb{N}$, then $G(Sx_{3n}, Tx_{3n+1}, Rx_{3n+2}) = 0$, a contradiction to our assumption.
Therefore, we take $G(Sx_{3n}, Tx_{3n+1}, Ru) > 0$, for all n.
From (4.1), we get

$$\phi(2s^4 G(Sx_{3n}, Tx_{3n+1}, Ru)) \leq \psi(\phi(M_2(x_{3n}, x_{3n+1}, u))), \tag{4.16}$$

where

$$\begin{aligned}&M_2(x_{3n}, x_{3n+1}, u)\\ &= \max\{G(x_{3n}, x_{3n+1}, u), G(x_{3n}, Sx_{3n}, Tx_{3n+1}),\\ &\qquad G(x_{3n+1}, Tx_{3n+1}, Ru), G(u, Ru, Sx_{3n}),\\ &\qquad \frac{1}{4s}[G(Sx_{3n}, x_{3n+1}, u) + G(x_{3n}, Tx_{3n+1}, u) + G(x_{3n}, x_{3n+1}, Ru)]\}.\end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\begin{aligned}\lim_{n\to\infty} M_2(x_{3n}, x_{3n+1}, u) &= \max\{G(u,u,u), G(u,u,Ru), \frac{1}{4s}G(u,u,Ru)\}\\ &= G(u,u,Ru).\end{aligned}$$

Taking limit as $n \to \infty$ in (4.16), we get

$$\phi(2s^4G(u,u,Ru)) \leq \psi(\phi(G(u,u,Ru))) < \phi(G(u,u,Ru)),$$

implies that

$$2s^4G(u,u,Ru) \leq G(u,u,Ru),$$

a contradiction. Hence, $u = Ru$, that is u is a fixed point of R.
Similarly, we can prove u is a fixed point of S and T both. Therefore, u is a common fixed point of S, T and R.

To prove that u is the unique common fixed point of S, T and R. Let v be another common fixed point of S, T and R. Then, $Su = Tu = Ru = u$ and $Sv = Tv = Rv = v$. We have $G(u,u,v) = G(Su,Tu,Rv) > 0$ and $G(u,v,v) = G(Su,Tv,Rv) > 0$.
From (4.1), we have

$$\phi(2s^4G(Su,Tu,Rv)) \leq \psi(\phi(M_2(u,u,v))) < \phi(M_2(u,u,v)), \tag{4.17}$$

where

$$M_2(u,u,v) = \max\{G(u,u,v), G(u,v,v)\}.$$

If $M_2(u,u,v) = G(u,v,v)$, then from (4.17), we get

$$\phi(2s^4G(u,u,v)) < \phi(G(u,v,v))$$

implies

$$2s^4G(u,u,v) < G(u,v,v) \leq 2sG(u,u,v),$$

a contradiction.

Similarly, if $M_2(u,u,v) = G(u,u,v)$, then from (4.17), we get

$$\phi(2s^4 G(u,u,v)) < \phi(G(u,u,v))$$

implies

$$2s^4 G(u,u,v) < G(u,u,v),$$

a contradiction.
Hence, S, T and R have a unique common fixed point in X.
Further, we prove that at least one of S, T and R is not continuous at u if and only if

$$\lim_{x\to u} M_2(x,u,u) \neq 0 \text{ or } \lim_{y\to u} M_2(u,y,u) \neq 0 \text{ or } \lim_{z\to u} M_2(u,u,z) \neq 0.$$

Equivalently, we prove that S, T and R are continuous at u if and only if

$$\lim_{x\to u} M_2(x,u,u) = 0 \text{ and } \lim_{y\to u} M_2(u,y,u) = 0 \text{ and } \lim_{z\to u} M_2(u,u,z) = 0.$$

We suppose that

$$\lim_{x\to u} M_2(x,u,u) = 0 \text{ and } \lim_{y\to u} M_2(u,y,u) = 0 \text{ and } \lim_{z\to u} M_2(u,u,z) = 0.$$

Now,

$$\begin{aligned}
&\lim_{x_n\to u} M_2(x_n,u,u)\\
&= \lim_{x_n\to u} \max\Big\{ G(x_n,u,u), G(x_n,Sx_n,Tu), G(u,Tu,Ru), G(u,Ru,Sx_n),\\
&\qquad \frac{1}{4s}[G(Sx_n,u,u) + G(x_n,Tu,u) + G(x_n,u,Ru)]\Big\} = 0.
\end{aligned}$$

Thus, $\lim_{x_n\to u} G(x_n,Sx_n,u) = 0$. This implies that $Sx_n \to u = Su$, that is, S is continuous at u.
Similarly, we can prove T and R are continuous at u.
On the other hand, if S, T and R are continuous at their common fixed point u, that is $\lim_{x_n\to u} Sx_n = Su$, $\lim_{x_n\to u} Tx_n = Tu$ *and* $\lim_{x_n\to u} Rx_n = Ru$.
Then

$$\begin{aligned}
&\lim_{x_n\to u} M_2(x_n,u,u)\\
&= \lim_{x_n\to u} \max\Big\{ G(x_n,u,u), G(x_n,Sx_n,Tu), G(u,Tu,Ru), G(u,Ru,Sx_n),
\end{aligned}$$

$$\left.\frac{1}{4s}[G(Sx_n,u,u)+G(x_n,Tu,u)+G(x_n,u,Ru)]\right\}=0,$$

$$\lim_{x_n\to u} M_2(u,x_n,u)$$
$$=\lim_{x_n\to u}\max\Big\{G(u,x_n,u),G(u,Su,Tx_n),G(x_n,Tx_n,Ru),G(u,Ru,Su),$$
$$\left.\frac{1}{4s}[G(Su,x_n,u)+G(u,Tx_n,u)+G(u,x_n,Ru)]\right\}=0,$$

$$\lim_{x_n\to u} M_2(u,u,x_n)$$
$$=\lim_{x_n\to u}\max\Big\{G(u,u,x_n),G(u,Su,Tu),G(u,Tu,Rx_n),G(x_n,Rx_n,Su),$$
$$\left.\frac{1}{4s}[G(Su,u,x_n)+G(u,Tu,x_n)+G(u,u,Rx_n)]\right\}=0.$$

□

The subsequent example affirms our obtained result.

Example 4.3.1. Let $X=[0,\infty)$ and define $G:X^3\to[0,\infty)$ by

$$G(x,y,z)=\begin{cases}0, & \text{if } x=y=z,\\ \max\{x,y,z\}, & \text{otherwise.}\end{cases}$$

Then (X,G) is a complete G_b-metric space with $s=1$.
We define $S,T,R:X\to X$ by

$$Sx=\begin{cases}\frac{x}{16}, & x\in[0,1],\\ 0, & x\in(1,\infty);\end{cases}$$

$$Tx=\begin{cases}\frac{x}{12}, & x\in[0,1],\\ 0, & x\in(1,\infty);\end{cases}$$

$$Rx=\begin{cases}\frac{x}{10}, & x\in[0,1],\\ 0, & x\in(1,\infty).\end{cases}$$

Also, take $\phi(t)=t$ and $\psi(t)=\frac{t}{2}$.
Then, S,T,R satisfy all the conditions of Theorem 4.3.3 and $x=0$ is the only common fixed point of S,T and R.

Corollary 4.3.4. *Let $S:X\to X$ be $(\psi,\phi)-G_b$-Wardowski contraction in a complete G_b-metric space. Then S has a unique fixed point, say u and $S^n x\to u$,* *for each $x \in X$. Further, S is discontinuous at u if and only if*

$$\lim_{x \to u} M_1(x, u, u) \neq 0.$$

Proof. By taking $S = T = R$ in Theorem 4.3.3, we get the result. □

Corollary 4.3.5. *Let (X, G_b) be a complete G_b-metric space and $S : X \to X$ satisfies*

$$G(Sx, Sy, Sz) > 0 \implies \phi(2s^4 G(Sx, Sy, Sz)) \leq \psi(\phi(G(x, y, z))),$$

for all $x, y, z \in X$, where $\phi \in \Phi$ and $\Psi \in F_{com}$. Then S has a unique fixed point, say u and $S^n x \to u$, for each $x \in X$. Further, S is discontinuous at u if and only if

$$\lim_{x \to u} G(x, u, u) \neq 0.$$

Proof. Taking $M_2(x, y, z) = G(x, y, z)$, the conclusion follows from Corollary 4.3.4. □

The following result is for Wardowski type contraction in G_b-metric spaces.

Corollary 4.3.6. *Let (X, G_b) be a complete G_b-metric space and $S : X \to X$ satisfies*

$$G(Sx, Sy, Sz) > 0 \implies \tau + F(2s^4 G(Sx, Sy, Sz)) \leq F(G(x, y, z)),$$

for all $x, y, z \in X$. Then S has a unique fixed point, say u and $S^n x \to u$, for each $x \in X$. Further, S is discontinuous at u if and only if

$$\lim_{x \to u} G(x, u, u) \neq 0.$$

Proof. In Corollary 4.3.4, we take $M_2(x, y, z) = G(x, y, z)$ and $\psi(t) = e^{-\tau} t$, where $\tau > 0$ and $\phi(t) = e^{F(t)}$, where F is an F-contraction, then we get the result. □

4.4 Application to neural networks

In fixed point theorems, contractive mappings that admit discontinuity at the fixed point have found applications in neural networks with discontinuous activation functions (e.g. Özgür and Tas [45] and Rashid et al. [51]). Here, an application of Theorem 4.3.3 is provided by taking into account discontinuous activation functions in neural networks. The class of discontinuous activation functions was generalized by Nie and Zheng [44] as follows.

$$S_i(x) = \begin{cases} u_i, & -\infty < x < p_i, \\ l_{i,1}x + c_{i,1}, & p_i \le x \le r_i, \\ l_{i,2}x + c_{i,2}, & r_i < x \le q_i, \\ v_i, & q_i < x < +\infty, \end{cases}$$

where $p_i, r_i, q_i, u_i, v_i, l_{i,1}, l_{i,2}, c_{i,1}, c_{i,2}$ are constants with

$$\begin{aligned} & -\infty < p_i < r_i < q_i < +\infty, \\ & l_{i,1} > 0, \quad l_{i,2} < 0, \\ & u_i = l_{i,1}p_i + c_{i,1} = l_{i,2}q_i + c_{i,2}, \\ & l_{i,1}r_i + c_{i,1} = l_{i,2}r_i + c_{i,2}, \\ & v_i > S_i(r_i), \quad i = 1, 2, ..., n. \end{aligned}$$

The function S_i is continuous at every real number except the value $x = q_i$.

Here, we consider the discontinuous activation functions S, T and R:

$$S(x) = \begin{cases} 4, & -\infty < x < -2, \\ x + 6, & -2 \le x \le 1, \\ -x + 8, & 1 < x \le 4, \\ 8, & 4 < x < +\infty, \end{cases}$$

where

$$\begin{aligned} & p_i = -2, \ r_i = 1, \ q_i = 4, \ u_i = 4, \ v_i = 3, \\ & l_{i,1} = 1, \ c_{i,1} = 6, \ l_{i,2} = -1, \ c_{i,2} = 8, \end{aligned}$$

$$T(x) = \begin{cases} -3, & -\infty < x < -2, \\ 2x + 1, & -2 \le x \le -\frac{1}{2}, \\ -2x - 1, & -\frac{1}{2} < x \le 1, \\ 4, & 1 < x < +\infty, \end{cases}$$

where

$$p_i = -2, \ r_i = -\frac{1}{2}, \ q_i = 1, \ u_i = -3, \ v_i = 4,$$

$$l_{i,1} = 2,\ c_{i,1} = 1,\ l_{i,2} = -2,\ c_{i,2} = -1$$

and

$$R(x) = \begin{cases} -2, & -\infty < x < -4, \\ 2x + 6, & -4 \leq x \leq -3, \\ -2x - 6, & -3 < x \leq -2, \\ 4, & -2 < x < +\infty, \end{cases}$$

where

$$p_i = -4,\ r_i = -3,\ q_i = -2,\ u_i = -2,\ v_i = 4,$$
$$l_{i,1} = 2,\ c_{i,1} = 6,\ l_{i,2} = -2,\ c_{i,2} = -6.$$

The function T has four fixed points, $u_1 = -3, u_2 = -1, u_3 = \frac{-1}{3}$ *and* $u_4 = 4$, and the functions S and R has only one fixed point at $x = 4$. So, $x = 4$ is the common fixed point of S, T and R.

$$\lim_{x \to 4} M_2(x, 4, 4) \neq 0,$$

implies S is discontinuous at $x = 4$.