CHAPTER 4

ON THE BLOCK STRUCTURE OF CERTAIN PBIB DESIGNS

4.1 Introduction

Here we consider (i) SRGD designs, (ii) certain triangular designs, (iii) certain L_2 designs and (iv) certain rectangular designs and obtain upper bound for the number of blocks having a given number of treatments common with a given block of the designs mentioned above. Further in the four classes of the designs mentioned above, we derive (i) conditions under which either no two blocks are disjoint or a given block has only one disjoint block and (ii) conditions under which no two blocks are the same set.

4.2 SRGD designs

For the description of a SRGD design, we refer to Section 2.2 of Chapter 2. We prove the following theorem.

Theorem 4.2.1. If in a SRGD design with $b > v - m + 1$, a given block has d blocks all having a given number $l(\leq k)$ of treatments common with it, then

$$d \leq b - 1 - [k(r-1) - l(b-1)]^2 Q^{-1},$$

where

$$Q = P - 2lk(r-1) + l^2(b-1) \quad \text{and}$$

$$P = k^2[(v-k)(b-r)-(v-rk)(v-m)]/v(v-m).$$

Further, if for some block, $d = b-1-[k(r-1)-l(b-1)]^2 Q^{-1}$, then $c = [P-lk(r-1)]/[k(r-1)-l(b-1)]$ is a positive integer and that block has c treatments common with each of the remaining $(b-d-1)$ blocks.

Proof. We denote the blocks as $B_1, B_2, \ldots, B_b$. Let x_i denote the number of treatments common between B_1 and B_i $(i = 2, 3, \ldots, b)$. Let $x_i = l$, for $i = 2, 3, \ldots, d+1$.

From the results (2.2.4) and (2.2.6), we have

$$(4.2.1) \qquad \sum_{i=d+2}^{b} x_i = k(r-1) - dl,$$

$$(4.2.2) \qquad \sum_{i=d+2}^{b} (x_i - \bar{x})^2 = P - dl^2 - \frac{[k(r-1)-dl]^2}{(b-d-1)},$$

where $P = k^2[(v-k)(b-r)-(v-rk)(v-m)]/v(v-m)$ and $\bar{x} = [k(r-1)-dl]/(b-d-1)$.

As $\sum_{i=d+2}^{b} (x_i - \bar{x})^2 \geq 0$, from (4.2.2) we get

$$dQ \leq (b-1)Q - [k(r-1)-l(b-1)]^2, \tag{4.2.3}$$

where $Q = P - 2lk(r-1) + l^2(b-1)$.

Since, Q can be written as

$$Q = \frac{k^2[(v-k)(b-r)-(v-rk)(v-m)]}{v(v-m)} - \frac{k^2(r-1)^2}{(b-1)} + \frac{[k(r-1)-l(b-1)]^2}{(b-1)}$$

$$= \frac{k^2(v-k)(b-r)(b-v+m-1)}{v(v-m)(b-1)} + \frac{[k(r-1)-l(b-1)]^2}{(b-1)},$$

it follows from the result (2.2.10) that when $b = v-m+1$, $Q = 0$ and when $b > v-m+1$, $Q > 0$. As for this design, $b > v-m+1$, we have $Q > 0$. Hence, from (4.2.3), we get

$$d \leq b - 1 - [k(r-1) - l(b-1)]^2 Q^{-1}. \tag{4.2.4}$$

If the sign of equality holds in (4.2.4), then $\sum_{i=d+2}^{b} (x_i - \bar{x})^2 = 0$ and hence all x_i's, $i = d+2, \ldots, b$ are equal to $\bar{x}$ and hence

$$\bar{x} = \frac{[k(r-1)-dl]}{(b-d-1)} = \frac{[P-lk(r-1)]}{[k(r-1)-l(b-1)]} = c$$

is a positive integer and the given block B_1 has c treatments common with each of the remaining $(b-d-1)$ blocks.

Theorem 2.2.2 follows as a corollary from Theorem 4.2.1 when $l = 0$.

It can be shown (Appendix 4.1) that a SRGD design with parameters $b = v-m+r$ and $v = 2k$, where k is an odd integer does not exist. Hence, we consider here a SRGD design in which $b = v-m+r$ and $v = 2k$, where k is an even integer. Putting $l = 0$, $b = v-m+r$ and $v = 2k$ in (4.2.4), we get $d \leq 1$. If $d = 1$, then the given block has $k/2$ treatments common with each of the remaining $(b-2)$ non-disjoint blocks.

Next, putting $l = k$, $b = v-m+r$ and $v = 2k$ in (4.2.4), we get $d \leq (r-1)/(r+1) < 1$, which shows that $d = 0$. Thus, we derive the following corollary.

Corollary 4.2.1. If in a SRGD design, $b = v-m+r$ and $v = 2k$, where k is an even integer, then (i) either no two blocks are disjoint or a given block has only one block disjoint with it in which case, it has $k/2$ treatments common with each of the remaining $(b-2)$

non-disjoint blocks, and (ii) no two blocks are the same set.

The SRGD design with the following set of parameters

$$v = 4m'n', \qquad b = 4m'(2n' - 1),$$

$$r = 2m'(2n' - 1), \qquad k = 2m'n',$$

$$\lambda_1 = 2m'(n' - 1), \qquad \lambda_2 = m'(2n' - 1),$$

$$n_1 = 2n' - 1, \qquad n_2 = 2n'(2m' - 1),$$

wherein m' and n' are integers ≥ 1, satisfy the conditions of the Corollary 4.2.1. Designs of this family have been constructed for $n' = 1$ and $m' = 2, 3, 4, 5$; $m' = 1$ and $n' = 2$ and 3, which are found in Table IIA of Bose, Clatworthy and Shrikhande [5]. The Corollary 4.2.1 asserts that in a design of this family, (i) either no two blocks are disjoint or a given block has only one block disjoint with it in which case it has $m'n'$ treatments common with each of the remaining (b-2) non-disjoint blocks and (ii) no two blocks are the same set.

4.3 Triangular designs

For the description of a triangular design, we refer to Section 2.3 of Chapter 2. We consider here the triangular design in which $b > v - n + 1$ and $rk - v\lambda_1 = n(r - \lambda_1)/2$. Then, proceeding exactly on the same lines as in Theorem 4.2.1, we get the following theorem.

Theorem 4.3.1. If in a triangular design with $b > v - n + 1$ and $rk - v\lambda_1 = n(r - \lambda_1)/2$, a given block has d blocks all having a given number $l(\leq k)$ of treatments common with it, then

$$d \leq b - 1 - [k(r-1) - l(b-1)]^2 Q^{-1},$$

where $Q = P - 2lk(r-1) + l^2(b-1)$ and $P = k^2[(v-k)(b-r)-(v-rk)(v-n)]/v(v-n)$. Further, if for some block, $d = b - 1 - [k(r-1) - l(b-1)]^2 Q^{-1}$, then $c = [P - lk(r-1)]/[k(r-1)-l(b-1)]$ is a positive integer and that block has c treatments common with each of the remaining $(b-d-1)$ blocks.

Theorem 2.3.2 follows as a corollary from the above theorem by taking $l = 0$.

It can be shown (Appendix 4.2) that a triangular design with parameters satisfying the relations $rk - v\lambda_1 = n(r - \lambda_1)/2$, $b = v-n+r$ and $v = 2k$ does not exist. Hence, corollary similar to Corollary 4.2.1 cannot be given here.

4.4 L_2 designs

For the description of a L_2 design, we refer to Section 2.4 of Chapter 2. We consider here a L_2 design with $b > v - 2s + 2$ and $rk - v\lambda_1 = s(r - \lambda_1)$. Then, proceeding exactly on the same lines as in Theorem 4.2.1, we get the following theorem.

Theorem 4.4.1. If in a L_2 design with $b > v - 2s + 2$ and $rk - v\lambda_1 = s(r - \lambda_1)$, a given block has d blocks all having a given number $l(\leq k)$ of treatments common with it, then

$$d \leq b - 1 - [k(r-1) - l(b-1)]^2 Q^{-1},$$

where $Q = P - 2lk(r-1) + l^2(b-1)$ and $P = k^2[(v-k)(b-r)-(v-rk)(s-1)^2]/v(s-1)^2$. Further, if for some block $d = b - 1 - [k(r-1) - l(b-1)]^2 Q^{-1}$, then $c = [P - lk(r-1)]/[k(r-1) - l(b-1)]$ is a positive integer and that block has c treatments common with each of the remaining $(b-d-1)$ blocks.

Theorem 2.4.2 follows as a corollary from the above theorem by taking $l = 0$.

It can be shown (Appendix 4.3) that a L_2 design with parameters satisfying the relation $v = 2k$, where k is an odd integer, does not exist. Hence, we consider a L_2 design with parameters satisfying the relations

$rk - v\lambda_1 = s(r - \lambda_1)$, $b = v - 2s + r + 1$ and $v = 2k$, where k is an even integer. Proceeding on exactly the same lines as in Corollary 4.2.1, we get the following corollary.

Corollary 4.4.1. If in a L_2 design, $rk - v\lambda_1 = s(r - \lambda_1)$, $b = v - 2s + r + 1$, $v = 2k$, where k is an even integer, then (i) either no two blocks are disjoint or a given block has only one block disjoint with it in which case it has $k/2$ treatments common with each of the remaining (b-2) non-disjoint blocks and (ii) no two blocks are the same set.

The L_2 design with the following set of parameters

$$v = 4t^2, \qquad b = 2(2t-1)^2, \qquad r = (2t-1)^2,$$

$$k = 2t^2, \qquad \lambda_1 = (t-1)(2t-1), \qquad \lambda_2 = 2t^2-2t+1,$$

$$n_1 = 2(2t-1), \qquad n_2 = (2t-1)^2,$$

where t is any positive integer, satisfy the conditions of Corollary 4.4.1. Hence, Corollary 4.4.1 asserts that in a design of this family (i) either no two blocks are disjoint or a given block has only one block disjoint with it in which case it has $k/2$ treatments common with each of the remaining (b-2) non-disjoint blocks and (ii) no two blocks are the same set.

4.5 Rectangular designs

For the description of a rectangular design, we refer to Section 2.5 of Chapter 2. Here, we consider the rectangular designs in which $\theta_1 = 0 = \theta_2$, where θ_1 and θ_2 are the characteristic roots of NN', N being the incidence matrix of this design. Proceeding exactly on the same lines as in Theorem 4.2.1, we get the following theorem.

Theorem 4.5.1. If in a rectangular design with $\theta_1 = 0 = \theta_2$ and $b > p + 1$, a given block has d blocks all having a given number $l(\leq k)$ of treatments common with it, then

$$d \leq b - 1 - [k(r-1) - l(b-1)]^2 Q^{-1},$$

where $Q = P - 2lk(r-1) + l^2(b-1)$ and
$P = k^2[(v-k)(b-r) - p(v-rk)]/vp$, p being equal to $(v_1 - 1)(v_2 - 1)$. Further, if for some block,
$d = b - 1 - [k(r-1) - l(b-1)]^2 Q^{-1}$, then
$c = [P - lk(r-1)]/[k(r-1) - l(b-1)]$ is a positive integer and that block has c treatments common with each of the remaining $(b-d-1)$ blocks.

Theorem 2.5.2 follows as a corollary from the above theorem by taking $l = 0$.

It can be shown (Appendix 4.4) that a rectangular design with parameters satisfying the relations $\theta_1 = 0 = \theta_2$, $b = p + r$ and $v = 2k$, where k is an odd integer does not exist. We, therefore, consider here a rectangular design with parameters satisfying the relations $\theta_1 = 0 = \theta_2$, $b = p + r$ and $v = 2k$, where k is an even integer.

Proceeding on exactly the same lines as in Corollary 4.2.1, we get the following corollary.

Corollary 4.5.1. If in a rectangular design, $\theta_1 = 0 = \theta_2$, $b = p + r$ and $v = 2k$, where k is an even integer then (i) either no two blocks are disjoint or a given block has only one block disjoint with it in which case it has $k/2$ treatments common with each of the remaining (b-2) non-disjoint blocks and (ii) no two blocks are the same set.

The rectangular design with the following set of parameters

$$v = v_1 v_2 = 4v_1' v_2', \qquad b = 2(2v_1'-1)(2v_2'-1),$$

$$r = p = (2v_1'-1)(2v_2'-1), \qquad k = 2v_1' v_2',$$

$$\lambda_1 = (2v_1'-1)(v_2'-1), \qquad \lambda_2 = (2v_2'-1)(v_1'-1),$$

$$\lambda_3 = 2v_1' v_2' - v_1' - v_2' + 1, \qquad n_1 = 2v_2'-1,$$

$$n_2 = 2v_1'-1, \qquad n_3 = (2v_1'-1)(2v_2'-1),$$

wherein $v_1 = 2v_1'$, $v_2 = 2v_2'$, v_1' and $v_2' \geq 1$, satisfy the conditions of corollary 4.5.1. Hence the corollary 4.5.1 asserts that in a design of this family (i) either no two blocks are disjoint or a given block has only one block disjoint with it in which case it has $k/2$ treatments common with each of the remaining (b-2) non-disjoint blocks and (ii) no two blocks are the same set.

Note. The non-existence of the designs mentioned in Sections 4.2, 4.3, 4.4 and 4.5 was suggested to the author by Professor W. H. Clatworthy, State University of New York at Buffalo and the author wishes to express his sincere thanks to him.