CHAPTER 2

AN UPPER BOUND FOR THE NUMBER OF DISJOINT BLOCKS IN CERTAIN PBIB DESIGNS

2.1 Introduction

An upper bound for the number of disjoint blocks in balanced incomplete block design was obtained by Majumdar [27]. In this chapter, upper bounds for the number of disjoint blocks in certain partially balanced incomplete block (PBIB) designs are obtained. The PBIB designs considered here are (i) semi-regular group divisible (SRGD) designs, (ii) certain PBIB designs with two associate classes having triangular association scheme (certain triangular designs), (iii) certain PBIB designs with two associate classes having a L_2 association scheme (certain L_2 designs) and (iv) certain PBIB designs with three associate classes having rectangular association scheme (certain rectangular designs). The upper bounds are derived by using the results proved by (i) Bose and Connor [6], (ii)

Raghavarao [34] and Vartak [54].

2.2 An upper bound for the number of disjoint blocks in SRGD designs.

An incomplete block design with v treatments each treatment being replicated r times, arranged in b blocks of k plots each is said to be group divisible (GD) (Bose and Shimamoto [8]), if the number of treatments in $v = mn$ and the treatments can be divided into m groups each with n treatments, so that the treatments belonging to the same group occur together in λ_1 blocks and the treatments belonging to different groups occur together in λ_2 blocks ($\lambda_1 \neq \lambda_2$). This is a PBIB design with two associate classes and the first associates of any treatment are the treatments belonging to the same group. The primary parameters of this design are $v = mn$, r, k, λ_1, λ_2, $n_1 = n-1$, $n_2 = n(m-1)$. The parameters obviously satisfy the relations

(2.2.1) $$bk = vr,$$

(2.2.2) $$r(k-1) = n_1 \lambda_1 + n_2 \lambda_2,$$

(2.2.3) $$r \geq \lambda_1 , \quad r \geq \lambda_2.$$

Bose and Connor [6] characterised semi-regular group divisible (SRGD) designs by $r - \lambda_1 \geq 0$ and

$rk - v\lambda_2 = 0$. The following result was proved by Bose and Connor [6] for SRGD designs.

Theorem 2.2.1. For a SRGD design, k is divisible by m. If $k = cm$, then every block must contain c treatments from every group.

We use Theorem 2.2.1 to obtain an upper bound for the number of disjoint blocks which have no treatments in common with a given block of SRGD design. The result is given in Theorem 2.2.2.

Theorem 2.2.2. A given block of a SRGD design cannot have more than

$$b - 1 - \frac{v(v-m)(r-1)^2}{[\,(v-k)(b-r) - (v-rk)(v-m)\,]}$$

disjoint blocks with it and if some block has that many disjoint blocks, then

$$c = k[\,(v-k)(b-r) - (v-rk)(v-m)\,]/v(v-m)(r-1)$$

is a positive integer and each non-disjoint block has c treatments common with that given block.

Proof. Let the given block have d disjoint blocks. Let it have x_i treatments common with the ith of the remaining $(b - d - 1)$ blocks. Then considering the treatments of the given block singly, we have

$$(2.2.4) \qquad \sum_{i=1}^{b-d-1} x_i = k(r-1).$$

The given block, by virtue of Theorem 2.2.1, contains k/m treatments from each group which form pairs of first associates. Hence considering the treatments of the given block pairwise, we get

$$(2.2.5) \qquad \sum_{i=1}^{b-d-1} x_i(x_i-1) = k[\lambda_1(k-m)+k\lambda_2(m-1)-m(k-1)] / m.$$

Now for a SRGD design, $\lambda_2 = rk/v$. Then, from $n_1\lambda_1 + n_2\lambda_2 = r(k-1)$, we get $\lambda_1 = r(k-m)/(v-m)$. Substituting these values of λ_1 and λ_2 in (2.2.5) and defining $\bar{x} = \sum_{i=1}^{b-d-1} x_i / (b-d-1)$, we get from (2.2.4) and (2.2.5)

$$(2.2.6) \qquad \sum_{i=1}^{b-d-1} (x_i - \bar{x})^2 = \frac{k^2[(v-k)(b-r)-(v-rk)(v-m)]}{v(v-m)} - \frac{k^2(r-1)^2}{(b-d-1)}.$$

As $\sum_{i=1}^{b-d-1} (x_i - \bar{x})^2 \geq 0$, and $[(v-k)(b-r) - (v-rk)(v-m)] > 0$, (Appendix 2.1), it follows from (2.2.6) that

$$(2.2.7) \qquad d \leq b - 1 - \frac{v(v-m)(r-1)^2}{[(v-k)(b-r) - (v-rk)(v-m)]} ,$$

which proves the first part of the theorem. If, however

$$(2.2.8) \qquad d = b - 1 - \frac{v(v-m)(r-1)^2}{[(v-k)(b-r) - (v-rk)(v-m)]} ,$$

then $\sum_{i=1}^{b-d-1} (x_i - \bar{x})^2 = 0$, showing that all x_i's are equal to c, where

$$(2.2.9) \qquad c = \frac{k[(v-k)(b-r) - (v-rk)(v-m)]}{v(v-m)(r-1)} ,$$

and the given block has c treatments common with each of the remaining $(b-d-1)$ non-disjoint blocks.

The following are the companion theorems to Theorem 2.2.2.

Theorem 2.2.3. The necessary and sufficient condition that a block of a SRGD design has the same number of treatments common with each of the remaining blocks is that (i) $b = v-m+1$ and (ii) $k(r-1)/(v-m)$ is an integer.

Proof. Let a block of the given design have x_i treatments common with the ith of the remaining $(b-1)$

blocks. Then, putting $d = 0$ in (2.2.6), we get

$$(2.2.10) \qquad \sum_{i=1}^{b-1} (x_i - \bar{x})^2 = \frac{k^2(v-k)(b-r)(b-v+m-1)}{v(v-m)(b-1)} ,$$

where $\bar{x} = k(r-1)/(b-1)$. All factors on the r.h.s. of (2.2.10) except $(b-v+m-1)$ are positive. Hence, we get the result from (2.2.10).

Theorem 2.2.4. If a block of a SRGD design with parameters $v = mn = tk$, $b = tr$, (t an integer greater than 1), has $(t-1)$ blocks disjoint with it, then the necessary and sufficient condition that it has the same number of treatments common with each of the non-disjoint blocks is that (i) $b = v - m + r$ and (ii) k/t is an integer.

Proof. Let a block of the given design have x_i treatments common with the ith of the remaining $(b-t) = t(r-1)$ non-disjoint blocks. Then, we have from (2.2.6), noting that $d = t-1$,

$$(2.2.11) \qquad \sum_{i=1}^{b-t} (x_i - \bar{x})^2 = \frac{k^2(v-k)(b-v+m-r)}{v(v-m)} ,$$

where $\bar{x} = k/t$. The theorem follows from (2.2.11).

We get the following two corollaries from the above theorem.

Corollary 2.2.1. For a resolvable SRGD design, $b \geq v-m+r$.

This is also proved by Bose and Connor [6].

Corollary 2.2.2. The necessary and sufficient condition that a resolvable SRGD design be affine resolvable is that it has a block which has the same number of treatments common with each block not belonging to its own replication.

2.3 An upper bound for the number of disjoint blocks in certain triangular designs

A PBIB design with two associate classes is said to have a triangular association scheme (Bose and Shimamoto [8]), if the number of treatments is $v = n(n-1)/2$ and the association scheme is an array of n rows and n columns with the following properties:

(a) the positions in the principal diagonal are blank,

(b) the $n(n-1)/2$ positions above the principal diagonal are filled by the numbers 1, 2, ..., $n(n-1)/2$, corresponding to the treatments,

(c) the array is symmetric about the principal diagonal,

(d) for any treatment θ, the first associates are exactly those treatments which lie in the same row and the same column as θ.

The design will be called as triangular design in short. The primary parameters of this design are $v = n(n-1)/2$, b, r, k, λ_1, λ_2, $n_1 = 2n-4$, $n_2 = (n-3)(n-2)/2$. We consider here triangular designs in which $rk - v\lambda_1 = n(r - \lambda_1)/2$. The following theorem has been proved by Raghavarao [34].

Theorem 2.3.1. If in a triangular design, $rk - v\lambda_1 = n(r - \lambda_1)/2$, then $2k$ is divisible by n. Further every block of this design contains $2k/n$ treatments from each of the n rows of the association scheme.

We use Theorem 2.3.1 to obtain an upper bound for the number of disjoint blocks which have no treatments common with a given block of the triangular design, in which $rk - v\lambda_1 = n(r - \lambda_1)/2$. The result is given in Theorem 2.3.2.

Theorem 2.3.2. A given block of a triangular design with $rk - v\lambda_1 = n(r - \lambda_1)/2$ cannot have more than

$$b - 1 - \frac{v(v-n)(r-1)^2}{[(v-k)(b-r) - (v-rk)(v-n)]}$$

disjoint blocks and if some block has that many disjoint blocks, then

$$c = k[(v-k)(b-r) - (v-rk)(v-n)] \,/\, v(v-n)(r-1)$$

is a positive integer and each non-disjoint block has c treatments common with that given block.

Proof. Let the given block have d disjoint blocks. Let it have x_i treatments common with the ith of the remaining (b-d-1) non-disjoint blocks. Then, considering the treatments of the given block singly, we have

$$\sum_{i=1}^{b-d-1} x_i = k(r-1). \tag{2.3.1}$$

Considering treatments of the given block pairwise and using Theorem 2.3.1, we have

$$\begin{aligned}\sum_{i=1}^{b-d-1} x_i(x_i - 1) &= n(2k/n)(2k/n - 1)(\lambda_1 - 1) \\ &\quad + \{k(k-1) - n(2k/n)(2k/n - 1)\}(\lambda_2 - 1).\end{aligned} \tag{2.3.2}$$

Let $v = v_1 v_2$, where $v_1 = n/2$ and $v_2 = (n-1) = 2v_1 - 1$.

From $rk - v\lambda_1 = n(r - \lambda_1)/2$, we get $\lambda_1 = r(k - v_1)/2v_1(v_1 - 1)$. Also, we have $n_1 = 4(v_1 - 1)$, $n_2 = (v_1 - 1)(v_2 - 2)$, and $\lambda_2 = r(kv_1 + v_1 - 2k)/v_1(v_1 - 1)(v_2 - 2)$. Putting $n = 2v_1$ and substituting the values of λ_1 and λ_2 in (2.3.2), we get

$$\sum_{i=1}^{b-d-1} x_i(x_i-1)$$

$$= \frac{k^2[v_1(b-2r+1)-(v-rk)(v_1-1)]}{v_1(v-2v_1)} - k(r-1)$$

$$(2.3.3) \qquad = \frac{k^2[n(b-2r+1)-(v-rk)(n-2)]}{n(v-n)} - k(r-1)$$

$$= \frac{k^2[n(n-1)(b-2r+1)-(v-rk)(n-2)(n-1)]}{n(n-1)(v-n)} - k(r-1)$$

$$= \frac{k^2[(v-k)(b-r)-(v-rk)(v-n)]}{v(v-n)} - k(r-1).$$

Let $\bar{x} = k(r-1)/(b-d-1)$. Then, from (2.3.1) and (2.3.3), we have

$$\sum_{i=1}^{b-d-1} (x_i - \bar{x})^2$$

(2.3.4)
$$= \frac{k^2[(v-k)(b-r)-(v-rk)(v-n)]}{v(v-n)} - \frac{k^2(r-1)^2}{(b-d-1)} \geq 0.$$

As $[(v-k)(b-r)-(v-rk)(v-n)] > 0$, (Appendix 2.1), it follows from (2.3.4), that

$$d \leq b - 1 - \frac{v(v-n)(r-1)^2}{(v-k)(b-r)-(v-rk)(v-n)} . \tag{2.3.5}$$

If, however, $d = b - 1 - \dfrac{v(v-n)(r-1)^2}{(v-k)(b-r)-(v-rk)(v-n)}$, then $\sum_{i=1}^{b-d-1} (x_i - \bar{x})^2 = 0$, showing that

$$x_i = \frac{k[(v-k)(b-r)-(v-rk)(v-n)]}{v(v-n)(r-1)} = c, \tag{2.3.6}$$

for all i. The theorem then follows from (2.3.5) and (2.3.6).

The following are the companion theorems to Theorem 2.3.2.

Theorem 2.3.3. The necessary and sufficient condition that a block of a triangular design with $rk - v\lambda_1 = n(r - \lambda_1)/2$, has the same number of treatments common with each of the remaining blocks is that (i) $b = v - n + 1$ and (ii) $k(r-1)/(v-n)$ is an integer.

Proof. Let a block of the given design have x_i treatments commonw with the ith remaining (b-1) blocks. Then, from (2.3.4), noting that $d = 0$, we get

$$\sum_{i=1}^{b-1} (x_i - \bar{x})^2 = \frac{k^2(b-r)(v-k)(b-v+n-1)}{v(v-n)(b-1)} . \tag{2.3.7}$$

The theorem, then, follows from (2.3.7).

Theorem 2.3.4. If a block of a triangular design with parameters $v = n(n-1)/2 = tk$, (t an integer greater than 1), $b = tr$ and $rk - v\lambda_1 = n(r - \lambda_1)/2$ has (t-1) blocks disjoint with it, then the necessary and sufficient condition that it has the same number of treatments common with each of the remaining non-disjoint blocks is that (i) $b = v + r - n$ and (ii) k/t is an integer.

Proof. Let a block of the given design have x_i treatments common with the ith of the remaining $b-t = t(r-1)$ non-disjoint blocks. Then, we have from (2.3.4), noting that $d = t-1$,

$$\sum_{i=1}^{b-t} (x_i - \bar{x})^2 = \frac{k^2(v-k)(b-v-r+n)}{v(v - n)} , \tag{2.3.8}$$

where $\bar{x} = k/t$. The theorem follows from the consideration of (2.3.8).

We get the following corollaries from the above theorem.

Corollary 2.3.1. For a resolvable triangular design with $rk - v\lambda_1 = n(r - \lambda_1)/2$, $b \geq v + r - n$.

Corollary 2.3.2. The necessary and sufficient condition that a resolvable triangular design with $rk - v\lambda_1 = n(r - \lambda_1)/2$ be affine resolvable is that it has a block which has the same number of common treatments with each block not belonging to its own replication.

2.4 An upper bound for the number of disjoint blocks in certain L_2 designs

A PBIB design with two associate classes is said to have a L_2 association scheme (Bose and Shimamoto [8]), if the number of treatments is $v = s^2$, where s is a positive integer and the treatments can be arranged in an sxs square such that treatments in the same row or column are first associates, while others are second associates. The primary parameters of this design are $v = s^2$, $b, r, k, \lambda_1, \lambda_2$, $n_1 = 2(s - 1)$, $n_2 = (s - 1)^2$. We call this design as L_2 design in short. We shall consider here L_2 designs in which $rk - v\lambda_1 = s(r - \lambda_1)$. The following theorem has been proved by Raghavarao [34].

Theorem 2.4.1. If in a L_2 design, $rk - v\lambda_1 = s(r - \lambda_1)$, then k is divisible by s. Further every block of this design contains k/s treatments from each of the s rows (or columns) of the association scheme.

We use Theorem 2.4.1 to obtain an upper bound for the number of disjoint blocks which have no treatments common with a given block of a L_2 design in which $rk - v\lambda_1 = s(r - \lambda_1)$. The result is given in Theorem 2.4.2.

Theorem 2.4.2. A given block of a L_2 design with $rk - v\lambda_1 = s(r - \lambda_1)$ cannot have more than

$$b - 1 - \frac{v(r-1)^2\ (s-1)^2}{(v-k)(b-r)-(v-rk)(s-1)^2}$$

disjoint blocks with it and if some block has that many disjoint blocks, then

$$c = k\left[(v-k)(b-r)-(v-rk)(s-1)^2\right] / v(r-1)(s-1)^2$$

is a positive integer and each non-disjoint block has c treatments common with that given block.

Proof. Let the given block have d disjoint blocks. Let it have x_i treatments common with the ith of the remaining $(b-d-1)$ non-disjoint blocks, $i = 1, 2, \ldots, (b-d-1)$. Then, considering the treatments of the given block singly, we have

$$\sum_{i=1}^{b-d-1} x_i = k(r-1). \tag{2.4.1}$$

Considering the treatments of the given block pairwise and using Theorem 2.4.1, we have

$$\sum_{i=1}^{b-d-1} x_i(x_i - 1) = k\left[2(k-s)\lambda_1 + (sk+s-2k)\lambda_2 - s(k-1)\right]/s. \tag{2.4.2}$$

Now, $rk - v\lambda_1 = s(r - \lambda_1)$ gives $\lambda_1 = r(k - s)/s(s-1)$. Also, $n_1\lambda_1 + n_2\lambda_2 = r(k-1)$ gives $\lambda_2 = r(sk+s-2k)/s(s-1)^2$. Hence, substituting the values of λ_1 and λ_2 in (2.4.2), we get

$$\sum_{i=1}^{b-d-1} x_i(x_i-1) = \frac{k\left[2r(k-s)^2(s-1)+r(sk+s-2k)^2+v(s-1)^2(r-k)\right]}{v(s-1)^2} - k(r-1) \tag{2.4.3}$$

$$= \frac{k^2\left[(v-k)(b-r)-(v-rk)(s-1)^2\right]}{v(s-1)^2} - k(r-1).$$

From (2.4.1) and (2.4.3), we get

$$\sum_{i=1}^{b-d-1} (x_i - \bar{x})^2$$

(2.4.4)

$$= \frac{k^2\left[(v-k)(b-r)-(v-rk)(s-1)^2\right]}{v(s-1)^2} - \frac{k^2(r-1)^2}{b-d-1} \geq 0,$$

where $\bar{x} = k(r-1)/(b-d-1)$. As $\left[(v-k)(b-r)-(v-rk)(s-1)^2\right] > 0$, (Appendix 2.1), it follows from (2.4.4) that

(2.4.5) $$d \leq b - 1 - \frac{v(s-1)^2(r-1)^2}{(v-k)(b-r)-(v-rk)(s-1)^2}.$$

This proves the first part of the theorem. If, however

$$d = b - 1 - \frac{v(s-1)^2(r-1)^2}{(v-k)(b-r)-(v-rk)(s-1)^2},$$

then $\sum_{i=1}^{b-d-1} (x_i - \bar{x})^2 = 0$, giving

$$x_i = \frac{k\left[(v-k)(b-r)-(v-rk)(s-1)^2\right]}{v(r-1)(s-1)^2} = \bullet$$

for all i. Hence the result.

The following are the companion theorems to Theorem 2.4.2.

Theorem 2.4.3. The necessary and sufficient condition that a block of a L_2 design with $rk - v\lambda_1 = s(r - \lambda_1)$ has the same number of treatments common with each of the remaining blocks is that (i) $b = v - 2s + 2$ and (ii) $k(r-1)/(s-1)^2$ is an integer.

Proof. Let a block of the given design have x_i treatments common with the ith of the remaining (b-1) blocks. Then, from (2.4.4), noting that $d = 0$, we get

$$(2.4.6) \qquad \sum_{i=1}^{b-1} (x_i - \bar{x})^2 = \frac{k^2(b-r)(v-k)(b-v+2s-2)}{v(b-1)(s-1)^2} .$$

Theorem 2.4.3 follows from (2.4.6).

Theorem 2.4.4. If a block of a L_2 design with parameters $v = s^2 = tk$, $b = tr$, (t an integer greater than 1), and $rk - v\lambda_1 = s(r - \lambda_1)$ has (t-1) blocks disjoint with it, then the necessary and sufficient condition that it has a block which has the same number of treatments common with each of the remaining non-disjoint blocks is that (i) $b = v - 2s + r + 1$ and (ii) k/t is an integer.

Proof. Let a block of the given design have x_i

treatments common with the ith of the remaining $(b-t) = t(r-1)$ non-disjoint blocks. Then, from (2.4.4), noting that $d = t-1$, we have

$$\sum_{i=1}^{b-t} (x_i - \bar{x})^2 = \frac{k^2(v-k)(b-v-r+2s-1)}{v(s-1)^2} . \tag{2.4.7}$$

The result follows from the consideration of (2.4.7).

We get the following corollaries from the above theorem.

Corollary 2.4.1. For a resolvable L_2 design with $rk - v\lambda_1 = s(r-\lambda_1)$, $b \geq v - 2s + r + 1$.

Corollary 2.4.2. The necessary and sufficient condition that a resolvable L_2 design with $rk - v\lambda_1 = s(r - \lambda_1)$ be affine resolvable is that it has a block which has the same number of treatments common with each block not belonging to its own replication.

2.5 An upper bound for the number of disjoint blocks in certain rectangular designs

A PBIB design with three associate classes is said to have a rectangular association scheme (Vartak [53]), if the number of treatments is $v = v_1 v_2$ and the treatments can be arranged in the form of a rectangle of v_1 rows and v_2 columns, so that the first associates of any treatment are the other $(v_2 - 1)$ treatments of the same row, the second associates are the other $(v_1 - 1)$ treatments of the same column; while the remaining $(v_1 - 1)(v_2 - 1)$ treatments are the third associates. The primary parameters of this design are $v = v_1 v_2$, b, r, k, $n_1 = v_2 - 1$, $n_2 = v_1 - 1$, $n_3 = n_1 n_2$, λ_1, λ_2 and λ_3. We shall call this design as rectangular design in short. Vartak [53] has proved that the characteristic roots of NN' (N being the incidence matrix of the design) of this design are

$$\theta_0 = rk,$$

$$\theta_1 = r - \lambda_1 + (v_1 - 1)(\lambda_2 - \lambda_3),$$

$$\theta_2 = r - \lambda_2 + (v_2 - 1)(\lambda_1 - \lambda_3),$$

$$\theta_3 = r - \lambda_1 - \lambda_2 + \lambda_3.$$

Here, we consider the rectangular designs in which $\theta_1 = \theta = \theta_2$. The following theorems were proved by Vartak [54].

Theorem 2.5.1. If in a rectangular design, $\theta_1 = 0$, then k is divisible by v_2 and every block of this design contains k/v_2 treatments from every column of the association scheme.

Theorem 2.5.2. If in a rectangular design, $\theta_2 = 0$, then k is divisible by v_1 and every block of this design contains k/v_1 treatments from every row of the association scheme.

We use Theorems 2.5.1 and 2.5.2 to obtain an upper bound for the number of disjoint blocks which have no treatments common with a given block of a rectangular design in which $\theta_1 = 0 = \theta_2$. The result is given in Theorem 2.5.3.

Theorem 2.5.3. A given block of a rectangular design with $\theta_1 = 0 = \theta_2$ cannot have more than

$$b - 1 - \frac{vp(r-1)^2}{(v-k)(b-r)-p(v-rk)}$$

disjoint blocks with it and if some block has that many disjoint blocks, then

$$c = k\ \left[(v-k)(b-r)-p(v-rk)\right] / vp(r-1)$$

is a positive integer and each non-disjoint block has c treatments common with that given block, where $p = (v_1 - 1)(v_2 - 1)$.

Proof. Let a block of the given design have d disjoint blocks and let it have x_i treatments common with the ith of the remaining (b-d-1) non-disjoint blocks. Then, considering the treatments of the given block singly, we have

$$\sum_{i=1}^{b-d-1} x_i = k(r-1). \tag{2.5.1}$$

Considering the treatments of the given block pairwise and using Theorems 2.5.1 and 2.5.2, we have

$$\sum_{i=1}^{b-d-1} x_i(x_i-1)$$

$$= k\left[v_2(k-v_1)(\lambda_1-\lambda_3)+v_1(k-v_2)(\lambda_2-\lambda_3) +v(k-1)(\lambda_3-1)\right] / v. \tag{2.5.2}$$

Next, we have

$$\theta_1 = r - \lambda_1 + (v_1 - 1)(\lambda_2 - \lambda_3) = 0, \tag{2.5.3}$$

$$\theta_2 = r - \lambda_2 + (v_2 - 1)(\lambda_1 - \lambda_3) = 0, \tag{2.5.4}$$

$$r(k - 1) = \lambda_1(v_2 - 1) + \lambda_2(v_1 - 1) + \lambda_3 p. \tag{2.5.5}$$

Solving equations (2.5.3), (2.5.4) and (2.5.5) for λ_1, λ_2 and λ_3, we obtain

$$\lambda_1 = rv_2(k - v_1)(v_1 - 1)/vp,$$

$$\lambda_2 = rv_1(k - v_2)(v_2 - 1)/vp,$$

$$\lambda_3 = r(v + kv - kv_1 - kv_2)/vp.$$

Substituting the values of λ_1 , λ_2 and λ_3 in (2.5.2), we get

(2.5.6)
$$\sum_{i=1}^{b-d-1} x_i(x_i-1) = k^2\left[(v-k)(b-r)-p(v-rk)\right]-k(r-1).$$

From (2.5.1) and (2.5.6), we get

(2.5.7)
$$\sum_{i=1}^{b-d-1} (x_i-\bar{x})^2 = k^2\left[(v-k)(b-r)-p(v-rk)\right] - \frac{k^2(r-1)^2}{(b-d-1)} \geq 0.$$

As $\left[(v-k)(b-r)-p(v-rk)\right] > 0$, (Appendix 2.1), it follows from (2.5.7) that

(2.5.8)
$$d \leq b - 1 - \frac{vp(r-1)^2}{(v-k)(b-r)-p(v-rk)} .$$

This proves the first part of the theorem. If, however

$$d = b - 1 - \frac{vp(r-1)^2}{(v-k)(b-r)-p(v-rk)} ,$$

then $\sum_{i=1}^{b-d-1} (x_i - \bar{x})^2 = 0$, giving

$$x_i = \frac{k|(v-k)(b-r)-p(v-rk)|}{vp(r-1)} = c, \tag{2.5.9}$$

for all i. Hence the result.

The following are the companion theorems to Theorem 2.5.3.

Theorem 2.5.4. The necessary and sufficient condition that a block of a rectangular design with $\theta_1 = 0 = \theta_2$ has the same number of treatments common with each of the remaining blocks is that (i) $b = p+1$ and (ii) $k(r-1)/p$ is an integer.

Proof. Let a block of the given design have x_i treatments common with the ith of the remaining $(b-1)$ blocks. Then, from (2.5.7), noting that $d = 0$, we get

$$\sum_{i=1}^{b-1} (x_i - \bar{x})^2 = \frac{k^2(v-k)(b-r)(b-p-1)}{vp(b-1)}, \tag{2.5.10}$$

from which the result follows.

Theorem 2.5.5. If a block of a rectangular design with $\theta_1 = 0 = \theta_2$ and parameters $v = v_1 v_2 = tk$, $b = tr$, (t an integer greater than 1) has $(t-1)$ blocks disjoint with it, then the necessary and sufficient condition that it has the same number of treatments common with each of the non-disjoint blocks is that (i) $b = p + r$ and (ii) k/t is an integer.

Proof. Let a block of the given design have x_i treatments common with each of the remaining $b-t = t(r-1)$ non-disjoint blocks. Then from (2.5.7), noting that $d = t-1$, we have

$$\sum_{i=1}^{b-t} (x_i - \bar{x})^2 = k^2(v - k)(b - r - p)/vp, \tag{2.5.11}$$

from which the result follows.

We get the following corollaries from the above theorem.

Corollary 2.5.1. For a resolvable rectangular design with $\theta_1 = 0 = \theta_2$, $b \geq p + r$.

Corollary 2.5.2. The necessary and sufficient condition that a resolvable rectangular design with $\theta_1 = 0 = \theta_2$ be affine resolvable is that it has a block which has the same number of treatments common with each block not belonging to its own replication.