Chapter 4

SINGLE SAMPLE SHRINKAGE TESTIMATORS UNDER GENERAL ENTROPY LOSS FUNCTION

Chapter – 4

SINGLE SAMPLE SHRINKAGE TESTIMATORS UNDER GENERAL ENTROPY LOSS FUNCTION

4.1 Introduction

The present chapter deals with one sample shrinkage testimators under General Entropy Loss Function (GELF) for single parameter Exponential distribution and Normal distribution.

The aim of systems reliability is to forecast of various system performance measures such as mean life time, guarantee period and reliability etc. In general, the type of failure distribution depends on the failure mechanism of components. If the failure rate is constant, which is mostly true for electronic components during the major part of their useful life, the failure time follows an exponential distribution with the p.d.f.

$$f(x;\theta) = \begin{cases} \frac{1}{\theta}\exp(-x/\theta), & x \geq 0, \theta > 0 \\ 0 & , otherwise \end{cases} \qquad \text{\_\_\_\_\_\_(4.1.1)}$$

In the context of life testing and reliability estimation, numerous data have been examined and it has been found that exponential distribution fits well for most of the cases. Several authors have proposed estimators, testimators with different choices of shrinkage factors (S.F.) under different loss functions. The choice of an appropriate loss function is guided by financial consideration apart from other considerations such as over estimation being more serious than under-estimation or vice-versa.

Shrinkage testimators for the mean μ of a Normal distribution $N(\mu, \sigma^2)$ when variance σ^2 is known or unknown, have been proposed by Waiker, Schuurman and Raghunandan (1984). Recently Pandey et. al. (1987) considered some shrinkage testimators for the variance estimator under Mean Square Error criterion (MSE). Parisan and Farsipour (1999), Misra and Meuten (2003), Pandey et. al. (2004), Ahmadi et. al. (2005), Xiao et. al. (2005), Prakash and Singh (2006), Prakash and Pandey (2007) and others have considered the estimation procedures under the LINEX loss function in various contexts. Pandey et. al. (2007) have proposed shrinkage testimator(s) variance and have studied the properties of these under the Asymmetric loss function (ASL). The present work is an attempt to study the risk properties of shrinkage testimator(s) for the variance of Normal distribution under a more general loss function viz. (GELF). Pandey et. al. (2007) have studied the risk properties of the same for positive degree of asymmetry only, under ASL. Where as this study attempts to find the range for positive as well as negative degrees of asymmetry under GEL where the shrinkage testimator of variance performs better than the UMVUE.

4.1.1 General Entropy Loss Function (GELF)

A suitable alternative to modified LINEX loss is the General Entropy Loss (GEL) proposed by Calabria and Pulcini (1996) given by:

$$L_E(\hat{\theta},\theta) \propto \left\{(\hat{\theta}/\theta)^p - p \ln(\hat{\theta}/\theta) - 1\right\}, \quad p \neq 0 \qquad \text{\_\_\_\_\_\_} (4.1.1.1)$$

Whose minimum occurs at $\hat{\theta} = \theta$.

This loss is a generalization of the entropy loss used by several authors (for example, Dey and Liu, 1992) where the shape parameter 'p' is equal to unity (1). The more general version of (4.1.1.1) allows different shapes of the loss function to be considered (say when p > 0, a positive error $(\hat{\theta}>\theta)$ causes more serious consequence than a negative error and when p < 0, then negative error is more serious). If we are considering prior distributions, then the Bayes estimate of θ under GELF is in a closed form and is given by

$$\hat{\theta}^E = \left[E_\theta\left(\theta^{-p}\right)\right]^{-1/p} \qquad \text{\_\_\_\_\_\_(4.1.1.2)}$$

provided that $E_\theta\left(\theta^{-p}\right)$ exists and is finite.

4.1.2 Incorporating a Point Guess and $\hat{\theta}_{ST}$

In many real life situations the experimenter may have some prior information regarding the parameter being estimated due to some past experience or similar kind of studies and it is thought to apply this information to inference procedures of the original model. If the prior information is available only in the form of a point (a single) value (say) θ_0 for θ. For example a medical practitioner knows that in how many days the patient may get cured (say) 7 days or 10 days due to his past experience of treatment. Here we may take θ_0 = 7 days. For such situations it is suggested to start with the current (sample) information, construct an estimator $\hat{\theta}$ (MVUE or UMVUE) and modify it by incorporating the guess θ_0 (sometimes called natural origin) so that the resulting estimator or testimator though perhaps biased, has smaller risk than that of $\hat{\theta}$ in some interval around θ_0.

In this chapter an attempt has been made to demonstrate that how shrinkage testimation procedure works under GELF.

We have proposed the shrinkage testimators for the scale parameter of an Exponential distribution in section 4.2. The risks of the proposed testimators have been derived in section 4.3. The section 4.4 deals with the relative risk(s) of these two estimators. Section 4.5 concludes with the comparison of UMVUE and the proposed shrinkage testimators in terms of their relative risks. Suggestion for the choice of shrinkage factor is made and recommendations regarding the choice of level of significance and degree of asymmetry have been made.

In section 4.6 we have proposed the two different shrinkage testimators for the variance of a Normal distribution and we have studied the risk properties of these two shrinkage testimators under General Entropy Loss Function. Section 4.7 deals with the derivation of the risk(s) of these two estimators. Section 4.8 deals with the relative risk(s) of these two estimators. Section 4.9 concludes with the comparison of UMVUE and the proposed shrinkage testimatiors in terms of their relative risks. Further in the same section a suggestion for the shrinkage factor is made, along with the choices of degrees of asymmetry and level of significance.

4.2 Shrinkage Testimator(s) for Scale Parameter of an Exponential Distribution.

Let x have the distribution defined in (4.1.1). It is assumed that the prior knowledge about θ is available in the form of an initial estimate θ_0. We are interested in constructing an estimator of θ possibly using the information about θ and the sample observations: $x_1, x_2 \ldots\ldots, x_n$. The proposed shrinkage testimator can be described as follows:

(i) Compute the sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ which is the 'best' estimator of θ in absence of any information about θ. (ii) Test the hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ at α level using the test statistic $\frac{2n\bar{x}}{\theta_0}$ which follows χ^2 – distribution with 2n degrees of freedom.

We define the shrinkage testimator $\hat{\theta}_{ST_1}$ and $\hat{\theta}_{ST_2}$ of θ as follows:

$$\hat{\theta}_{ST_1} = \begin{cases} k\bar{x} + (1-k)\theta_0 & ; \text{ if } \chi_1^2 \le (2n\bar{x}/\theta_0) \le \chi_2^2 \\ \bar{x} & ; \text{ otherwise} \end{cases} \qquad \text{\_\_\_\_\_\_\_(4.2.1)}$$

where k being dependent on test statistic is given by $k = 2n\bar{x}/\theta_0\,\chi^2$ and $\chi^2 = (\chi_2^2 - \chi_1^2)$

Now, taking the 'square' of k (i.e. $k = k^2$), another testimator is defined as

$$\hat{\theta}_{ST_2} = \begin{cases} (2n\bar{x}/\theta_0\,\chi^2)^2\,\bar{x} + \left[1-(2n\bar{x}/\theta_0\,\chi^2)^2\right]\theta_0 & ; \text{ if } H_0 \text{ is accepted} \\ \bar{x} & ; \text{ otherwise} \end{cases} \qquad \text{\_\_\_\_\_(4.2.2)}$$

4.3 Risk of Testimators

In this section we derive the risk of these two testimators which are defined in the previous section.

4.3.1 Risk of $\hat{\theta}_{ST1}$

The risk of $\hat{\theta}_{ST_1}$ under $L_E(\hat{\theta},\theta)$ is defined by

$$R(\hat{\theta}_{ST_1}) = E[\hat{\theta}_{ST_1} \mid L_E(\hat{\theta},\theta)]$$

$$= E\left[k\bar{x} + (1-k)\theta_0 / \chi_1^2 < 2n\bar{x}/\theta_0 < \chi_2^2\right]\cdot p\left[\chi_1^2 < 2n\bar{x}/\theta_0 < \chi_2^2\right]$$
$$+ E\left[\bar{x} \mid 2n\bar{x}/\theta_0 < \chi_1^2 \cup 2n\bar{x}/\theta_0 > \chi_2^2\right]\cdot p\left[2n\bar{x}/\theta_0 < \chi_1^2 \cup 2n\bar{x}/\theta_0 > \chi_2^2\right]$$

$$\text{\_\_\_\_\_\_\_(4.3.1.1)}$$

$$= \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} \left[\left(\frac{2n\bar{x}}{\theta_0\chi^2}\right)(\bar{x}-\theta_0)+\theta_0 \Big/ \theta\right]^p f(\bar{x})\,d\bar{x} - \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} p\ln\left[\frac{2n\bar{x}}{\theta_0\chi^2}(\bar{x}-\theta_0)+\theta_0 \Big/ \theta\right] f(\bar{x})\,d\bar{x}$$
$$- \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\,d\bar{x} + \int_0^{\frac{\chi_1^2\theta_0}{2n}} \left[(\bar{x}/\theta)^p - p\ln(\bar{x}/\theta)-1\right] f(\bar{x})\,d\bar{x} + \int_{\frac{\chi_2^2\theta_0}{2n}}^{\infty} \left[(\bar{x}/\theta)^p - p\ln(\bar{x}/\theta)-1\right] f(\bar{x})\,d\bar{x}$$

$$\text{\_\_\_\_\_\_\_(4.3.1.2)}$$

Where $f(\bar{x}) = (1/\Gamma n)\,(n/\theta)^n (\bar{x})^{n-1} \exp(-n\bar{x}/\theta)$

A Straight forward integration of (4.3.1.2) gives

$$R(\hat{\theta}_{ST_1}) = I_1 - I_2 - \left\{ I\left(\frac{x_2^2\phi}{2}, n\right) - I\left(\frac{x_1^2\phi}{2}, n\right) \right\} + (1/n)^p \,\frac{\Gamma(p+n)}{\Gamma n} \left\{ I\left(\frac{x_1^2\phi}{2}, n+p\right) - I\left(\frac{x_2^2\phi}{2}, n+p\right) + 1 \right\} - \left\{ I\left(\frac{x_1^2\phi}{2}, n\right) - I\left(\frac{x_2^2\phi}{2}, n\right) + 1 \right\} - I_3 - I_4 \qquad (4.3.1.3)$$

Where $I(x;p) = (1/\Gamma p) \int_0^x e^{-x} x^{p-1}\, dx$ refers to the standard incomplete gamma function and

$$I_1 = \int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} \left[(2t^2/n\phi\chi^2) - (2t/\chi^2) + \phi\right]^p (1/\Gamma n)\, e^{-t}\, t^{n-1}\, dt$$

$$I_2 = \int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} p \ln\left[(2t^2/n\phi\chi^2) - (2t/\chi^2) + \phi\right] (1/\Gamma n)\, e^{-t}\, t^{n-1}\, dt$$

$$I_3 = \int_0^{\frac{x_1^2\phi}{2}} p \ln(t/n)\,(1/\Gamma n)\, e^{-t}\, t^{n-1}\, dt$$

$$I_4 = \int_{\frac{x_2^2\phi}{2}}^{\infty} p \ln(t/n)\,(1/\Gamma n)\, e^{-t}\, t^{n-1}\, dt$$

4.3.2 Risk of $\hat{\theta}_{ST2}$

Similarly, we obtain the risk of $\hat{\theta}_{ST_2}$ under $L_E(\hat{\theta}, \theta)$ given by

$$R(\hat{\theta}_{ST_2}) = E[\hat{\theta}_{ST_2} \mid L_E(\hat{\theta},\theta)]$$

$$= E\left[\left(2n\bar{x}/\theta_0\,\chi^2\right)^2(\bar{x}-\theta_0)+\theta_0/\chi_1^2 < 2n\bar{x}/\theta_0 < \chi_2^2\right]\cdot p\left[\chi_1^2 < 2n\bar{x}/\theta_0 < \chi_2^2\right]$$
$$+E\left[\bar{x} \mid 2n\bar{x}/\theta_0 < \chi_1^2 \cup 2n\bar{x}/\theta_0 > \chi_2^2\right]\cdot p\left[2n\bar{x}/\theta_0 < \chi_1^2 \cup 2n\bar{x}/\theta_0 > \chi_2^2\right]$$

________(4.3.2.1)

$$= \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} \left[\left(\frac{2n\bar{x}}{\theta_0\,\chi^2}\right)^2(\bar{x}-\theta_0)+\theta_0\Big/\theta\right]^p f(\bar{x})\,d\bar{x} - \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} p\ln\left[\left(\frac{2n\bar{x}}{\theta_0\,\chi^2}\right)^2(\bar{x}-\theta_0)+\theta_0\Big/\theta\right] f(\bar{x})\,d\bar{x}$$
$$-\int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\,d\bar{x} + \int_0^{\frac{\chi_1^2\theta_0}{2n}} \left[(\bar{x}/\theta)^p - p\ln(\bar{x}/\theta)-1\right]f(\bar{x})\,d\bar{x} + \int_{\frac{\chi_2^2\theta_0}{2n}}^{\infty} \left[(\bar{x}/\theta)^p - p\ln(\bar{x}/\theta)-1\right]f(\bar{x})\,d\bar{x}$$

________(4.3.2.2)

Where $f(\bar{x}) = (1/\Gamma n)\,(n/\theta)^n(\bar{x})^{n-1}\exp(-n\bar{x}/\theta)$

A straight forward integration of (4.3.2.3) gives:

$$R(\hat{\theta}_{ST_2}) = I_1 - I_2 - \left\{I\left(\frac{x_2^2\,\phi}{2},n\right)-I\left(\frac{x_1^2\,\phi}{2},n\right)\right\} +$$
$$(1/n)^p\,\frac{\Gamma(p+n)}{\Gamma n}\left\{I\left(\frac{x_1^2\,\phi}{2},n+p\right)-I\left(\frac{x_2^2\,\phi}{2},n+p\right)+1\right\} -$$
$$\left\{I\left(\frac{x_1^2\,\phi}{2},n\right)-I\left(\frac{x_2^2\,\phi}{2},n\right)+1\right\} - I_3 - I_4$$

________(4.3.2.3)

Where $I(x;p) = (1/\Gamma p)\int_0^x e^{-x}\,x^{p-1}\,dx$ refers to the standard incomplete gamma function and

$$I_1 = \int_{\frac{x_1^2\,\phi}{2}}^{\frac{x_2^2\,\phi}{2}} \left[\left(4t^3/n\phi^2\left(\chi^2\right)^2\right)-\left(4t^2/\phi\left(\chi^2\right)^2\right)+\phi\right]^p (1/\Gamma n)\,e^{-t}\,t^{n-1}\,dt$$

$$I_2 = \int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} p\ln\left[\left(4t^3/n\phi^2\left(\chi^2\right)^2\right)-\left(4t^2/\phi\left(\chi^2\right)^2\right)+\phi\right](1/\Gamma n)\, e^{-t}\, t^{n-1}\, dt$$

$$I_3 = \int_{0}^{\frac{x_1^2\phi}{2}} p\ln(t/n)\,(1/\Gamma n)\, e^{-t}\, t^{n-1}\, dt$$

$$I_4 = \int_{\frac{x_2^2\phi}{2}}^{\infty} p\ln(t/n)\,(1/\Gamma n)\, e^{-t}\, t^{n-1}\, dt$$

4.4 Relative Risks of $\hat{\theta}_{ST_i}$

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator $\bar{x}$ in this case. For this purpose, we obtain the risk of $\bar{x}$ under $L_E(\hat{\theta},\theta)$ as:

$$R_E(\bar{x}) = E[\bar{x} \mid L_E(\hat{\theta},\theta)]$$

$$= \int_0^{\infty} \left[(\bar{x}/\theta)^p - p\ln(\bar{x}/\theta) - 1\right] f(\bar{x})\, d\bar{x} \qquad \text{\_\_\_\_\_\_\_\_(4.4.1)}$$

A straightforward integration of (4.4.1) gives

$$R_E(\bar{x}) = \left[\frac{\Gamma(n+P)}{\Gamma n\,(n^P)} - p\{\psi(n) - \ln(n)\}\right] - 1 \qquad \text{\_\_\_\_\_\_\_\_(4.4.2)}$$

Where $\psi(n) = (d/dn)\ln\Gamma n$ refers to the Euler's psi function.

Now, we define the Relative Risk of $\hat{\theta}_{ST1}$ with respect to $\bar{x}$ under $L_E(\hat{\theta},\theta)$ as follows:

$$RR_1 = \frac{R_E(\bar{x})}{R(\hat{\theta}_{ST_1})} \qquad \text{\_\_\_\_\_\_\_\_(4.4.3)}$$

Using (4.4.2) and (4.3.1.3) the expression for RR_1 given in (4.4.3) can be obtained;

Similarly, we define the Relative Risk of $\hat{\theta}_{ST_2}$ by

$$RR_2 = \frac{R_E(\bar{x})}{R(\hat{\theta}_{ST_2})} \qquad \text{\_\_\_\_\_\_\_(4.4.4)}$$

The expression for RR_2 is given by (4.4.4) which can be obtained by using equations (4.4.2) and (4.3.2.3).

Now, it is observed that both RR_1 and RR_2 are functions of 'ϕ', 'n', 'α' and 'p'.

4.5 Recommendations for $\hat{\theta}_{ST_i}$

In this section we provide the comparison of UMVUE and the proposed shrinkage testimators in terms of their relative risks. Recommendations regarding the applications of proposed testimators are provided.

- In order to study the behaviour of $\hat{\theta}_{ST_1}$ and $\hat{\theta}_{ST_2}$ and the effect of shrinkage factor (S.F.) on the proposed testimators we have computed the values of Relative Risk (RR_1) for the following set of values. n = 5, 8, 10, 12; α = 1%, 5%, 10% ; p = -3, -2, -1 and p = 2, 3, 4. In all there will be several tables of RR for different variations in 'p', 'α' and 'n'. We have considered ϕ = 0.2 (0.2) 1.6. Some of the tables have been assembled in the appendix by (i) keeping 'α' to be fixed and varying 'p' (ii) keeping 'p' to be fixed and varying 'α' as we wish to recommend for these two values.

- For n = 5, α = 1% and for different values of 'p' (positive as well as negative) $\hat{\theta}_{ST_1}$ performs better than the conventional estimator for all the values of 'p' with its best performance for p = -3 and p = 2 for the whole range of ϕ. Considered here i.e. $0.2 \le \phi \le 1.6$.

- Next we have changed to α = 5%. Similar pattern of behaviour is observed for the relative risk and p = -3 and p = 2 provide the best results. However the magnitude of RR is small compared to α = 1% values.

- We have also considered α = 10%. In order to observe the behaviour for still higher level of significance just to confirm whether under different loss function the value of 'α' gets changed or not. We found that $\hat{\theta}_{ST_1}$ performs still better than the conventional estimator but the magnitude of RR values is still small though in all the cases it is above unity.

- So, a small value of α = 1% is recommended. Also by varying 'n' it is observed that RR values are higher for n = 5 compared to its other values of 8, 10 and 12. Hence a smaller 'n' is suggested. A higher RR_1 value indicates a 'better' control over the risk. So, by choosing appropriate value of 'p' and 'α' a better gain in terms of performance of $\hat{\theta}_{ST_1}$ can be achieved.

- $\hat{\theta}_{ST_2}$, is another testimator proposed by taking the 'SQUARE' of shrinkage factor. We have again prepared the relevant tables of Relative Risk (RR_2) of $\hat{\theta}_{ST_2}$ with respect to the conventional estimator for the same set of values as we have considered to study the behaviour of $\hat{\theta}_{ST_1}$. We observe the following:

- For $\hat{\theta}_{ST_2}$ where we have considered the square of S.F. Following behaviour of RR is observed. For almost the entire range of ϕ i.e. $0.2 \leq \phi \leq 1.4$ the values of RR (in terms of magnitude) are higher than those for S.F.(without square).

- Almost similar pattern of RR for different values of 'p' and 'α' has been observed for the values of n considered here. The S.F. can be made small either by taking smaller values of α or by fixing α and taking higher powers of 'k'.

- So, the proposed testimator is having smaller risk than the conventional estimator provided n is small, α is small and square of S.F. is considered.

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 4.5.1.1 **Relative Risk of** $\hat{\theta}_{ST_1}$ $\alpha = 1\%$, n = 5

Φ	**p = -3**	**p = -2**	**p = -1**	**p = 2**	**p = 3**	**p = 4**
0.20	0.959	0.777	0.568	1.002	0.953	0.968
0.40	1.327	0.918	0.595	1.935	1	1
0.60	1.893	1.327	0.745	2.369	1.144	1.071
0.80	2.183	1.821	0.949	3.448	1.966	1.476
1.00	3.003	1.934	1.048	4.583	3.359	2.257
1.20	1.669	1.641	1.626	3.008	2.301	1.453
1.40	1.383	1.291	1.362	1.772	1.654	1.464
1.60	1.175	1.026	1.113	0.744	0.723	0.741

Table : 4.5.1.2 **Relative Risk of** $\hat{\theta}_{ST_1}$ $\alpha = 1\%$, n = 8

Φ	**p = -3**	**p = -2**	**p = -1**	**p = 2**	**p = 3**	**p = 4**
0.20	0.995	0.995	0.984	0.957	0.976	0.968
0.40	1.046	0.998	1	1.004	0.977	0.986
0.60	1.429	1.215	1.087	1.006	1.002	1.001
0.80	2.149	1.742	1.371	2.11	1.394	1.197
1.00	2.435	2.124	1.603	4.259	3.894	2.992
1.20	1.943	1.839	1.505	3.227	2.768	1.824
1.40	1.411	1.351	1.211	1.408	1.166	1.096
1.60	1.071	1.002	0.942	0.48	0.513	0.528

Table : 4.5.1.3 **Relative Risk of $\hat{\theta}_{ST_1}$ $\alpha = 5\%$, n = 5**

φ	p = -3	p = -2	p = -1	p = 2	p = 3	p = 4
0.20	1.12	1.026	0.991	1.049	1.022	1.012
0.40	1.371	1.179	1.051	1.09	1.04	1.021
0.60	1.449	1.437	1.197	1.314	1.122	1.058
0.80	1.575	1.589	1.36	2.17	1.375	1.17
1.00	1.587	1.63	1.404	3.488	2.793	1.404
1.20	1.28	1.391	1.299	2.844	1.771	1.35
1.40	1.139	1.182	1.132	1.524	1.233	1.136
1.60	1.035	1.02	0.979	0.722	0.739	0.76

Table : 4.5.1.4 **Relative Risk of $\hat{\theta}_{ST_1}$ $\alpha = 5\%$, n = 8**

φ	p = -3	p = -2	p = -1	p = 2	p = 3	p = 4
0.20	1.033	1.017	1.007	1.008	1.004	1.002
0.40	1.151	1.088	1.047	1.096	1.053	1.035
0.60	1.341	1.203	1.097	1.11	1.047	1.023
0.80	1.545	1.403	1.216	1.303	1.089	1.007
1.00	1.555	1.487	1.304	2.055	1.396	1.18
1.20	1.324	1.326	1.232	1.998	1.319	1.111
1.40	1.099	1.094	1.063	1.062	0.992	0.951
1.60	0.947	0.918	0.907	0.556	0.606	0.627

Table : 4.5.2.1 **Relative Risk of $\hat{\theta}_{ST_2}$ $\alpha = 1\%$, n = 5**

φ	p = -3	p = -2	p = -1	p = 2	p = 3	p = 4
0.20	0.919	0.883	0.916	0.82	0.883	0.923
0.40	1.302	1.027	0.932	0.968	0.983	0.99
0.60	2.034	1.608	1.231	1.261	1.091	1.035
0.80	2.215	2.396	1.671	3.366	2.363	1.621
1.00	2.463	2.508	1.843	6.484	5.819	3.158
1.20	1.786	1.993	1.617	4.733	4.083	3.031
1.40	1.449	1.5	1.288	1.617	1.273	1.237
1.60	1.219	1.169	1.023	0.495	0.513	0.54

Table : 4.5.2.2 **Relative Risk of $\hat{\theta}_{ST_2}$** $\alpha = 1\%$, n = 8

ф	**p = -3**	**p = -2**	**p = -1**	**p = 2**	**p = 3**	**p = 4**
0.20	0.979	0.943	0.952	0.874	0.881	0.886
0.40	0.978	0.994	1.001	0.906	0.948	0.969
0.60	1.405	1.183	1.057	1.008	1.004	1.003
0.80	2.361	1.878	1.431	2.305	1.385	1.144
1.00	2.628	2.306	1.697	5.258	3.935	2.094
1.20	1.883	1.774	1.467	2.069	2.689	1.952
1.40	1.29	1.205	1.09	0.768	0.759	0.764
1.60	0.963	0.871	0.816	0.291	0.322	0.331

4.6 Shrinkage Testimator for the Variance of a Normal Distribution

Shrinkage testimators for the mean μ of a Normal distribution N(μ, σ^2) when variance σ^2 is known or unknown, have been proposed by Waiker, Schuurman and Raghunandan (1984). Recently Pandey et. al. (2007) have studied the risk properties for the positive degree of asymmetry. Where as this study finds the range for positive as well as negative degrees of asymmetry where the shrinkage testimator perform better than the UMVUE.

Let X be Normally distributed with mean μ and variance σ^2. We have proposed a single sample shrinkage testimator. It is assumed that the prior knowledge about σ^2 is available in the form of an initial estimate σ_0^2. Using the sample observations $x_1, x_2 ...., x_n$ and possibly the given information we wish to construct a shrinkage testimator. The procedure is as follows:

1. First test with a sample of size n, the null hypothesis $H_0 : \sigma^2 = \sigma_0^2$ against the alternative $H_1 : \sigma^2 \neq \sigma_0^2$ using the test statistics $\frac{\nu s^2}{\sigma_0^2}$, where ν = (n -1)

and $s^2 = \frac{1}{n-1}\sum(x_i - \bar{x})^2$. The test statistics is distributed as χ^2 with ν degrees of freedom.

2. If H_0 is accepted at α level of significance i.e. $x_1^2 < \frac{\nu s^2}{\sigma_0^2} < x_2^2$ where x_1^2 and x_2^2 are the lower and upper points of the uniformly most powerful unbiased (UMPU) test of H_0, use the conventional shrinkage estimator with shrinkage factor $k = \frac{\nu s^2}{\sigma_0^2 x^2}$, which is inversely proportional to χ^2 and it depends on the test statistic, so the arbitrariness in the choice of shrinkage factor has been removed by making it dependent on the test statistic.
3. If H_0 is rejected, use s^2, the Uniformly Minimum Variance Unbiased Estimator (UMVUE) as the estimator of σ^2.

Now, the proposed shrinkage testimator $\hat{\sigma}_{ST1}^2$ of σ^2 is

$$\sigma_{ST1}^2 = \begin{cases} k\, s^2 + (1-k)\sigma_0^2 & , \textit{if } H_0 \textit{ is accepted} \\ s^2 & , \textit{otherwise} \end{cases}$$

The next proposed shrinkage testimator $\hat{\sigma}_{ST2}^2$ of σ^2 is

$$\hat{\sigma}_{ST2}^2 = \begin{cases} k_1\, s^2 + (1-k_1)\sigma_0^2 & , \textit{if } H_0 \textit{ is accepted} \\ s^2 & , \textit{otherwise} \end{cases}$$

Where $k_1 = \frac{\nu s^2}{\sigma_0{}^2\, x^2}$

Estimators of this type with and arbitrary k (0 ≤ k ≤ 1) have been proposed by Pandey and Srivastava (1987) and others. In all such studies it has been found that the shrinkage estimators work well if k is near zero and 'n' is small and 'α' is also small. The present work deal with the shrinkage factor dependent on the test statistic and arbitrary 'k'.

We have studied the risk properties for several choices of level of significance, sample sizes, a wide range of λ and several values of degrees of asymmetry.

4.7 <u>Risk of Testimators</u>

In this section we derive the risk of these two testimators which are defined in the previous section.

4.7.1 <u>Risk of</u> $\hat{\sigma}^2_{ST1}$

The risk of $\hat{\sigma}^2{}_{ST1}$ under $L_E(\hat{\theta},\theta)$ is defined by

$$R(\hat{\sigma}^2{}_{ST_1}) = E[\hat{\sigma}^2{}_{ST_1} | L_E(\hat{\theta},\theta)]$$

$$= E\left[ks^2 + (1-k)\sigma_0^2 \Big/ \chi_1^2 < \frac{\upsilon s^2}{\sigma_0^2} < \chi_2^2\right] \cdot p\left[\chi_1^2 < \frac{\upsilon s^2}{\sigma_0^2} < \chi_2^2\right]$$
$$+ E\left[s^2 \,\middle|\, \frac{\upsilon s^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon s^2}{\sigma_0^2} > \chi_2^2\right] \cdot p\left[\frac{\upsilon s^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon s^2}{\sigma_0^2} > \chi_2^2\right]$$

_______(4.7.1.1)

$$= \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} \left[\frac{k(s^2-\sigma_0^2)+\sigma_0^2}{\sigma^2}\right]^p f(s^2)\,ds^2 - \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} p\,\ln\left[\frac{k(s^2-\sigma_0^2)+\sigma_0^2}{\sigma^2}\right] f(s^2)\,ds^2$$

$$- \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} f(s^2)\,ds^2 + \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon}} \left[\left(\frac{s^2}{\sigma^2}\right)^p - p\,\ln\left(\frac{s^2}{\sigma^2}\right) - 1\right] f(s^2)\,ds^2 + \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon}}^{\infty} \left[\left(\frac{s^2}{\sigma^2}\right)^p - p\,\ln\left(\frac{s^2}{\sigma^2}\right) - 1\right] f(s^2)\,ds^2$$

_______(4.7.1.2)

Where $f(s^2) = \frac{1}{2^{\upsilon/2}\,\Gamma(\upsilon/2)} (s^2)^{\frac{\upsilon}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon s^2}{\sigma^2}\right)} ds^2$

Straight forward integration of (4.7.1.2) gives

$$R\left(\hat{\sigma}^2_{ST1}\right) = \left(\frac{\sigma^2}{\nu}\right)^{\nu/2} \left[\begin{array}{c} I_1 - I_2 - \left\{I\left(\chi_2^2\lambda, \frac{\nu}{2}\right) - I\left(\chi_1^2\lambda, \frac{\nu}{2}\right)\right\} \\ -\left[I\left(\chi_1^2\lambda, \frac{\nu}{2}\right) - I\left(\chi_2^2\lambda, \frac{\nu}{2}\right) + 1\right] \\ + \frac{\Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}\right)\left(\frac{\nu}{2}\right)^p} \left[I\left(\chi_1^2\lambda, \frac{\nu}{2}+p\right) - I\left(\chi_2^2\lambda, \frac{\nu}{2}+p\right) + 1\right] \\ -I_3 - I_4 \end{array}\right]$$

_____(4.7.1.3)

Where $I(x;p) = (1/\Gamma p) \int_0^x e^{-x} x^{p-1} dx$ refers to the standard incomplete gamma function, $\lambda = \frac{\sigma_0^2}{\sigma^2}$, and

$$I_1 = \frac{1}{2^{\nu/2}\ \Gamma\left(\frac{\nu}{2}\right)} \int_{x_1^2\lambda}^{x_2^2\lambda} \left(k\left(\frac{t}{\nu} - \lambda\right) + \lambda\right)^p\ e^{-\left(\frac{1}{2}\right)t}\ t^{\frac{\nu}{2}-1}\ dt$$

$$I_2 = \frac{p}{2^{\nu/2}\ \Gamma\left(\frac{\nu}{2}\right)} \int_{x_1^2\lambda}^{x_2^2\lambda} ln\left(k\left(\frac{t}{\nu} - \lambda\right) + \lambda\right)\ e^{-\left(\frac{1}{2}\right)t}\ t^{\frac{\nu}{2}-1}\ dt$$

$$I_3 = \frac{p}{2^{\nu/2}\ \Gamma\left(\frac{\nu}{2}\right)} \int_0^{x_1^2\lambda} ln\left(\frac{t}{\nu}\right)\ e^{-\left(\frac{1}{2}\right)t}\ t^{\frac{\nu}{2}-1}\ dt$$

$$I_4 = \frac{p}{2^{\nu/2}\ \Gamma\left(\frac{\nu}{2}\right)} \int_{x_2^2\lambda}^{\infty} ln\left(\frac{t}{\nu}\right)\ e^{-\left(\frac{1}{2}\right)t}\ t^{\frac{\nu}{2}-1}\ dt$$

4.7.2 Risk of $\hat{\sigma}^2_{ST2}$

Again, we obtain the risk of $\hat{\sigma}^2{}_{ST_2}$ under $L_E\left(\hat{\theta},\theta\right)$ with respect to s^2, given by

$$R(\hat{\sigma}^2{}_{ST_2}) = E[\,\hat{\sigma}^2{}_{ST_2} \mid L_E\left(\hat{\theta},\theta\right)\,]$$

$$= E\left[k_1 s^2 + (1-k_1)\sigma_0^2 \middle/ \chi_1^2 < \frac{\upsilon s^2}{\sigma_0^2} < \chi_2^2\right] \cdot p\left[\chi_1^2 < \frac{\upsilon s^2}{\sigma_0^2} < \chi_2^2\right]$$
$$+ E\left[s^2 \,\middle|\, \frac{\upsilon s^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon s^2}{\sigma_0^2} > \chi_2^2\right] \cdot p\left[\frac{\upsilon s^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon s^2}{\sigma_0^2} > \chi_2^2\right]$$

___(4.7.2.1)

$$= \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} \left[\frac{\frac{\upsilon s^2}{\sigma_0^2\chi^2}\left(s^2-\sigma_0^2\right)+\sigma_0^2}{\sigma^2}\right]^p f(s^2)\,ds^2 - \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} p\,\ln\left[\frac{\frac{\upsilon s^2}{\sigma_0^2\chi^2}\left(s^2-\sigma_0^2\right)+\sigma_0^2}{\sigma^2}\right] f(s^2)\,ds^2$$

$$- \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} f(s^2)\,ds^2 + \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon}} \left[\left(\frac{s^2}{\sigma^2}\right)^p - p\ln\left(\frac{s^2}{\sigma^2}\right)-1\right] f(s^2)\,ds^2 + \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon}}^{\infty} \left[\left(\frac{s^2}{\sigma^2}\right)^p - p\ln\left(\frac{s^2}{\sigma^2}\right)-1\right] f(s^2)\,ds^2$$

_______(4.7.2.2)

Where $f(s^2) = \frac{1}{2^{\upsilon/2}\,\Gamma\left(\upsilon/2\right)} \left(s^2\right)^{\frac{\upsilon}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon s^2}{\sigma^2}\right)} ds^2$

Straight forward integration of (4.7.2.2) gives

$$R\left(\hat{\sigma}_{ST1}^2\right) = \left(\frac{\sigma^2}{\nu}\right)^{\nu/2} \left[\begin{array}{c} I_1 - I_2 - \left\{I\left(\chi_2^2\lambda,\frac{\nu}{2}\right) - I\left(\chi_1^2\lambda,\frac{\nu}{2}\right)\right\} \\ -\left[I\left(\chi_1^2\lambda,\frac{\nu}{2}\right) - I\left(\chi_2^2\lambda,\frac{\nu}{2}\right)+1\right] \\ +\frac{\Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}\right)\left(\frac{\nu}{2}\right)^p}\left[I\left(\chi_1^2\lambda,\frac{\nu}{2}+p\right) - I\left(\chi_2^2\lambda,\frac{\nu}{2}+p\right)+1\right] \\ -I_3 - I_4 \end{array}\right]$$

_____(4.7.2.3)

Where $I(x;p) = \left(1/\Gamma p\right)\int_0^x e^{-x}x^{p-1}\,dx$ refers to the standard incomplete gamma function, $\lambda = \frac{\sigma_0^2}{\sigma^2}$, and

$$I_1 = \frac{1}{2^{\nu/2}\,\Gamma\left(\frac{\nu}{2}\right)} \int_{x_1^2\lambda}^{x_2^2\lambda} \left(\frac{t^2}{\nu\lambda\chi^2} - \frac{t}{\chi^2} + \lambda\right)^p e^{-\left(\frac{1}{2}\right)t}\, t^{\frac{\nu}{2}-1}\,dt$$

$$I_2 = \frac{p}{2^{\nu/2}\,\Gamma\left(\frac{\nu}{2}\right)} \int_{x_1^2\lambda}^{x_2^2\lambda} ln\left(\frac{t^2}{\nu\lambda\chi^2} - \frac{t}{\chi^2} + \lambda\right) e^{-\left(\frac{1}{2}\right)t}\, t^{\frac{\nu}{2}-1}\,dt$$

$$I_3 = \frac{p}{2^{\nu/2}\,\Gamma\left(\frac{\nu}{2}\right)} \int_0^{x_1^2\lambda} ln\left(\frac{t}{\nu}\right) e^{-\left(\frac{1}{2}\right)t}\, t^{\frac{\nu}{2}-1}\,dt$$

$$I_4 = \frac{p}{2^{v/2}\ \Gamma\left(\frac{v}{2}\right)} \int_{x_2^2 \lambda}^{\infty} ln\left(\frac{t}{v}\right)\ e^{-\left(\frac{1}{2}\right)t}\ t^{\frac{v}{2}-1}\ dt$$

4.8 Relative Risk of $\hat{\sigma}^2_{STi}$

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator s^2 in this case. For this purpose, we obtain the risk of s^2 under $L_E\left(\hat{\sigma}^2, \sigma^2\right)$ as:

$$R_E(s^2) = E[s^2 \,|\, L(\hat{\sigma}^2, \sigma^2)]$$

$$= \int_0^\infty \left[\left(s^2/\sigma^2\right)^p - p \ln\left(s^2/\sigma^2\right) - 1\right] f(s^2)\, ds^2 \qquad \text{\_\_\_\_\_\_(4.8.1)}$$

Where $f(s^2) = \dfrac{1}{2^{\upsilon/2}\,\Gamma\left(\upsilon/2\right)} \left(s^2\right)^{\frac{\upsilon}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon s^2}{\sigma^2}\right)}$

A straightforward integration of (4.8.1) gives

$$R_E(s^2) = \left[\frac{\Gamma\left(\frac{\nu}{2}+P\right)}{\Gamma\left(\frac{\nu}{2}\right)\left(\frac{\nu}{2}\right)^p} - p\left\{\psi\left(\frac{\nu}{2}\right) - \ln\left(\frac{\nu}{2}\right)\right\}\right] - 1 \qquad \text{\_\_\_\_\_\_(4.8.2)}$$

Where $\psi(n) = \left(d/dn\right) \ln \Gamma n$ refers to the Euler's psi function.

Now, we define the Relative Risk of $\hat{\sigma}^2{}_{ST_i}$, $i = 1,2$ with respect to s^2 under $L\left(\hat{\sigma}^2, \sigma^2\right)$ as follows:

$$RR_1 = \frac{R_E(s^2)}{R(\hat{\sigma}^2{}_{ST1})} \qquad \text{\_\_\_\_\_\_(4.8.3)}$$

Using (4.8.2) and (4.7.1.3) the expression for RR_1 given in (4.8.3) can be obtained; it is observed that RR_1 is a function of 'λ', ' ν ', 'α', 'k' and 'p'.

Finally, we define the Relative Risk of $\hat{\sigma}^2{}_{ST_2}$ by

$$RR_2 = \frac{R_E(s^2)}{R(\hat{\sigma}^2{}_{ST2})} \qquad \text{\_\_\_\_\_\_\_\_(4.8.4)}$$

The expression for RR_2 is given by (4.8.4) which can be obtained by using (4.8.2) and (4.7.2.3). Again we observed that RR_2 is a function of 'λ', 'ν', 'α' and 'p'.

4.9 Recommendations for $\hat{\sigma}^2_{STi}$

In this section we wish to compare the performance of $\hat{\sigma}^2{}_{ST_1}$ and $\hat{\sigma}^2{}_{ST_2}$ with respect to the best available (unbiased) estimator of σ^2.

4.9.1 Recommendations for $\hat{\sigma}^2_{ST1}$

It is observed that RR_1 is a function of ν, α, λ, k and the degrees of asymmetry "p". In order to study the behaviour of $\hat{\sigma}^2{}_{ST_1}$ with respect to the best available estimator we have considered several values of above mentioned quantities viz. k = 0.2 (0.2) 1.0, λ = 0.2 (0.2) 2.0, ν = 5,8,10,12, and p = -2, -1.75, -1.5, -1.25, -1.0 , 1.0 and smaller values of α = 1% and 0.1%. As we have observed that RR_1 values start getting negative even for p = +1, so other higher values of 'p' are not considered with a view that for positive values of 'p' the usual estimator may perform better than the proposed one. Also, several studies have pointed out that smaller level of significance should be taken, this motivated us to consider smaller values of α^s considered as above. There will be several tables of RR_1 . Some of these have been assembled at the end of the chapter. However our recommendations based on all these tables are as follows.

1. $\hat{\sigma}^2{}_{ST_1}$ performs better than $\hat{\sigma}^2$ at α = 1% for the whole range of 'λ' for p = -2 i.e. the values of RR_1 are greater than unity for $0.4 \le \lambda \le 1.8$. In this situation the range of 'k' is $0.2 \le k \le 0.8$. It is observed that as p assume

other negative values upto p = -1, still the performance is better but the range of 'λ' changes and for p = -1 it is $0.8 \leq \lambda \leq 1.2$. i.e. it reduces. These values are obtained for $\nu = 5$. However, for other values of ν i.e. 8, 10 and 12 again a similar pattern is observed but now the recommended values of p are upto -1.50.

2. The positive values of 'p' (p = +1 reported here) are indicative of better performance of σ^2, so it is suggested that the use of GEL would be beneficial for under estimation situations.
3. We have considered $\alpha = 0.1\%$ also to observe the behaviour of $\hat{\sigma}^2_{ST_1}$, here the range of 'λ' is increased as now it is $0.4 \leq \lambda \leq 2.0$ which holds even for 'p' upto -1.25 again when p = -1 the range changes slightly and becomes $0.4 \leq \lambda \leq 1.8$. As ν is increased to '8' the range of 'λ' decreases for different negative values of 'p' and it is now $0.6 \leq \lambda \leq 1.8$ for p = -2 and $0.8 \leq \lambda \leq 1.2$ for p = -1.
4. Still increasing ν to 10 and 12 we have observed that the range of λ reduces to $0.6 \leq \lambda \leq 1.6$ and now the values are better upto p = -1.50.
5. For both the values of α^s considered here the RR_1 values are more than '1' but the magnitude of these values are higher for $\alpha = 0.1\%$ and the range of shrinkage factor for all the above recommendations is $0.2 \leq k \leq 0.8$.
6. So, it is recommended to consider higher degrees of underestimation with a small sample size and smaller level of significance. i.e. take $\nu = 5$, p = -2, $\alpha = 0.1\%$ than $\hat{\sigma}^2_{ST_1}$ performs better than $\hat{\sigma}^2$ for $0.4 \leq \lambda \leq 2.0$ and $0.2 \leq k \leq 0.8$.

4.9.2 Recommendations for $\hat{\sigma}^2_{ST2}$

As the arbitrariness in the choice of 'k' is removed by making it dependent on test statistic, now the relative risk of $\hat{\sigma}^2_{ST_2}$ with respect to $\hat{\sigma}^2$ is a function of p, λ, c, and α. In order to study the behaviour of RR_2 we have considered p = -2, -1.75, -1.50, -1.25, -1.0 and 1.0, λ = 0.2 (0.2) 2.0, ν = 5,8,10 and 12, α = 1% and 0.1%. Again the reason for considering only one positive value for degree is that RR_2 values turn negative even at p = +1. Again there will be several tables of RR_2 some of these have been assembled at the end of the chapter however our recommendations for $\hat{\sigma}^2_{ST_2}$ are as follows:

1. For $0.2 \leq \lambda \leq 1.6$, p =-2, ν = 5 and α = 1% $\hat{\sigma}^2_{ST_2}$ dominates $\hat{\sigma}^2$. However the range of 'λ' decreases as 'p' becomes -1.75, now it is $0.2 \leq \lambda \leq 1.4$ and it remains true upto -1.25. But for p = -1 the range of 'λ' is shorter as it is now $0.8 \leq \lambda \leq 1.2$. These values of RR_2 were observed for ν = 5. For the other values of 'ν' almost similar pattern of RR_2 values is observed but the values become smaller as ν increase.

2. Here also for positive values of 'p' $\hat{\sigma}^2$ the usual estimator performs better than $\hat{\sigma}^2_{ST_2}$ as the RR_2 values are negative in this case.

3. For another lower level of significance i.e. α = 0.1% the values of RR_2 are higher in magnitude as compared to those at α = 1%. Also the range of 'λ' increases and it becomes $0.2 \leq \lambda \leq 2.0$ upto p = -1.50, it slightly decreases and becomes $0.6 \leq \lambda \leq 1.6$ for p = -1. Again for p = +1 the RR_2 values are negative for the whole range of λ.

4. Changing $\nu = 8, 10, 12$ we observe that the range of 'λ' reduces further and it becomes $0.6 \leq \lambda \leq 1.6$. However for $\nu = 12$ none of the RR_2 values is greater than '1'.

5. For both the values of α^s considered here the RR_2 values are more than unity but the magnitude of RR_2 values is higher for lower level of significance.

6. So, it is recommended to consider the higher values of degree of asymmetry when under estimation is more serious than over estimation and a lower values of 'ν'.

<u>CONCLUSION</u>:

Two shrinkage testimators for the variance of Normal distribution have been proposed viz. $\hat{\sigma}^2_{ST_1}$ and $\hat{\sigma}^2_{ST_2}$.

The values of RR_1 (i.e. $\hat{\sigma}^2_{ST_1}$ with respect to $\hat{\sigma}^2$) and RR_2 (i.e. $\hat{\sigma}^2_{ST_2}$ with respect to $\hat{\sigma}^2$) are not much different in their magnitudes. However $\hat{\sigma}^2_{ST_2}$ is a shrinkage testimator based on test statistic, so it could be used. It is observed that the use of GELF does not provide good result for positive values of degrees of asymmetry (i.e. overestimation being more serious). So, it is recommended for the reverse situations.

A lower value $\nu = 5$ with $p = -2$, $\alpha = 0.1\%$ provide better result for almost the whole range of 'λ'. However both the estimators perform better than the usual estimator for other values also but the reported values are indicative of the best performance.

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 4.9.1.1 **Relative Risk of** $\hat{\sigma}^2_{ST_1}$ $\alpha = 0.1\%$, $\nu_1 = 5$, k = 0.2

λ	p = -2	p = -1.75	p = -1.50	p = -1.25	p = -1	p = 1
0.20	0.833	0.671	0.558	0.476	0.411	-2.82
0.40	1.432	1.298	1.079	0.869	0.687	-2.13
0.60	1.801	1.962	1.827	1.522	1.18	-1.37
0.80	1.831	2.171	2.214	1.95	1.527	-1.101
1.00	1.727	2.058	2.16	1.966	1.575	-1.028
1.20	1.599	1.856	1.932	1.775	1.448	-1.081
1.40	1.48	1.661	1.691	1.547	1.277	-1.268
1.60	1.379	1.494	1.485	1.346	1.117	-1.719
1.80	1.293	1.358	1.319	1.182	0.983	-3.129
2.00	1.221	1.247	1.186	1.051	0.873	-3.934

Table : 4.9.1.2 **Relative Risk of** $\hat{\sigma}^2_{ST_1}$ $\alpha = 0.1\%$, $\nu_1 = 5$, a = -1.75

λ	k = 0.2	k = 0.4	k = 0.6	k = 0.8	k = 1.0
0.20	0.671	0.867	0.994	1.027	0.839
0.40	1.298	1.428	1.411	1.255	0.821
0.60	1.962	1.911	1.733	1.429	0.818
0.80	2.171	2.05	1.84	1.502	0.807
1.00	2.058	1.976	1.818	1.517	0.796
1.20	1.856	1.829	1.738	1.5	0.786
1.40	1.661	1.674	1.639	1.466	0.777
1.60	1.494	1.533	1.538	1.423	0.769
1.80	1.358	1.411	1.444	1.376	0.762
2.00	1.247	1.308	1.358	1.328	0.756

Table : 4.9.1.3 **Relative Risk of** $\hat{\sigma}^2_{ST_1}$ $\alpha = 0.1\%$, $\nu_1 = 8$, a = -1.75

λ	k = 0.2	k = 0.4	k = 0.6	k = 0.8	k = 1.0
0.20	0.346	0.486	0.623	0.745	0.825
0.40	0.562	0.703	0.782	0.79	0.709
0.60	1.237	1.288	1.197	1.013	0.738
0.80	1.914	1.741	1.466	1.131	0.723
1.00	1.913	1.756	1.506	1.16	0.695
1.20	1.565	1.532	1.406	1.136	0.666
1.40	1.24	1.28	1.255	1.081	0.639
1.60	1.003	1.072	1.106	1.014	0.615
1.80	0.837	0.914	0.977	0.945	0.594
2.00	0.719	0.795	0.871	0.881	0.577

Table : 4.9.1.4 **Relative Risk of** $\hat{\sigma}^2_{ST_1}$ $\alpha = 1\%$, $\nu_1 = 5$, a = -2

λ	k = 0.2	k = 0.4	k = 0.6	k = 0.8	k = 1.0
0.20	0.859	0.991	1.058	1.047	0.863
0.40	1.127	1.172	1.157	1.073	0.816
0.60	1.277	1.263	1.21	1.099	0.81
0.80	1.273	1.247	1.196	1.093	0.805
1.00	1.21	1.193	1.156	1.072	0.801
1.20	1.138	1.132	1.111	1.048	0.798
1.40	1.074	1.078	1.07	1.025	0.797
1.60	1.02	1.031	1.033	1.003	0.797
1.80	0.976	0.991	1	0.984	0.798
2.00	0.939	0.957	0.973	0.967	0.801

Table : 4.9.2.1 **Relative Risk of** $\hat{\sigma}^2_{ST_2}$ $\alpha = 0.1\%$, $\nu_1 = 5$

λ	p = -0.2	p = -1.75	p = -1.5	p = -1.25	p = -1.0
0.20	1.242	1.057	0.876	0.728	0.601
0.40	1.632	1.558	1.317	1.053	0.815
0.60	1.839	2.029	1.899	1.579	1.215
0.80	1.824	2.151	2.179	1.907	1.489
1.00	1.714	2.026	2.106	1.904	1.521
1.20	1.584	1.824	1.879	1.714	1.396
1.40	1.464	1.627	1.639	1.488	1.226
1.60	1.361	1.459	1.433	1.288	1.065
1.80	1.273	1.321	1.267	1.125	0.931
2.00	1.2	1.209	1.134	0.995	0.822

Table : 4.9.2.2 **Relative Risk of** $\hat{\sigma}^2_{ST_2}$ $\alpha = 1\%$, $\nu_1 = 5$

λ	p = -0.2	p = -1.75	p = -1.5	p = -1.25	p = -1.0
0.20	1.169	1.085	0.967	0.846	0.727
0.40	1.256	1.217	1.084	0.908	0.723
0.60	1.3	1.347	1.282	1.125	0.914
0.80	1.266	1.339	1.323	1.207	1.013
1.00	1.199	1.254	1.24	1.146	0.978
1.20	1.129	1.152	1.121	1.03	0.884
1.40	1.066	1.058	1.008	0.915	0.782
1.60	1.012	0.979	0.913	0.816	0.694
1.80	0.967	0.913	0.835	0.736	0.621
2.00	0.929	0.86	0.773	0.673	0.564