CHAPTER I

INTRODUCTION

1.1 BRIEF REVIEW

Fisher [20] formulated the principles of experimental design and analysis. Yates [39] , [40] developed further and he introduced factorial experiments in which all factors occur at 2 levels or 3 levels. Barnard [5] introduced the "Generalised Interaction" for a 2^m factorial experiment.

Bose and Kishen [6] and Bose [7] considered the problem of construction and analysis of symmetrical factorial designs of the type s^m where s is a prime power, with the help of Galois Fields and the Associated Finite Geometries.

Nair and Rao [27], [28], [29], Rao [35] , Nair and Rao [30] developed a set of sufficient conditions which led to the construction of balanced confounded designs for the $s_1 \times s_2 \times \ldots \times s_m$ type of factorial experiments.

Finney [19] gave fractional replicates of 2^m and 3^n factorial experiment and analysed them using higher factor interactions as error. Kempthorne [21], [22], Kishen [23] , Banerjee [4], Rao [34], further extended the ideas of fractional replication and confounding. Rao [33],[34] defined certain combinatorial arrangements called hypercubes of strength d and Rao [35] also introduced the idea of orthogonal arrays of strength d and using them, constructed confounded designs involving maximum number of factors and preserving at the same time main effects and interactions upto order (d-1). Plackett and Burman [32] introduced symmetrical factorial experiment so that the main effects are estimable with maximum precision.

Bose [8] gave a compact mathematical treatment of the problem of construction and analysis of symmetrical fractional factorial designs. Kempthorne suggested having a fractional replicate with respect to one group of factors and then combining this with assemblies of the other group of factors, a fractional replicate may be attained. Morrison [26] gave a series of fractional replicate designs for asymetrical factorial experiments with or without analysis.

Bose and Bush [9], Bush [12], Chakravarti [13], using orthogonal arrays, gave fractionally replicated designs of the type $s_1^{m_1} x\ s_2^{m_2} x \ldots x\ s_q^{m_q}$ in which by a proper choice of the orthogonal arrays of suitable strength, the main effects and interactions upto a certain order have been preserved. Rao [36] constructed fractional replicated for the special class of experiments $s_1^{n_1} \times s_2$ where $s_2 = s_1^t$ and s_1 is prime power. Besides, he also gave a number of fractionally replicated designs with and without blocks of the type $2^m \times 3^n$, 3×2^k, 2×3^k, etc. estimating the main effects and the mixed two factor interactions orthogonally.

Bose and Connor [10] gave methods of constructing and analysing fractionally replicated designs of the type $2^m \times 3^n$ by associating fractions from the 2^m factorial with those of the 3^n factorial using the method of symbolic direct product of matrices, so as to preserve the main effects and two factor interactions.

Connor and Young [15] gave a series of fractional designs ($2^m \times 3^n$) in which estimates for main effects and two factor interactions are either orthogonal or correlated. A lot of work has been done by Connor [14], Addelman [1],[2],[3], Srivastva [38], Daniel [17], Box and Hunter [11], Connor and Zelen [16], Das [18], Margolin [25], Raktoe [37], etc. on construction and analysis of various types of fractionally factorial designs using orthogonal arrays and other techniques.

1.2 SUMMARY OF THE THESIS

In this thesis, a technique has been developed to construct a fractional factorial design with or without blocks using orthogonal arrays where the main effect and the two factor interactions (assuming higher order interactions to be absent) can be estimated economically by reducing the total number of runs. It is expected that the use of this technique would result in less complicated computation.

Further the author has constructed Group Balanced Fractional Factorial Design (GBFF) of type 2^m. Here each group of main effects and/or some two factor interactions have the same variance. This property of having the same variance per group reduces considerably the computational work and also it is believed that this property is observed for the first time. Such a design with uniform variance group-wise is defined as GBFF.

The second chapter is on construction of economic and partially duplicated fractional factorial designs of type 2^m, including group balanced fractional factorial design of type 2^m.

The third chapter is related to construction of economic fractional factorial designs of type 3^n.

The fourth chapter is on construction of economic fractional factorial designs of type $2^m \times 3^n$.

1.3 TREATMENTS, TREATMENT COMBINATIONS AND THEIR RESPONSES

Let there be m factors $A_1, A_2, \ldots, A_m$ each of two levels and n factors $B_1, B_2, \ldots, B_n$ each at three levels. Then their $2^m \times 3^n$ treatment combinations or treatments are denoted by

$$a_1^{X_1} a_2^{X_2} \ldots a_m^{X_m} b_1^{Y_1} b_2^{Y_2} \ldots b_n^{Y_n} \qquad \ldots(1.3.1)$$

in which the factors $A_1, A_2, \ldots, A_m$ occur at the levels $X_1, X_2, \ldots, X_m$; $X_i = 0, 1$ $(i=1,2, \ldots, m)$ and the factors $B_1, B_2, \ldots, B_n$ occur at the levels $Y_1, Y_2, \ldots, Y_n$; $Y_j = 0,1,2$ $(j=1,2, \ldots, n)$.

Besides when all B factors are absent, the treatment will be denoted by

$$a_1^{X_1} a_2^{X_2} \ldots a_m^{X_m}, \qquad \ldots(1.3.2)$$

and when all A factors are absent, it will be denoted by

$$b_1^{Y_1} b_2^{Y_2} \ldots b_n^{Y_n}. \qquad \ldots(1.3.3)$$

For simplicity the treatment (1.3.1) will be denoted by

$$(X_1, X_2, \ldots, X_m; Y_1, Y_2, \ldots, Y_n), \qquad \ldots(1.3.4)$$

the treatment (1.3.2) by

$$(X_1, X_2, \ldots, X_m) \qquad \ldots(1.3.5)$$

and the treatment (1.3.3) by

$$(Y_1,\ Y_2,\ \ldots,\ Y_n)\ . \qquad \ldots(1.3.6)$$

They will also be referred to as assemblies. The assemblies (1.3.5) and (1.3.6) are called pure assemblies and the assembly (1.3.4), a mixed assembly.

We shall denote an assembly or the mean response to an assembly by the same symbol. Thus, if $Z(X_1, X_2, \ldots, X_m; Y_1, Y_2, \ldots, Y_n)$ is an observed response corresponding to an assembly (1.3.4), then,

$$E\left[Z(X_1,\ X_2,\ \ldots,\ X_m;\ Y_1,\ Y_2,\ \ldots,\ Y_n)\right]$$
$$= (X_1,\ X_2,\ \ldots,\ X_m;\ Y_1,\ Y_2,\ \ldots,\ Y_n)$$

where E stands for "Expectation". Similar remarks apply to the pure assemblies.

1.4 ORTHOGONAL ARRAYS

The assemblies of a 2^m factorial design can be identified with points $(X_1, X_2, \ldots, X_m)$; $X_i = 0,1,\ (i=1,2,\ldots,m)$ of EG(m,2) and the assemblies of a 3^n factorial design with points $(Y_1, Y_2, \ldots, Y_n)$; $Y_j = 0,1,2\ (j=1,2,\ldots,\ n)$ of EG(n,3).

Consider an EG (m,s). A p-flat $\sum_p$ in this geometry is defined by a set of (m-p) linearly independent equations

$$a_{\alpha 1} X_1 + a_{\alpha 2} X_2 + \dots + a_{\alpha m} X_m = d_\alpha \quad (\alpha = 1, 2, \dots, m-p) \qquad \dots (1.4.1)$$

where $d_\alpha = 0, 1, 2, \dots, s-1$.

There are s^p points on Σ_p. We shall say that an assembly of an s^m factorial design lies on the p-flat Σ_p if the corresponding point of EG(m,s) lies on it. Consider a linear form $L = a_1 x_1 + a_2 x_2 + \dots + a_m x_m$ of levels $x_1, x_2, \dots, x_m$ of the m factors, where $a_i = 0, 1,$; $i = 1, 2, \dots, m$. The number of non-zero a's is called the weight of the linear form L.

Two linear forms are said to be dependent if one is non-zero multiple of the other. If not, they will be independent.

Let L_α denote the linear form given by

$$L_\alpha = a_{\alpha 1} X_1 + a_{\alpha 2} X_2 + \dots + a_{\alpha m} X_m \qquad \dots (1.4.2)$$
$$\alpha = 1, 2, \dots, p.$$

The necessary and sufficient condition that the s^p points of Σ_p written as column vectors constitute an orthogonal array of strength (d+1) with s^p assemblies, m constraints, s levels and index λ denoted by $(s^p, m, s, d+1)$ is that every linear form $\sum_{\alpha=1}^{m-p} \lambda_\alpha L_\alpha$ has atleast (d+2) non-zero coefficients (i.e. $\sum \lambda_\alpha L_\alpha$ is of weight $\geqslant$ d+2) for $(\lambda_1, \lambda_2, \ldots, \lambda_{m-p}) \neq (0,0, \ldots, 0)$, λ being equal to s^{p-d-1}.

For convenience, we shall write the points as row vectors and the resulting matrix of dimension $(s^p \times m)$ will also be referred to as an orthogonal array of strength $(d+1)$. In this array, the rows will represent the assemblies and the columns represent the constraints.

If $\underline{X}' = (X_1, X_2, \ldots, X_m)$ and the s^p $(=u)$ points are $\underline{X}_1', \underline{X}_2', \ldots, \underline{X}_u'$ then the orthogonal array is of the form

$$\begin{bmatrix} \underline{X}_1' \\ \underline{X}_2' \\ \cdots \\ \cdots \\ \underline{X}_u' \end{bmatrix} = \begin{bmatrix} (X_{11}, X_{12}, \ldots, X_{1m}) \\ (X_{21}, X_{22}, \ldots, X_{2m}) \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ (X_{u1}, X_{u2}, \ldots, X_{um}) \end{bmatrix}$$

In this form, the array can be looked upon as a fractional design of u assemblies of an s^m factorial.

The above result was first proved by Rao [36] . The (m-p) linear forms L_α $(\alpha=1,2, \ldots, m-p)$, will be called the generating forms or the generators of the array. Giving all possible values 0,1,2, ..., s-1, to d_α (α=1,2, ..., m-p), we get s^{m-p} disjoint orthogonal arrays, all of the same strength, such that between them they exhaust all the s^m assemblies.

These arrays will be called pure arrays.

Arrays of strength (d+1) defined as above will be denoted by $S_1, S_2, \ldots, S_{2^{m-p}}$ and will be said to belong to the class Ω_{d+1} in EG(m,2). In EG(n,3) arrays of the same strength will be denoted by $T_1, T_2, \ldots, T_{3^{n-p}}$ and will be said to belong to the class $\emptyset_{d+1}$.

Each set of arrays is such that no two arrays in the same set have any assembly in common and between them they exhaust all the assemblies of the corresponding design.

1.5 KRONECKER PRODUCT OF TWO MATRICES

If A is (pxq) matrix and B is (rxs) matrix, then the Kronecker product (Direct product) is defined as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ b_{r1} & b_{r2} & \cdots & b_{rs} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1q}B \\ a_{21}B & a_{22}B & \cdots & a_{2q}B \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{p1}B & a_{p2}B & \cdots & a_{pq}B \end{bmatrix}$$

which is a (pr x qs) matrix. We shall denote either of the above product $A \otimes B$.

Also, the kronecker Product is defined as

$$\begin{bmatrix} A\,b_{11} & A\,b_{12} & \cdots & A\,b_{1s} \\ A\,b_{21} & A\,b_{22} & \cdots & A\,b_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ A\,b_{r1} & A\,b_{r2} & \cdots & A\,b_{rs} \end{bmatrix}$$

which is again a (pr x qs) matrix.

1.6 MIXED ARRAY AND FRACTIONALLY REPLICATED ASYMMETRICAL FACTORIAL DESIGN

Let S be a column vector of u assemblies, say

$$S = \begin{bmatrix} (X_{11}\ X_{12}\ \cdots\ X_{1m}) \\ (X_{21}\ X_{22}\ \cdots\ X_{2m}) \\ \cdots\cdots\cdots \\ (X_{u1}\ X_{u2}\ \cdots\ X_{um}) \end{bmatrix}$$

1 which also is an array of some strength in $EG(m,2)$ and T be a column vector of v assemblies, say

$$T = \begin{bmatrix} (Y_{11}\ Y_{12}\ \cdots\ Y_{1n}) \\ (Y_{21}\ Y_{22}\ \cdots\ Y_{2n}) \\ \cdots\cdots\cdots\cdots \\ (Y_{v1}\ Y_{v2}\ \cdots\ Y_{vn}) \end{bmatrix}$$

which is again an array of some strength in $EG(n,3)$. Then $S \otimes T$ will mean the symbolic direct product of two column vectors of assemblies S and T and is taken as

$$S \otimes T = \left[\begin{array}{ll} (X_{11}\ X_{12}\ \cdots\ X_{1m} & Y_{11}\ Y_{12}\ \cdots\ Y_{1n}) \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ (X_{11}\ X_{12}\ \cdots\ X_{1m} & Y_{v1}\ Y_{v2}\ \cdots\ Y_{vn}) \\ \hline \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ (X_{u1}\ X_{u2}\ \cdots\ X_{um} & Y_{11}\ Y_{12}\ \cdots\ Y_{1n}) \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ (X_{u1}\ X_{u2}\ \cdots\ X_{um} & Y_{v1}\ Y_{v2}\ \cdots\ Y_{vn}) \end{array}\right]$$

which is $(uv \times 1)$ column vector of mixed assemblies or a mixed array with uv assemblies and $(m+n)$ constraints. It can also be looked upon as a fractionally replicated design consisting of uv assemblies of a $2^m \times 3^n$ design.

1.7 EFFECTS OF THE 2^m FACTORIAL EXPERIMENT

Any interaction of a 2^m experiment will be denoted by

$$A_1^{\lambda_1} \; A_2^{\lambda_2} \; \ldots \; A_m^{\lambda_m} \qquad \ldots(1.7.1)$$
$$(\lambda_i = 0,1, \quad i = 1,2, \ldots, m)$$

where the interaction (1.7.1) also includes the main effects and the average response of all assemblies. If $\lambda_1 = 1$ and the rest of the λ_i's $(i=2,3, \ldots, m)$ are zero, then (1.7.1) represents the main effect A. If $\lambda_1 = \lambda_2 = 1$ and the rest of the λ_i's $(i=3,4, \ldots, m)$ are zero, then (1.7.1) represents the two factor interaction A_1A_2 and similarly for other factors.

Let

$$H = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & C(o) \\ 1 & C(1) \end{bmatrix} \qquad \ldots(1.7.2)$$

where the constants $C(X)$; $X = 0,1$, are defined by (1.7.2). Then we can easily varify that $H^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Let $$U^{(m)} = U \otimes U \otimes \ldots \otimes U \qquad \ldots(1.7.3)$$

denote the Kronecker product of U taken m times. Then we shall define all the interactions in (1.7.1) by the matrix identity

3 $$\begin{pmatrix}1\\a_1\end{pmatrix} \otimes \begin{pmatrix}1\\a_2\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}1\\a_m\end{pmatrix} = H^{(m)} \begin{pmatrix}I\\A_1\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}I\\A_m\end{pmatrix} \qquad \ldots(1.7.4)$$

Or

$$\begin{pmatrix}I\\A_1\end{pmatrix} \otimes \begin{pmatrix}I\\A_2\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}I\\A_m\end{pmatrix} = H^{-(m)} \begin{pmatrix}1\\a_1\end{pmatrix} \otimes \begin{pmatrix}1\\a_2\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}1\\a_m\end{pmatrix} \qquad \ldots(1.7.5)$$

with the convention

$$I.I = I, \quad I.A^{\lambda} = A^{\lambda} = A^{\lambda}.I, \quad A^{\circ} = I$$

where I is the average of the mean responses of all assemblies (Bose and Connor [10]). The effect defined in this way are known as the "Product Effects.".

From (1.7.4) and denoting the treatment combination, $a_1^{X_1} a_2^{X_2} \ldots a_m^{X_m}$ by $(X_1, X_2, \ldots, X_m)$ for simplicity, we deduce

$$(X_1, X_2, \ldots, X_m) = \prod_{i=1}^{m} \left[I + C(X_i) A_i \right] . \qquad \ldots(1.7.6)$$

Assuming three or more factor interactions as negligible (1.7.6) reduces to

$$(X_1 . X_2 \ldots X_m) = I + \sum_{i=1}^{m} C(X_i)\, A_i + \sum_{i < i'} C(X_i)\, C(X_{i'})\, A_i A_{i'} \qquad \ldots(1.7.7)$$

where $C(X_i)$, $X_i = 0$ or 1 $\;i = 1, 2, \ldots, m$ are as defined by

(1.7.2) and $\sum_{i < i'}$ means the summation over all pairs of indices i, i' for $i < i'$ (i, i' = 1,2,..., m).

1.8 EFFECTS OF THE 3^n FACTORIAL EXPERIMENT

Any interaction of a 3^n experiment will be denoted by

$$B_1^{\mu_1} \; B_2^{\mu_2} \; \ldots \; B_n^{\mu_n} \qquad \ldots(1.8.1)$$

μ_j = 0,1,2 and j = 1,2, ..., n.

when only one μ is different from zero, (1.8.1) represents a main effect. It is a linear effect if μ=1 and a quadratic effect if μ=2. When two μ's are different from zero, (1.8.1) represents a two-factor interaction. It is a linear x linear, linear x quadratic or quadratic x linear, quadratic x quadratic effect according as the couplet of non-zero μ's is (1,1), (1,2) or (2,1) and (2,2).

Let

$$K = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & d_1(o) & d_2(o) \\ 1 & d_1(1) & d_2(1) \\ 1 & d_1(2) & d_2(2) \end{bmatrix} \qquad \ldots(1.8.2)$$

Then we can easily varify that $K^{-1} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/2 & 0 & 1/2 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}$

where the constants $d_\mu(z)$; $z = 0,1,2$; $\mu=1,2$ are defined by (1.8.2). Then we shall define all interactions in (1.8.1) by the matrix identity

$$\begin{bmatrix}1\\ b_1\\ b_1^2\end{bmatrix} \otimes \begin{bmatrix}1\\ b_2\\ b_2^2\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}1\\ b_n\\ b_n^2\end{bmatrix} = K^{(n)} \begin{bmatrix}I\\ B_1\\ B_1^2\end{bmatrix} \otimes \begin{bmatrix}I\\ B_2\\ B_2^2\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}I\\ B_n\\ B_n^2\end{bmatrix} \qquad \ldots(1.8.3)$$

Or

$$\begin{bmatrix}I\\ B_1\\ B_1^2\end{bmatrix} \otimes \begin{bmatrix}I\\ B_2\\ B_2^2\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}I\\ B_n\\ B_n^2\end{bmatrix} = K^{-(n)} \begin{bmatrix}1\\ b_1\\ b_1^2\end{bmatrix} \otimes \begin{bmatrix}1\\ b_2\\ b_2^2\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}1\\ b_n\\ b_n^2\end{bmatrix}$$

with the convention

$$I.I=I, \quad I.B^\mu = B^\mu = B^\mu.I, \quad B^0 = I$$

where I is the average of the mean responses of all assemblies. (Bose and Connor [10]). The effects defined in this way are known as the "Product Effects".

Assuming three or more factor interactions as negligible and denoting $b_1^{Y_1} b_2^{Y_2} \ldots b_n^{Y_n}$ by $(Y_1 Y_2 \ldots Y_n)$ for simplicity, we obtain from (1.8.3)

$$(Y_1 . Y_2 \ldots Y_n) = I + \sum_{\mu=1}^{2} \sum_{j=1}^{n} d_\mu(Y_j)\, B_j^\mu$$

$$+ \sum_{\mu < \mu'} \sum_{j < j'} d_\mu(Y_j)\, d'_\mu (Y'_j)\, B_j^\mu\, B_{j'}^{\mu'}$$

where $d_{\mu}(Y_j)$; $\mu = 1,2$; $Y_j = 0,1,2$; $j=1,2, \ldots, n$ are as defined by (1.8.2) and $\sum_{j<j'}$ means the summation over all pairs of indices j, j' for $j < j'$ ($j,j' = 1,2,\ldots,n$) and $\sum_{\mu<\mu'}$ means the summation over all pairs of indices μ, μ' (μ, $\mu' = 1,2$).

1.9 EFFECTS OF THE $2^m \times 3^n$ FACTORIAL EXPERIMENT

Any interaction in this experiment will be denoted by

$$A_1^{\lambda_1} A_2^{\lambda_2} \ldots A_m^{\lambda_m} B_1^{\mu_1} B_2^{\mu_2} \ldots B_n^{\mu_n} \qquad \ldots(1.9.1)$$

which may be regarded as the symbolic product of $A_1^{\lambda_1} A_2^{\lambda_2} \ldots A_m^{\lambda_m}$ and $B_1^{\mu_1} B_2^{\mu_2} \ldots B_n^{\mu_n}$. If $\lambda_i = 1$ and $\mu_j=1$ and the rest of (λ, μ)'s are zero, then (1.9.1) represents the linear effect A_iB_j of the i^{th} factor of the 2^m factorial and the j^{th} factor of the 3^n factorial. If $\lambda_i=1$ and $\mu_j=2$ and rest of the (λ, μ)'s are zero, then (1.9.1) represents the quadratic effect $A_iB_j^2$ of the same factors ($i=1,2,\ldots,m$; $j=1,2,\ldots,n$).

All interactions in (1.9.1) will be defined by the matrix identity

7 $$\begin{bmatrix}1\\a_1\end{bmatrix} \otimes \begin{bmatrix}1\\a_2\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}1\\a_m\end{bmatrix} \otimes \begin{bmatrix}1\\b_1\\b_1^2\end{bmatrix} \otimes \begin{bmatrix}1\\b_2\\b_2^2\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}1\\b_n\\b_n^2\end{bmatrix}$$

$$= H^{(m)} \otimes K^{(n)} \begin{bmatrix}I\\A_1\end{bmatrix} \otimes \begin{bmatrix}I\\A_2\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}I\\A_m\end{bmatrix} \otimes \begin{bmatrix}I\\B_1\\B_1^2\end{bmatrix} \otimes \begin{bmatrix}I\\B_2\\B_2^2\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}I\\B_n\\B_n^2\end{bmatrix} \quad \ldots(1.9.2)$$

Or

$$\begin{bmatrix}I\\A_1\end{bmatrix} \otimes \begin{bmatrix}I\\A_2\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}I\\A_m\end{bmatrix} \otimes \begin{bmatrix}I\\B_1\\B_1^2\end{bmatrix} \otimes \begin{bmatrix}I\\B_2\\B_2^2\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}I\\B_n\\B_n^2\end{bmatrix}$$

$$= \bar{H}^{(m)} \otimes \bar{K}^{(n)} \begin{bmatrix}1\\a_1\end{bmatrix} \otimes \begin{bmatrix}1\\a_2\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}1\\a_m\end{bmatrix} \otimes \begin{bmatrix}1\\b_1\\b_1^2\end{bmatrix} \otimes \begin{bmatrix}1\\b_2\\b_2^2\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}1\\b_n\\b_n^2\end{bmatrix} \quad \ldots(1.9.3)$$

Where I denotes the average of the mean responses of all mixed assemblies (Bose and Connor [10]).

Assuming three or more factor interactions as negligible and denoting $a_1^{X_1} a_2^{X_2} \ldots a_m^{X_m} b_1^{Y_1} b_2^{Y_2} \ldots b_n^{Y_n}$ by $(X_1X_2 \ldots X_m Y_1Y_2 \ldots Y_n)$, We obtain from (1.9.2)

$$(X_1\ X_2\ \dots\ X_m,\ Y_1\ Y_2\ \dots\ Y_n) = I + \sum_{i=1}^{m} c(X_i)\ A_i$$

$$+ \sum_{i<i'} c(X_i)\ c(X'_i)\ A_i A'_i$$

$$+ \sum_{\mu=1}^{2} \sum_{j=1}^{n} d_\mu(Y_j)\ B_j^\mu$$

$$+ \sum_{\mu<\mu'} \sum_{j<j'} d_\mu(Y_j) d_{\mu'}(Y'_j) B_j^\mu\ B_{j'}^{\mu'}$$

$$+ \sum_{\mu=1}^{2} \sum_{i,j} c(X_i) d_\mu(Y_j) A_i B_j^\mu$$

where $c(X_i)$, $d_\mu(Y_j)$, $\sum_{i<i'}$, $\sum_{j<j'}$ are already defined earlier and $\sum_{i,j}$ means the summation over all pairs of indices i,j (i=1,2, ..., m, j=1,2, ..., n).

1.10 FRACTIONAL REPLICATE OF THE 2^m FACTORIAL DESIGN

In the usual theory of fractional replication, a $\frac{1}{2^p}$ fraction of the 2^m factorial consists of the assemblies $(X_1\ X_2\ \dots\ X_m)$ satisfying the p linearly independent equations

$$L_\alpha \equiv a_{\alpha 1} X_1 + a_{\alpha 2} X_2 + \dots + a_{\alpha m} X_m = d_\alpha \qquad \dots(1.10.1)$$

$$(\alpha = 1, 2, \dots, p)$$

in GF(2), where $(a_{\alpha 1}, a_{\alpha 2}, \dots, a_{\alpha m}) \neq (0,0, \dots, 0)$

We shall say that the interaction

$$A_1^{\lambda_1}\ A_2^{\lambda_2}\ \dots\ A_m^{\lambda_m} \qquad \dots(1.10.2)$$

9 corresponds to the linear form $\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_m X_m$ in GF(2) and carries 1 d.f.

The interaction (1.10.2) is estimated by the contrast

$$\frac{1}{2^m}\left[(\lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_m X_m = 1) - (\lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_m X_m = 0)\right], \qquad \ldots(1.10.3)$$

where the parenthesis ( ) means the sum of the observed responses of the assemblies satisfying the equation within it. If only one λ differs from zero, (1.10.2) represents a main effect. If two λ's differ from zero, it represents a two-factor interaction. The main effects and the interactions defined in this way agree with the definition given in Section 1.7 except that the interactions may differ in sign.

Consider any linear form L which is not of the form

$$\lambda_1 L_1 + \lambda_2 L_2 + \ldots + \lambda_p L_p \qquad \ldots(1.10.4)$$

$$\lambda_\alpha = 0,1; \quad \alpha = 1,2, \ldots, p; \quad (\lambda_1, \lambda_2, \ldots, \lambda_p) \neq (0,0,\ldots,0)$$

The interactions corresponding to the linear form

$$L + (\lambda_1 L_1 + \lambda_2 L_2 + \ldots + \lambda_p L_p) \qquad \ldots(1.10.5)$$

are said to be aliases of the interaction corresponding L.

It is known that each interaction not corresponding to any linear form $\lambda_1 L_1 + \lambda_2 L_2 + \ldots + \lambda_p L_p$ is a member of one and only one alias set.

It is clear that the assemblies of the fraction defined by (1.10.1) which satisfy $L=0$ also satisfy $L+(\lambda_1 L_1+\lambda_2 L_2+\dots+\lambda_p L_p)=\lambda_1 d_1+\lambda_2 d_2+\dots+\lambda_p d_p$ and those satisfying $L=1$, satisfy

$$L+(\lambda_1 L_1+\lambda_2 L_2+\dots+\lambda_p L_p)=1+\lambda_1 d_1+\lambda_2 d_2+\dots+\lambda_p d_p$$

where $\sum_{\alpha=1}^{p}\lambda_\alpha d_\alpha = 0$ or 1

Let Σ_{m-p} denote the (m-p)-flat determined by $L_\alpha = d_\alpha$; $\alpha=1,2,\dots,p$ in EG(m,2) and $\{\Sigma_{m-p}\}$, the set of points lying on Σ_{m-p}. Also let $\{L=d\}$ denote the set of points satisfying the equation $L=d$ in GF(2). Then $\{L=d\}\cap\{\Sigma_{m-p}\}$ will mean the set of points common between the two sets. Let $(\{L=d\}\cap\{\Sigma_{m-p}\})$ represents the sum of the observed responses of assemblies corresponding to points of $\{L=d\}\cap\{\Sigma_{m-p}\}$.

Then from (1.10.5) and succeding paragraphs, it follows

$$E\left[(\{L=1\}\cap\{\Sigma_{m-p}\}) - (\{L=0\}\cap\{\Sigma_{m-p}\})\right]$$

$$= 2^{m-p}\Big[\text{interactions corresponding to } L \pm \text{interactions corresponding to } L+(\lambda_1 L_1+\lambda_2 L_2+\dots+\lambda_p L_p)\Big] \quad \dots(1.10.6)$$

$$(\lambda_1,\lambda_2,\dots,\lambda_p)\neq(0,0,\dots,0)$$

1 The implication of (1.10.6) is that in the expected value of the column vector of observed response of the assemblies corresponding to points in $\sum_{m-p}$, the rank of the matrix formed by the column vectors of coefficients belonging to effects which correspond to $L + (\lambda_1 L_1 + \lambda_2 L_2 + \dots + \lambda_p L_p)$ is 1.

The identity relationship for the fraction defined by (1.10.1) is

$$I = G_1 = G_2 = \dots = G_p = G_1 G_2 = \dots = G_{p-1} G_p = \dots = G_1 G_2 \dots G_p \qquad \dots(1.10.7)$$

where G's are the interactions corresponding to the linear forms (1.10.4).

The relationship (1.10.7) is useful in determining the set of effects aliased with a given effect.

1.11 FRACTIONAL REPLICATE OF THE 3^n FACTORIAL DESIGN

A $\frac{1}{3^q}$ fractional replicate of a 3^n factorial consists of the assemblies $(Y_1, Y_2, \dots, Y_n)$ satisfying the q linearly independent equations

$$M_\theta \equiv b_{\theta 1} Y_1 + b_{\theta 2} Y_2 + \dots + b_{\theta n} Y_n = e_\theta \qquad \dots(1.11.1)$$

in GF(3).

Writing β_i for B_i; i = 1,2, ..., n we shall say that the interaction

$$\beta_1^{\mu_1} \beta_2^{\mu_2} \cdots \beta_n^{\mu_n} \qquad \ldots(1.11.2)$$

corresponds to the linear form $\mu_1 Y_1 + \mu_2 Y_2 + \cdots + \mu_n Y_n$ and carries 2 d.f. which may be assigned to the two orthogonal components, linear and quadratic denoted by $L(\beta_1^{\mu_1} \beta_2^{\mu_2} \cdots \beta_n^{\mu_n})$ and $Q(\beta_1^{\mu_1} \beta_2^{\mu_2} \cdots \beta_n^{\mu_n})$ respectively. The linear effect $L(\beta_1^{\mu_1} \beta_2^{\mu_2} \cdots \beta_n^{\mu_n})$ is estimated by the contrast

$$\frac{1}{2x3^{n-1}} \left[(\mu_1Y_1+\mu_2Y_2 + \cdots + \mu_nY_n=2) - (\mu_1Y_1+\mu_2Y_2 \cdots + \mu_nY_n=0)\right]$$

and the quadratic effect $Q(\beta_1^{\mu_1} \beta_2^{\mu_2} \cdots \beta_n^{\mu_n})$, by the contrast

$$\frac{1}{2x3^{n}} \Big[(\mu_1Y_1+\mu_2Y_2+ \cdots +\mu_nY_n=0) - 2(\mu_1Y_1+\mu_2Y_2+ \cdots +\mu_nY_n=1) + (\mu_1Y_1+ \mu_2Y_2+ \cdots + \mu_nY_n=2)\Big]$$

where ( ) has the same meaning as in (1.10.3).

This definition also includes the main effects. For any factor B_1, the linear effect $L(\beta_1)$ and the quadratic effect $Q(\beta_1)$ are the same as the effects β_1, β_1^2 defined in Section 1.8.

However, for any two factors B_1 and B_2 say, the effects $L(\beta_1\beta_2)$, $Q(\beta_1 \cdot \beta_2)$, $L(\beta_1 \cdot \beta_2^2)$, $Q(\beta_1\beta_2^2)$ are not the same as B_1B_2, $B_1B_2^2$, $B_1^2B_2$, $B_1^2B_2^2$, defined in Section 1.8, but are connected by the relation

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$$\begin{bmatrix} B_1B_2 \\ B_1B_2^2 \\ B_1^2B_2 \\ B_1^2B_2^2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -3 & -3 & -3 & 3 \\ 1 & -3 & 1 & 3 \\ 1 & -3 & -1 & -3 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} L(\beta_1\beta_2) \\ Q(\beta_1\cdot\beta_2) \\ L(\beta_1\beta_2^2) \\ Q(\beta_1\beta_2^2) \end{bmatrix} \qquad \ldots(1.11.3)$$

(Connor [14]).

The effects B_1B_2, $B_1B_2^2$, $B_1^2B_2$, $B_1^2B_2^2$ are called Product Effects and the effects $L(\beta_1\cdot\beta_2)$, $Q(\beta_1\cdot\beta_2)$, $L(\beta_1,\beta_2^2)$, $Q(\beta_1\cdot\beta_2^2)$ are called Geometric Effects. To distinguish between the two sets of effects, the factors, B_1, B_2, ..., B_n in the Product set are denoted by β_1, β_2, ..., β_n in the Geometric set.

Let M be any linear form which is not of the form

$$\mu_1M_1 + \mu_2M_2 + \cdots + \mu_qM_q \qquad \ldots(1.11.4)$$

$$\mu_\theta = 0,1,2,; \ \theta = 1,2, \ldots, q; \ (\mu_1,\mu_2, \ldots, \mu_q) \neq (0,0,..,0)$$

The interactions corresponding to the linear form

$$M+(\mu_1M_1 + \mu_2M_2 + \cdots + \mu_qM_q) \qquad \ldots(1.11.5)$$

are said to be aliases of the interaction corresponding to the linear form M.

Each interaction not corresponding to any linear form (1.11.4) is a member of one and only one alias set.

It is clear that the assemblies of the fraction defined by (1.11.1) satisfying M=0,1,2 also satisfy,

$$M+(\mu_1 M_1 + \mu_2 M_2 + \dots + \mu_q M_q) = \sum_{\theta=1}^{q} \mu_\theta e_\theta, \ \sum_{\theta=1}^{q} \mu_\theta e_\theta + 1,$$

$$\sum_{\theta=1}^{q} \mu_\theta e_\theta + 2$$

respectively where $\sum_{\theta=1}^{q} \mu_\theta e_\theta = 0,1,2.$

This means that each of the contrasts

$$\left[\left(\{M = 2\} \cap \{\Sigma_{n-q}\}\right) - \left(\{M=0\} \cap \{\Sigma_{n-q}\}\right)\right]$$

and $$\left[\left(\{M=0\} \cap \{\Sigma_{n-q}\}\right) - 2\left(\{M=1\} \cap \{\Sigma_{n-q}\}\right) + \left(\{M=2\} \cap \{\Sigma_{n-q}\}\right)\right]$$

where Σ_{n-q} is the (n-q)-flat determined by $M_\theta = e_\theta$ ($\theta = 1,2, \dots, q$) in GF(3), will estimate some linear function involving interactions corresponding to the linear form $M+(\mu_1 M_1 + \mu_2 M_2 + \dots + \mu_q M_q)$.

In other words, this implies that in the expected value of the column vector of responses of the assemblies corresponding to the points in Σ_{n-q}, the matrix formed by the column vectors of coefficients belonging to effects corresponding to $M+(\mu_1 M_1 + \mu_2 M_2 + \dots + \mu_q M_q)$ is of rank 2.

5 The identity relationship for the fraction defined by (1.11.1) is

$$
\begin{aligned}
I &= G_1' = \ldots = G_q' \\
&= G_1'G_2' = \ldots = G_{q-1}'G_q' \\
&= G_1'(G_2')^2 = \ldots = G_{q-1}'(G_q')^2 \\
&= \ldots \\
&= G_1'G_2' \ldots G_q' = \ldots = G_1'(G_2')^2 \ldots (G_q')^2, \qquad \ldots(1.11.6)
\end{aligned}
$$

where G's are the interactions corresponding to the linear forms (1.11.4). The relationship (1.11.6) is useful in determining the set of effects aliased with a given effect.

1.12 FRACTIONAL REPLICATE OF THE $2^m \times 3^n$ FACTORIAL DESIGN

Fractional replicate of this design is obtained by symbolic association of assemblies belonging to an array in EG(m,2) with those belonging to an array in EG(n,3) as indicated in section 1.6.

An interaction $A_1^{\lambda_1} A_2^{\lambda_2} \ldots A_m^{\lambda_m} B_1^{\mu_1} B_2^{\mu_2} \ldots B_n^{\mu_n}$ may be said to belong to two linear forms

$\lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_m X_m$ in GF(2) and

$\mu_1 Y_1 + \mu_2 Y_2 + \ldots + \mu_n Y_n$ in GF(3)

6 and will carry 2 d.f. which we may assign to linear and quadratic components.

$L(A_1^{\lambda_1}\ A_2^{\lambda_2} \ldots A_m^{\lambda_m}\ B_1^{\mu_1}\ B_2^{\mu_2} \ldots B_n^{\mu_n})$ and $Q(A_1^{\lambda_1}\ A_2^{\lambda_2} \ldots A_m^{\lambda_m}\ B_1^{\mu_1}\ B_2^{\mu_2} \ldots B_n^{\mu_n})$

Let the equations

$$\begin{aligned} \lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_m X_m &= d \\ \mu_1 Y_1 + \mu_2 Y_2 + \ldots + \mu_n Y_n &= e \end{aligned} \qquad \ldots(1.12.1)$$

be satisfied by points $(X_1 X_2 \ldots X_m\ Y_1 Y_2 \ldots Y_n)$ such that $(X_1 X_2 \ldots X_m)$ satisfies $\lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_m X_m = d$ in GF(2) and $(Y_1 Y_2 \ldots Y_n)$ satisfies $\mu_1 Y_1 + \mu_2 Y_2 + \ldots + \mu_n Y_n = e$ in GF(3), then Linear Component $L(A_1^{\lambda_1}\ A_2^{\lambda_2} \ldots A_m^{\lambda_m}\ B_1^{\mu_1}\ B_2^{\mu_2} \ldots B_n^{\mu_n})$ is estimated by the contrast

$$\left[\left\{\begin{pmatrix} \lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_m X_m = 1 \\ \mu_1 Y_1 + \mu_2 Y_2 + \ldots + \mu_n Y_n = 2 \end{pmatrix} - \begin{pmatrix} \lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_m X_m = 1 \\ \mu_1 Y_1 + \mu_2 Y_2 + \ldots + \mu_n Y_n = 0 \end{pmatrix}\right\} - \left\{\begin{pmatrix} \lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_m X_m = 0 \\ \mu_1 Y_1 + \mu_2 Y_2 + \ldots + \mu_n Y_n = 2 \end{pmatrix} - \begin{pmatrix} \lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_m X_m = 0 \\ \mu_1 Y_1 + \mu_2 Y_2 + \ldots + \mu_n Y_n = 0 \end{pmatrix}\right\}\right]$$

$$\div\ 2 \times 3^{n-1} \times 2^{m} \qquad \ldots(1.12.2)$$

7 and the quadratic component $Q(A_1^{\lambda_1}\ A_2^{\lambda_2} \dots A_m^{\lambda_m}\ B_1^{\mu_1}\ B_2^{\mu_2} \dots B_n^{\mu_n})$ is estimated by the contrast

$$\left[\begin{pmatrix} \lambda_1 X_1 + \lambda_2 X_2 + \dots + \lambda_m X_m = 1 \\ \mu_1 Y_1 + \mu_2 Y_2 + \dots + \mu_n Y_n = 0 \end{pmatrix} - 2\begin{pmatrix} \lambda_1 X_1 + \lambda_2 X_2 + \dots + \lambda_m X_m = 1 \\ \mu_1 Y_1 + \mu_2 Y_2 + \dots + \mu_n Y_n = 1 \end{pmatrix}\right.$$

$$+\begin{pmatrix} \lambda_1 X_1 + \lambda_2 X_2 + \dots + \lambda_m X_m = 1 \\ \mu_1 Y_1 + \mu_2 Y_2 + \dots + \mu_n Y_n = 2 \end{pmatrix} - \begin{pmatrix} \lambda_1 X_1 + \lambda_2 X_2 + \dots + \lambda_m X_m = 0 \\ \mu_1 Y_1 + \mu_2 Y_2 + \dots + \mu_n Y_n = 0 \end{pmatrix}$$

$$\left. -2\begin{pmatrix} \lambda_1 X_1 + \lambda_2 X_2 + \dots + \lambda_m X_m = 0 \\ \mu_1 Y_1 + \mu_2 Y_2 + \dots + \mu_n Y_n = 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 X_1 + \lambda_2 X_2 + \dots + \lambda_m X_m = 0 \\ \mu_1 Y_1 + \mu_2 Y_2 + \dots + \mu_n Y_n = 2 \end{pmatrix}\right]$$

where ( ) has the same meaning as in (1.10.3).

The two components can also be defined by interchanging the roles of the linear forms

$$\lambda_1 X_1 + \lambda_2 X_2 + \dots + \lambda_m X_m \text{ and } \mu_1 Y_1 + \mu_2 Y_2 + \dots + \mu_n Y_n.$$

If only λ_1, μ_1 are non-zero, and the rest are zero, then the interaction $A_1^{\lambda_1}\ A_2^{\lambda_2} \dots A_m^{\lambda_m}\ B_1^{\mu_1}\ B_2^{\mu_2} \dots B_n^{\mu_n}$

(writing β's in place of B's)

represents a mixed two-factor interaction carrying 2 d.f. belonging to $L(A_1, \beta_1)$ and $Q(A_1, \beta_1)$ say.

The interactions defined in this way do not agree with the definition given in section 1.9, but the two-factor interactions do have the same meaning as in that section.

3 The identity relationship for the fractional replicate of $2^m \times 3^n$ design can be obtained from those of 2^m and 3^n designs.

If the fraction is given by the mixed assemblies obtained by combining the assemblies of the fraction defined by (1.10.1) and the assemblies of the fraction defined by (1.11.1) as in section 1.6, then the identity relationship is

$$
\begin{aligned}
I &= G_1 = \dots = G_p = G_1G_2 = \dots = G_{p-1}G_p = \dots = G_1G_2\dots G_p \\
&= G_1' = \dots = G_q' = G_1'G_2' = \dots = G_{q-1}'G_q' = G_1'(G_2')^2 = \dots \\
&\qquad\qquad = G_{q-1}'(G_q')^2 = \dots \\
&= G_1'G_2' \dots G_q' = \dots = G_1'(G_2')^2 \dots (G_q')^2 \\
&= G_1G_1' = \dots = G_1G_q' = \dots = G_pG_1' = \dots = G_pG_q' \\
&= \dots \\
&= G_1G_2 \dots G_pG_1'(G_2')^2 \dots (G_q')^2, \qquad \dots(1.12.4)
\end{aligned}
$$

where the interactions are obtained by taking all products of interactions from the identity relationship for the fractions defined by (1.10.1) and (1.11.1).

The relationship (1.12.4) is useful in determining the set of effects aliased with a given effect.