Chapter 5

DOUBLE STAGE SHRINKAGE TESTIMATORS UNDER GENERAL ENTROPY LOSS FUNCTION

Chapter – 5

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5.1 Introduction

In situations when there is no a priori knowledge is available for the parameter θ (scale parameter) the sample mean $\bar{x}$ is the BLUE (Best Linear Unbiased Estimator) of θ based on complete set of observations.

However in many real life situations such as mean life time of a component / system, average number of days required to get cured from a disease, etc. A guess value of θ in terms of a point (single) or interval is available to the experimenter either due to past studies or similar studies or his familiarity with behavior of the characteristic under study. Then this guess may be utilized to improve the estimation procedure. In order to use this information for constructing an estimator for θ, the use of preliminary test of significance has been suggested by Bancroft (1944). An extensive bibliography in this area is provided by Han and Bancroft (1977) and Han, Rao and Ravichandran (1988).

Several authors have proposed estimators / testimators for the mean life (scale parameter) with different shrinkage factors and under different loss functions mostly under Squared Error Loss Function (SELF). Recently Srivastava and Tanna (2007) have proposed a double stage shrinkage testimator under General Entropy Loss Function (GELF) and they have shown the superiority of the proposed testimators, over the usual estimator.

The shrinkage testimators are proposed when the shrinkage factor can take any arbitrary value between '0' and '1'. In the present paper this arbitrariness in the choice of shrinkage factor is removed by making it dependent on the test statistics and hence for a given level of significance and degrees of freedom, the shrinkage factor is no longer arbitrary. The choice of an appropriate loss function is often guided by economic considerations and the situation(s) under which the parameter is being estimated.

In this chapter the problem of estimation of the mean life θ of exponential population is considered when a guess θ_0 of θ is available to the experimenter. The double stage estimation for θ is to use the mean of the first sample and the guess value if $H_0 : \theta = \theta_0$ is accepted; otherwise use pooled mean $\bar{x}_p$ of the two samples if H_0 is rejected.

In section 5.2 we have proposed the two different shrinkage testimators for scale parameter of an Exponential Distribution and we have studied the risk properties of these two shrinkage testimators under General Entropy Loss Function defined in section 4.1.1. Section 5.3 deals with the derivation of the risk(s) of these two estimators. Section 5.4 deals with the relative risk(s) of these two estimators. Section 5.5 concludes with the comparison of unbiased pooled estimator and the proposed shrinkage testimators in terms of their relative risks. Further in the same section a suggestion for the shrinkage factor is made.

In section 5.6 we have proposed the two different shrinkage testimators for the variance of a Normal Distribution and we have studied the risk properties of these two shrinkage testimators under General Entropy Loss Function. Section 5.7 deals with the derivation of the risk(s) of these two estimators. Section 5.8 deals with the relative risk(s) of these two estimators. Section 5.9 concludes with the comparison of unbiased pooled estimator and the proposed shrinkage testimators in terms of their relative risks. Further in the same section a suggestion for the shrinkage factor is made.

5.2 Shrinkage Testimator(s) for Scale Parameter of an Exponential Distribution.

Let x_{11}, x_{12}, ---------, x_{1n1} be the first stage sample of size n_1 from the exponential population

$$f(x;\theta) = \begin{cases} (1/\theta)\, e^{-x/\theta} \ ; & x,\, \theta > 0 \\ 0 & ;\ otherwise \end{cases} \qquad \text{\_\_(5.2.1)}$$

Let θ_0 be the guess value of θ. Compute the sample mean $\bar{x}_1 = \frac{1}{n}\sum_{i=1}^{n} x_{1i}$ and test the preliminary hypothesis $H_p: \ \theta = \ \theta_0$, using the test statistic $(2n_1\bar{x}_1/\theta_0)$ which has $\chi^2_{2n_1}$ distribution. It is to be noted that H_P is accepted if $x_1^2 \le \frac{2n_1\bar{x}_1}{\theta_0} \le x_2^2$ and H_P is rejected, otherwise where x_1^2 and x_2^2 being given by $P[x^2_{2n_1} \ge x_2^2] + P[x^2_{2n_1} \le x_1^2] = \alpha$ where α is the pre-assigned level of significance.

Now, if H_p is accepted, take the estimator $k\,(\bar{x}_1 - \theta_0) + \theta_0 \ (0 \le k \le 1)$ and if it is rejected then take $n_2 = n - n_1$ additional observations x_{21}, x_{22},_ _ _ _, X_{2n2} and use the pooled estimator $\bar{x}_p = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{(n_1 + n_2)}$ as the estimator of θ. The properties of such estimators have been studied by Srivastava and Tanna (2007) under General Entropy Loss Function.

Now, we define the shrinkage testimator $\hat{\theta}_{DST_1}$ and $\hat{\theta}_{DST_2}$ of θ as follows:

$$\hat{\theta}_{DST_1} = \begin{cases} k\,\bar{x}_1 + (1-k)\theta_0 & ;\ if\ \chi_1^2 \le (2n_1\bar{x}_1/\theta_0) \le \chi_2^2 \\ \bar{x}_p & ;\ otherwise \end{cases} \qquad \text{\_\_\_\_\_(5.2.2)}$$

where k being dependent on test statistic is given by $k = 2n_1\,\bar{x}_1/\theta_0\,\chi^2$ and $\chi^2 = (\chi_2^2 - \chi_1^2)$

Finally, taking the 'square' of k (i.e. $k_1 = k^2$), another testimator is defined as

$$\hat{\theta}_{DST_2} = \begin{cases} \left(2n_1\,\bar{x}_1/\theta_0\,\chi^2\right)^2 \bar{x} + \left[1-\left(2n_1\,\bar{x}_1/\theta_0\,\chi^2\right)^2\right]\theta_0 & ;\ if\ H_0\ is\ accepted \\ \bar{x}_p & ;\ otherwise \end{cases} \qquad \text{\_\_\_\_(5.2.3)}$$

5.3 Risk of Testimators

In this section we derive the risk of all the two testimators which are defined in the previous section.

5.3.1 Risk of $\hat{\theta}_{DST1}$

The risk of $\hat{\theta}_{DST_1}$ under $L_E(\hat{\theta},\theta)$ is defined by

$$R(\hat{\theta}_{DST_1}) = E[\hat{\theta}_{DST_1} \mid L_E(\hat{\theta},\theta)]$$

$$\begin{aligned} &= E\left[k\bar{x}_1 + (1-k)\theta_0 / \chi_1^2 < 2n_1\bar{x}_1/\theta_0 < \chi_2^2\right]\cdot p\left[\chi_1^2 < 2n_1\bar{x}_1/\theta_0 < \chi_2^2\right] \\ &\quad + E\left[\bar{x}_p \mid 2n_1\bar{x}_1/\theta_0 < \chi_1^2 \cup 2n_1\bar{x}_1/\theta_0 > \chi_2^2\right]\cdot p\left[2n_1\bar{x}_1/\theta_0 < \chi_1^2 \cup 2n_1\bar{x}_1/\theta_0 > \chi_2^2\right] \end{aligned} \qquad \text{\_\_\_\_\_\_\_(5.3.1.1)}$$

$$\begin{aligned} &= \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} \left[\left(\frac{2n_1\bar{x}_1}{\theta_0\,\chi^2}\right)(\bar{x}_1-\theta_0)+\theta_0 \Big/ \theta\right]^p f(\bar{x}_1)\,d\bar{x}_1 - \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} p\ln\left[\frac{2n_1\bar{x}_1}{\theta_0\,\chi^2}(\bar{x}_1-\theta_0)+\theta_0 \Big/ \theta\right] f(\bar{x}_1)\,d\bar{x}_1 - \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} f(\bar{x}_1)\,d\bar{x}_1 \\ &\quad + \int_0^{\frac{\chi_1^2\theta_0}{2n_1}} \int_0^{\infty} \left[\left(\frac{\bar{x}_p}{\theta}\right)^p - p\ln\left(\frac{\bar{x}_p}{\theta}\right) - 1\right] f(\bar{x}_1) f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2 + \int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty} \int_0^{\infty} \left[\left(\frac{\bar{x}_p}{\theta}\right)^p - p\ln\left(\frac{\bar{x}_p}{\theta}\right) - 1\right] f(\bar{x}_1) f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2 \end{aligned} \qquad \text{\_\_\_\_\_\_\_(5.3.1.2)}$$

Where $f(\bar{x}_1) = (1/\Gamma n_1)\ (n_1/\theta)^{n_1} (\bar{x}_1)^{n_1-1} \exp(-n_1\bar{x}_1/\theta)$

$$f(\bar{x}_2) = (1/\Gamma n_2)\ (n_2/\theta)^{n_2} (\bar{x}_2)^{n_2-1} \exp(-n_2\bar{x}_2/\theta)$$

A Straight forward integration of (5.3.1.2) gives

$$\begin{aligned} R(\hat{\theta}_{DST_1}) \;=\; & I_1 - I_2 - \left\{ I\left(\frac{x_2^2\phi}{2}, n_1\right) - I\left(\frac{x_1^2\phi}{2}, n_1\right) \right\} + \\ & (I_3 + I_4) - (I_5 + I_6) - \left\{ I\left(\frac{x_1^2\phi}{2}, n_1\right) - I\left(\frac{x_2^2\phi}{2}, n_1\right) + 1 \right\} \end{aligned} \qquad \text{\_\_\_\_\_\_\_\_(5.3.1.3)}$$

Where $I(x;p) \;=\; (1/\Gamma p) \int_0^x e^{-x} x^{p-1}\, dx$ refers to the standard incomplete gamma function and

$$I_1 = \int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} \left[\left(2t^2/n_1\, \phi\, \chi^2\right) - \left(2t/\,\chi^2\right) + \phi\right]^p \left(1/\Gamma n_1\right) e^{-t}\, t^{n_1-1}\, dt$$

$$I_2 = \int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} p \ln\left[\left(2t^2/n_1\, \phi\, \chi^2\right) - \left(2t/\,\chi^2\right) + \phi\right] \left(1/\Gamma n_1\right) e^{-t}\, t^{n_1-1}\, dt$$

$$I_3 = \int_0^{\frac{x_1^2\phi}{2}} \int_0^{\infty} \frac{1}{(\Gamma n_1)(\Gamma n_2)(n_1+n_2)^p} \cdot (t_1+t_2)^p\, e^{-t_1}\, t_1^{n_1-1}\, e^{-t_2}\, t_2^{n_2-1}\, dt_1\, dt_2$$

$$I_4 = \int_{\frac{x_2^2\phi}{2}}^{\infty} \int_0^{\infty} \frac{1}{(\Gamma n_1)(\Gamma n_2)(n_1+n_2)^p} \cdot (t_1+t_2)^p\, e^{-t_1}\, t_1^{n_1-1}\, e^{-t_2}\, t_2^{n_2-1}\, dt_1\, dt_2$$

$$I_5 = \int_0^{\frac{x_1^2\phi}{2}} \int_0^{\infty} \frac{p}{(\Gamma n_1)(\Gamma n_2)(n_1+n_2)^p} \cdot \ln(t_1+t_2)\, e^{-t_1}\, t_1^{n_1-1}\, e^{-t_2}\, t_2^{n_2-1}\, dt_1\, dt_2$$

$$I_6 = \int_{\frac{\chi_2^2 \phi}{2}}^{\infty} \int_0^{\infty} \frac{p}{(\Gamma n_1)(\Gamma n_2)(n_1+n_2)^p} \cdot \ln(t_1+t_2) e^{-t_1} t_1^{n_1-1} e^{-t_2} t_2^{n_2-1} dt_1\, dt_2$$

5.3.2 Risk of $\hat{\theta}_{DST2}$

Again, we obtain the risk of $\hat{\theta}_{DST_2}$ under $L_E(\hat{\theta},\theta)$ given by

$$R(\hat{\theta}_{DST_2}) = E[\hat{\theta}_{DST_2} \mid L_E(\hat{\theta},\theta)]$$

$$= E\left[\left(2n_1\bar{x}_1/\theta_0 \chi^2\right)^2 (\bar{x}_1-\theta_0)+\theta_0 / \chi_1^2 < 2n_1\bar{x}_1/\theta_0 < \chi_2^2\right] \cdot p\left[\chi_1^2 < 2n_1\bar{x}_1/\theta_0 < \chi_2^2\right]$$
$$+ E\left[\bar{x}_p \mid 2n_1\bar{x}_1/\theta_0 < \chi_1^2 \cup 2n_1\bar{x}_1/\theta_0 > \chi_2^2\right] \cdot p\left[2n_1\bar{x}_1/\theta_0 < \chi_1^2 \cup 2n_1\bar{x}_1/\theta_0 > \chi_2^2\right]$$

_______(5.3.2.1)

$$= \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} \left[\left(\frac{2n_1\bar{x}_1}{\theta_0 \chi^2}\right)^2 (\bar{x}_1-\theta_0)+\theta_0 \Big/ \theta\right]^p f(\bar{x}_1)\, d\bar{x}_1$$
$$- \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} p \ln\left[\left(\frac{2n_1\bar{x}_1}{\theta_0 \chi^2}\right)^2 (\bar{x}_1-\theta_0)+\theta_0 \Big/ \theta\right] f(\bar{x}_1)\, d\bar{x}_1$$
$$- \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} f(\bar{x}_1)\, d\bar{x}_1 + \int_0^{\frac{\chi_1^2\theta_0}{2n_1}} \int_0^{\infty} \left[\left(\frac{\bar{x}_p}{\theta}\right)^p - p\ln\left(\frac{\bar{x}_p}{\theta}\right) - 1\right] f(\bar{x}_1) f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2$$
$$+ \int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty} \int_0^{\infty} \left[\left(\frac{\bar{x}_p}{\theta}\right)^p - p\ln\left(\frac{\bar{x}_p}{\theta}\right) - 1\right] f(\bar{x}_1) f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2$$

_______(5.3.2.2)

Where $f(\bar{x}_1) = (1/\Gamma n_1)\,(n_1/\theta)^{n_1} (\bar{x}_1)^{n_1-1} \exp(-n_1\bar{x}_1/\theta)$

$$f(\bar{x}_2) = (1/\Gamma n_2)\,(n_2/\theta)^{n_2}\,(\bar{x}_2)^{n_2-1}\exp(-n_2\bar{x}_2/\theta)$$

A straight forward integration of (5.3.2.2) gives:

$$R(\hat{\theta}_{DST_1}) = I_1 - I_2 - \left\{ I\left(\frac{x_2^2\phi}{2}, n_1\right) - I\left(\frac{x_1^2\phi}{2}, n_1\right)\right\} +$$

$$(I_3 + I_4) - (I_5 + I_6) - \left\{ I\left(\frac{x_1^2\phi}{2}, n_1\right) - I\left(\frac{x_2^2\phi}{2}, n_1\right) + 1\right\}$$

________(5.3.2.3)

Where

$$I_1 = \int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} \left[\left(4t^3/n_1\phi^2\left(\chi^2\right)^2\right) - \left(4t^2/\phi\left(\chi^2\right)^2\right) + \phi\right]^p (1/\Gamma n_1)\, e^{-t}\, t^{n_1-1}\, dt$$

$$I_2 = \int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} p\ln\left[\left(4t^3/n_1\phi^2\left(\chi^2\right)^2\right) - \left(4t^2/\phi\left(\chi^2\right)^2\right) + \phi\right] (1/\Gamma n_1)\, e^{-t}\, t^{n_1-1}\, dt$$

$$I_3 = \int_0^{\frac{x_1^2\phi}{2}} \int_0^{\infty} \frac{1}{(\Gamma n_1)(\Gamma n_2)(n_1+n_2)^p} \cdot (t_1+t_2)^p\, e^{-t_1}\, t_1^{\,n_1-1}\; e^{-t_2}\, t_2^{\,n_2-1}\, dt_1\, dt_2$$

$$I_4 = \int_{\frac{x_2^2\phi}{2}}^{\infty} \int_0^{\infty} \frac{1}{(\Gamma n_1)(\Gamma n_2)(n_1+n_2)^p} \cdot (t_1+t_2)^p\, e^{-t_1}\, t_1^{\,n_1-1}\; e^{-t_2}\, t_2^{\,n_2-1}\, dt_1\, dt_2$$

$$I_5 = \int_0^{\frac{x_1^2\phi}{2}} \int_0^{\infty} \frac{p}{(\Gamma n_1)(\Gamma n_2)(n_1+n_2)^p} \cdot \ln(t_1+t_2)\, e^{-t_1}\, t_1^{\,n_1-1}\; e^{-t_2}\, t_2^{\,n_2-1}\, dt_1\, dt_2$$

$$I_6 = \int_{\frac{x_2^2\phi}{2}}^{\infty} \int_0^{\infty} \frac{p}{(\Gamma n_1)(\Gamma n_2)(n_1+n_2)^p} \cdot \ln(t_1+t_2)\, e^{-t_1}\, t_1^{\,n_1-1}\; e^{-t_2}\, t_2^{\,n_2-1}\, dt_1\, dt_2$$

5.4 Relative Risks of $\hat{\theta}_{DST_i}$

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator $\bar{x}_p$ in this case. For this purpose, we obtain the risk of $\bar{x}_p$ under $L_E(\hat{\theta},\theta)$ as:

$$R_E(\bar{x}_p) = E[\bar{x}_p \mid L_E(\hat{\theta},\theta)]$$

$$= \int_0^\infty \int_0^\infty \left[\left(\frac{\bar{x}_p}{\theta}\right)^p - p\ln\left(\frac{\bar{x}_p}{\theta}\right) - 1\right] f(\bar{x}_1) f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2$$

______(5.4.1)

A straightforward integration of (5.4.1) gives

$$R_E(\bar{x}) = \left[\frac{\Gamma(n+P)}{\Gamma n\,(n^P)} - p\{\psi(n) - \ln(n)\}\right] - 1 \qquad \text{\_\_\_\_\_\_(5.4.2)}$$

Where $\psi(n) = (d/dn)\ln\Gamma n$ refers to the Euler's psi function.

Now, we define the Relative Risk of $\hat{\theta}_{DST1}$ with respect to $\bar{x}_p$ under $L_E(\hat{\theta},\theta)$ as follows:

$$RR_1 = R_E(\bar{x}_p)/R(\hat{\theta}_{DST_1}) \qquad \text{\_\_\_\_\_\_\_(5.4.3)}$$

Using (5.4.2) and (5.3.1.3) the expression for RR_1 given in (5.4.3) can be obtained;

Similarly, we define the Relative Risk of $\hat{\theta}_{DST_2}$ under $L_E(\hat{\theta},\theta)$ as follows

$$RR_2 = R_E(\bar{x}_p)/R(\hat{\theta}_{DST_2}) \qquad \text{\_\_\_\_\_\_\_(5.4.4)}$$

The expression for RR_2 given in (5.4.4) which can be obtain by using equation (5.4.2) and (5.3.2.3).

Now, it is observed that both RR_1 and RR_2 are a function of '$\emptyset$', 'n_1', 'n_2', 'α' and 'p'. To observe the behavior of the risk(s) of $\hat{\theta}_{DST_1}$ and $\hat{\theta}_{DST_2}$, we have taken several values of these viz α = 1%, 5%, 10%, (n_1, n_2) = (4,6), (4,8), (6,10), (4,12), $\emptyset$ = 0.2 (0.2) 1.6 and p = -3, -2, -1, 2, 3, 4 ; 'p' is the prime important factor and decides about the seriousness of over/under estimation in the real life situation. The recommendations regarding the applications of proposed testimators are provided as follows:

5.5 Recommendations for $\hat{\theta}_{DST_i}$

In this section we wish to compare the performance of $\hat{\theta}_{DST_1}$ and $\hat{\theta}_{DST_2}$ with respect to the best available (unbiased) estimator of $\bar{x}_p$.

(1) Taking $n_1 = 4$, $n_2 = 6$ and fixing α = 1% we have allowed the variation in 'p' which represents the degree of asymmetry. As the shrinkage factor depends on test statistics and hence on 'α'. It has been observed that the RR_1 values are higher than 1 (unity) in the whole range of ϕ, demonstrating that $\hat{\theta}_{DST_1}$ performs better than $\bar{x}_p$. For p = -3 (negative) and p = 2 (positive) its performance is 'best' however it performs better for the other values of 'p' also.

(2) It is also observed that $\hat{\theta}_{DST1}$ performs still better for $n_1 = 4$, $n_2 = 8$ ($n_2 = 2n_1$) i.e. perhaps second sample should be twice as much compared to the 1st stage sample.

(3) For α = 5% and α = 10% a similar pattern of performance is observed however the magnitude of RR_1 is highest at α = 1%.

(4) For $\alpha = 10\%$ and $n_1 = 4$, $n_2 = 6$, it observed in particular that RR_1 is highest for $p = 2$ (positive) and then followed by $p = -3$ (negative) a trend not observed earlier. However for other values of (n_1, n_2) considered here, $p = -3$ shows larger values of RR_1.

(5) In the next comparison stage, we have fixed $p = -3$ and have allowed variation in α values. Maximum gain in RR_1 is observed at $\phi = 1.0$. So, we have fixed $\phi = 1.0$ again for the whole range and for all the combination(s) of (n_1, n_2). $\hat{\theta}_{DST1}$ fairs better than the usual estimator. It is also observed that there is a minor difference in the values of RR_1 for (4, 8) and (4, 12). So again second stage sample may be chosen in this light.

(6) It is observed that (6,10) sample combination does not give better control over risk as the values of RR_1 are smaller in magnitude compared to other RR_1 values.

(7) The data set considered here is $n_1 = 4$, $n_2 = 10$ and $\phi = 0.8$ (different from $\phi = 1.0$ i.e. $\theta \neq \theta_0$) again for $\alpha = 1\%$. We have allowed the variation in the values of shape parameter 'p' and it has been observed that $\hat{\theta}_{DST1}$ dominates the usual unbiased estimator for all values of 'p' and the performance is at its best for $p = -3$.

(8) It has been observed that positive values of 'p' considered here, the maximum RR_1 values have been observed at $p = 2$, for different values of α^s. The highest values in terms of magnitude are observed at $\alpha = 1\%$ for different n_1 and n_2 combination values.

The present investigation has also considered the square of S.F. and we have proposed another testimator viz $\hat{\theta}_{DST2}$. We have also studied the behaviour of Relative risk(s) of $\hat{\theta}_{DST2}$ with respect to $\bar{x}_p$ and have computed RR_2 values, to observe the behaviour of $\hat{\theta}_{DST2}$. For all the values of (n_1, n_2), ϕ, α and p considered for RR_1, we have computed RR_2 values for the same set of values. Following observations have been made.

1) It is observed that $\hat{\theta}_{DST2}$ performs better than the usual estimator $\bar{x}_p$. For all the values considered here. However the magnitude of RR_2 values are higher than RR_1 values, indicating a **better** control over risk by the proposed estimator $\hat{\theta}_{DST2}$.
2) Almost similar recommendations as above in case of $\hat{\theta}_{DST1}$ (1 - 8) follow here also. But definitely $\hat{\theta}_{DST2}$ has better performance compared to $\hat{\theta}_{DST1}$.

CONCLUSIONS:

The present chapter studies the risk properties of double stage shrinkage testimator(s) of the scale parameter (average life) of exponential life model using General Entropy Loss Function. Two choices of the shrinkage factor have been made making it dependent on the test statistics, hence the choice of 'α' plays an important role. We conclude that a lower value of level of significance i.e. α = 1% is suitable for almost all values of 'shape' parameter of the loss function but in particular when p = -3, at α = 1% its performance best for (n_1 = 4, n_2 = 8) and similar recommendation holds for p = 2 (positive).

The 'square' of S.F. gives better control over the relative risk as has been observed by Comparing the relative risk values. So, to conclude take α = 1% square of the shrinkage factor, p = -3 or p = 2 and (n_1 = 4, n_2 = 8).

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 5.5.1.1 **Relative Risk of** $\hat{\theta}_{DST_1}$ $\alpha = 1\%$, $n_1 = 4$, $n_2 = 8$

∅	p = -3	p = -2	p = -1	p = 2	p = 3	p = 4
0.20	1.324	0.684	0.321	0.873	0.836	0.94
0.40	2.225	1.741	1.218	1.147	1.53	1.187
0.60	4.729	3.217	2.278	2.453	2.241	1.664
0.80	6.102	4.69	3.54	3.62	3.33	2.444
1.00	9.883	6.898	5.369	5.147	5.883	5.062
1.20	4.447	4.8	3.533	3.998	3.576	3.01
1.40	2.284	3.01	2.017	2.712	2.016	1.833
1.60	1.614	1.688	1.786	1.089	1.032	0.916

Table : 5.5.1.2 **Relative Risk of** $\hat{\theta}_{DST_1}$ $\alpha = 1\%$, $n_1 = 6$, $n_2 = 10$

∅	p = -3	p = -2	p = -1	p = 2	p = 3	p = 4
0.20	0.953	0.431	0.153	0.159	0.483	0.831
0.40	1.224	0.445	0.209	0.282	0.643	0.897
0.60	2.002	1.096	1.388	1.904	1.618	1.783
0.80	5.362	4.12	3.789	4.027	3.218	2.866
1.00	9.259	8.591	6.399	7.714	6.367	5.165
1.20	5.908	4.197	4.218	5.025	3.737	3.165
1.40	2.954	2.985	2.58	3.073	1.633	1.421
1.60	1.722	1.329	1.766	1.513	0.823	0.698

Table : 5.5.1.3 **Relative Risk of** $\hat{\theta}_{DST_1}$ $\alpha = 5\%$, $n_1 = 4$, $n_2 = 8$

∅	p = -3	p = -2	p = -1	p = 2	p = 3	p = 4
0.20	1.103	0.564	0.461	0.598	0.641	0.86
0.40	1.237	0.76	0.987	0.819	0.753	1.689
0.60	2.103	1.771	1.639	1.313	1.468	2.096
0.80	4.103	3.631	3.238	2.819	2.694	2.437
1.00	6.684	6.075	5.758	5.097	4.771	4.413
1.20	4.04	3.068	2.587	3.111	2.581	1.988
1.40	2.15	2.019	1.627	2.098	1.627	1.077
1.60	1.04	0.957	0.804	1.203	0.787	0.653

Table : 5.5.1.4 **Relative Risk of** $\hat{\theta}_{DST_1}$ $\alpha = 5\%$, $n_1 = 6$, $n_2 = 10$

Ø	**p = -3**	**p = -2**	**p = -1**	**p = 2**	**p = 3**	**p = 4**
0.20	0.702	0.52	0.147	0.177	0.248	0.845
0.40	0.806	0.834	0.258	0.886	0.548	1.053
0.60	1.702	1.44	1.249	1.202	1.023	1.241
0.80	2.891	2.484	2.841	2.514	2.702	2.311
1.00	5.888	5.073	4.069	4.667	3.992	3.893
1.20	3.214	2.334	2.23	2.871	2.667	2.378
1.40	1.066	0.417	1.077	1.911	1.754	1.716
1.60	0.025	0.16	0.678	0.363	0.429	0.927

Table : 5.5.1.5 **Relative Risk of** $\hat{\theta}_{DST_1}$ $\alpha = 10\%$, $n_1 = 4$, $n_2 = 6$

Ø	**p = -3**	**p = -2**	**p = -1**	**p = 2**	**p = 3**	**p = 4**
0.20	0.685	0.716	0.338	0.784	0.236	0.669
0.40	1.404	0.938	0.535	0.809	1.092	1.585
0.60	1.949	1.789	1.331	1.409	1.37	2.07
0.80	2.554	2.537	2.486	3.054	2.418	2.382
1.00	3.934	3.443	3.3	4.881	3.789	3.148
1.20	2.282	2.631	2.737	2.097	2.087	2.07
1.40	1.136	1.306	1.206	1.598	1.34	1.286
1.60	0.075	0.168	0.591	0.784	0.672	0.94

Table : 5.5.2.1 **Relative Risk of** $\hat{\theta}_{DST_2}$ $\alpha = 1\%$, $n_1 = 4$, $n_2 = 8$

Ø	**p = -3**	**p = -2**	**p = -1**	**p = 2**	**p = 3**	**p = 4**
0.20	1.154	0.626	0.311	0.739	0.683	0.939
0.40	2.047	1.697	1.213	0.815	1.399	1.084
0.60	3.742	1.212	2.277	1.715	2.221	1.785
0.80	4.696	2.893	4.553	2.974	3.468	1.866
1.00	6.42	7.955	5.443	4.653	4.094	2.279
1.20	4.318	5.692	3.556	2.173	2.259	2.267
1.40	3.548	2.514	2.147	1.516	1.416	1.441
1.60	1.491	1.63	1.882	0.868	0.768	0.763

5.6 Shrinkage Testimator for the Variance of a Normal Distribution

Let X be normally distributed with mean μ and variance σ^2, both unknown. It is assumed that the prior knowledge about σ^2 is available in the form of an initial estimate σ_0^2. We are interested in constructing an estimator of σ^2 using the sample observations and possibly the guess value σ_0^2. We define a double stage shrinkage testimator of σ^2 as follows:

1. Take a random sample x_{1i} $(i = 1,2,\_\_, n_1)$ of size n_1 from $N(\mu, \sigma^2)$ and compute $\bar{x}_1 = \frac{1}{n_1}\sum x_{1i}$, $s_1^2 = \frac{1}{n_1-1}\sum(x_{1i} - \bar{x}_1)^2$.
2. Test the hypothesis $H_0 : \sigma^2 = \sigma_0^2$ against the alternative $H_1 : \sigma^2 \neq \sigma_0^2$ at level α using the test statistic $\frac{v_1 s_1^2}{\sigma_0^2}$, which is distributed as χ^2 with $v_1 = (n_1 - 1)$ degrees of freedom.
3. If H_0 is accepted at α level of significance i.e. $x_1^2 < \frac{v_1 s_1^2}{\sigma_0^2} < x_2^2$,where x_1^2 and x_2^2 refer to lower and upper critical points of the unbiased portioning of the test statistic at a given level of significance α, take $k_1 s_1^2 + (1 - k_1)\sigma_0^2$ as the shrinkage estimator of σ^2 with shrinkage factor k_1 dependent on the test statistic.
4. If H_0 is rejected, take a second sample x_{2j} $(j = 1,2,\_\_, n_2)$ of size $n_2 = (n - n_1)$ compute $\bar{x}_2 = \frac{1}{n_2}\sum x_{2j}$, $s_2^2 = \frac{1}{n_2-1}\sum(x_{2j} - \bar{x}_2)^2$ and take $(v_1 s_1^2 + v_2 s_2^2)/(v_1 + v_2)$ where $v_2 = (n_2 - 1)$ as the estimator of σ^2.

To summarize, we define the double- stage shrinkage testimators $\hat{\sigma}_{DST1}^2$ and $\hat{\sigma}_{DST2}^2$ of σ^2 as follows:

$$\hat{\sigma}^2_{DST1} = \begin{cases} k\, s_1^2 + (1-k)\sigma_0^2\ , & if\ \ H_0\ \ is\ accepted \\ s_p^2 = \dfrac{(v_1 s_1^2 + v_2 s_2^2)}{(v_1 + v_2)}, & if\ H_0\ \ is\ \ rejected \end{cases}$$

Estimators of this type with k arbitrary and lying between 0 and 1 have been proposed by Katti (1962), Shah(1964), Arnold and Al-Bayyati (1970), Waikar and Katti (1971), Pandey (1979) and k being dependent on the test statistics by Waikar, Schuurman and Raghunandan (1984), Pandey, Srivastava and Malik (1988).

$$\hat{\sigma}^2_{DST2} = \begin{cases} k_1\, s_1^2 + (1-k_1)\sigma_0^2\ , & if\ \ H_0\ \ is\ accepted \\ s_p^2 = \dfrac{(v_1 s_1^2 + v_2 s_2^2)}{(v_1 + v_2)}, & if\ H_0\ \ is\ \ rejected \end{cases}$$

Where k_1 being dependent on test statistic and is given by $k_1 = \frac{v_1 s_1^2}{\sigma_0{}^2\, \chi^2}$

We have studied the risk properties of these testimators under GELF defined in section 4.1.1.

5.7 Risk of Testimators

In this section we derive the risk of proposed testimators which are defined in the previous section.

5.7.1 Risk of $\hat{\sigma}^2_{DST1}$

The risk of $\hat{\sigma}^2{}_{DST1}$ under $L_E\left(\hat{\sigma}^2, \sigma^2\right)$ is defined by

$$R(\hat{\sigma}^2{}_{DST_1}) = E[\,\hat{\sigma}^2{}_{DST_1} |\ L_E\left(\hat{\sigma}^2, \sigma^2\right)]$$

$$= E\left[ks_1^{\,2} + (1-k)\sigma_0^2 \Big/ \chi_1^2 < \frac{\upsilon_1 s_1^{\,2}}{\sigma_0^2} < \chi_2^2 \right] \cdot p\left[\chi_1^2 < \frac{\upsilon_1 s_1^{\,2}}{\sigma_0^2} < \chi_2^2 \right]$$

$$+ E\left[s_p^{\,2} \;\middle|\; \frac{\upsilon_1 s_1^{\,2}}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon_1 s_1^{\,2}}{\sigma_0^2} > \chi_2^2 \right] \cdot p\left[\frac{\upsilon_1 s_1^{\,2}}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon_1 s_1^{\,2}}{\sigma_0^2} > \chi_2^2 \right]$$

______(5.7.1.1)

$$= \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} \left[\frac{k\left(s_1^{\,2} - \sigma_0^2\right) + \sigma_0^2}{\sigma^2} \right]^p f(s_1^{\,2})\, ds_1^{\,2} - \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} p \ \ln\left[\frac{k\left(s_1^{\,2} - \sigma_0^2\right) + \sigma_0^2}{\sigma^2} \right] f(s_1^{\,2})\, ds_1^{\,2}$$

$$- \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} f(s_1^{\,2})\, ds_1^{\,2} + \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon}} \int_0^{\infty} \left[\left(\frac{s_p^{\,2}}{\sigma^2} \right)^p - p \ \ln\left(\frac{s_p^{\,2}}{\sigma^2} \right) - 1 \right] f(s_1^{\,2}) f(s_2^{\,2})\, ds_1^{\,2} ds_2^{\,2}$$

$$+ \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon}}^{\infty} \int_0^{\infty} \left[\left(\frac{s_p^{\,2}}{\sigma^2} \right)^p - p \ \ln\left(\frac{s_p^{\,2}}{\sigma^2} \right) - 1 \right] f(s_1^{\,2}) f(s_2^{\,2})\, ds_1^{\,2} ds_2^{\,2}$$

______(5.7.1.2)

Where $$f(s_1^{\,2}) = \frac{1}{2^{\upsilon_1/2}\, \Gamma\left(\upsilon_1/2\right)} \left(s_1^{\,2}\right)^{\frac{\upsilon_1}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_1 s_1^{\,2}}{\sigma^2}\right)} ds_1^{\,2}$$

$$f(s_2^{\,2}) = \frac{1}{2^{\upsilon_2/2}\, \Gamma\left(\upsilon_2/2\right)} \left(s_2^{\,2}\right)^{\frac{\upsilon_2}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_2 s_2^{\,2}}{\sigma^2}\right)} ds_2^{\,2}$$

Straight forward integration of (5.7.1.2) gives

$$R\left(\hat{\sigma}^2_{DST1}\right)=\left(\frac{\sigma^2}{\nu_1}\right)^{\nu_1/2}\left(\frac{\sigma^2}{\nu_2}\right)^{\nu_1/2}\begin{bmatrix} I_1 - I_2 - \left\{ I\left(\chi_2^2\lambda,\frac{\nu_1}{2}\right) - I\left(\chi_1^2\lambda,\frac{\nu_1}{2}\right)\right\} \\ -\left[I\left(\chi_1^2\lambda,\frac{\nu_1}{2}\right) - I\left(\chi_2^2\lambda,\frac{\nu_1}{2}\right)+1\right] \\ +\frac{\Gamma\left(\frac{\nu_1}{2}+p\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\left(\frac{\nu_1}{2}\right)^p}\left[I\left(\chi_1^2\lambda,\frac{\nu_1}{2}+p\right) - I\left(\chi_2^2\lambda,\frac{\nu_1}{2}+p\right)+1\right] \\ - I_3 - I_4 \end{bmatrix}$$

_____(5.7.1.3)

Where $I(x;p) = (1/\Gamma p)\int_0^x e^{-x}x^{p-1}\,dx$ refers to the standard incomplete gamma function, $\lambda = \frac{\sigma_0^2}{\sigma^2}$, and

$$I_1 = \frac{1}{2^{\frac{\nu_1}{2}}\,\Gamma\left(\frac{\nu_1}{2}\right)}\int_{x_1^2\lambda}^{x_2^2\lambda}\left(k\left(\frac{t_1}{\nu_1}-\lambda\right)+\lambda\right)^p e^{-\left(\frac{1}{2}\right)t_1}\,t_1^{\frac{\nu_1}{2}-1}\,dt_1$$

$$I_2 = \frac{p}{2^{\frac{\nu_1}{2}}\,\Gamma\left(\frac{\nu_1}{2}\right)}\int_{x_1^2\lambda}^{x_2^2\lambda} ln\left(k\left(\frac{t_1}{\nu_1}-\lambda\right)+\lambda\right) e^{-\left(\frac{1}{2}\right)t_1}\,t_1^{\frac{\nu_1}{2}-1}\,dt_1$$

$$I_3 = \frac{p}{2^{\nu_1/2}\,\Gamma\left(\frac{\nu_1}{2}\right)}\int_0^{x_1^2\lambda} ln\left(\frac{t_1}{\nu_1}\right) e^{-\left(\frac{1}{2}\right)t_1}\,t^{\frac{\nu_1}{2}-1}\,dt_1$$

$$I_4 = \frac{p}{2^{\nu_1/2}\,\Gamma\left(\frac{\nu_1}{2}\right)}\int_{x_2^2\lambda}^{\infty} ln\left(\frac{t_1}{\nu_1}\right) e^{-\left(\frac{1}{2}\right)t_1}\,t^{\frac{\nu_1}{2}-1}\,dt_1$$

5.7.2 <u>Risk of</u> $\hat{\sigma}^2_{DST2}$

Again, we obtain the risk of $\hat{\sigma}^2{}_{DST_2}$ under $L_E\left(\hat{\theta},\theta\right)$ with respect to s_p^2, given by

$$R(\hat{\sigma}^2{}_{DST_2}) = E[\hat{\sigma}^2{}_{DST_2}\big| L_E\left(\hat{\theta},\theta\right)]$$

$$= E\left[k_1 s_1{}^2 + (1-k_1)\sigma_0^2 \Big/ \chi_1^2 < \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} < \chi_2^2\right]\cdot p\left[\chi_1^2 < \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} < \chi_2^2\right]$$

$$+ E\left[s_p{}^2 \left| \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} > \chi_2^2\right.\right]\cdot p\left[\frac{\upsilon_1 s_1{}^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} > \chi_2^2\right]$$

_____(5.7.2.1)

$$= \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} \left[\frac{\frac{\upsilon_1 s_1^2}{\sigma_0^2\chi^2}\left(s_1^2-\sigma_0^2\right)+\sigma_0^2}{\sigma^2}\right]^p f(s_1^2)\,ds_1^2 - \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} p\,\ln\left[\frac{\frac{\upsilon_1 s_1^2}{\sigma_0^2\chi^2}\left(s_1^2-\sigma_0^2\right)+\sigma_0^2}{\sigma^2}\right] f(s_1^2)\,ds_1^2$$

$$-\int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} f(s_1^2)\,ds_1^2 + \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon}}\int_0^{\infty}\left[\left(\frac{s_p^2}{\sigma^2}\right)^p - p\ln\left(\frac{s_p^2}{\sigma^2}\right)-1\right] f(s_1^2)\,f(s_2^2)\,ds_1^2 ds_2^2$$

$$+\int_{\frac{\chi_2^2\sigma_0^2}{\upsilon}}^{\infty}\int_0^{\infty}\left[\left(\frac{s_p^2}{\sigma^2}\right)^p - p\ln\left(\frac{s_p^2}{\sigma^2}\right)-1\right] f(s_1^2)\,f(s_2^2)\,ds_1^2\,ds_2^2$$

_______(5.7.2.2)

Where $f(s_1^2) = \dfrac{1}{2^{\upsilon_1/2}\,\Gamma\left(\upsilon_1/2\right)}\left(s_1^2\right)^{\frac{\upsilon_1}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_1 s_1^2}{\sigma^2}\right)} ds_1^2$

and $f(s_2^2) = \dfrac{1}{2^{\upsilon_2/2}\,\Gamma\left(\upsilon_2/2\right)}\left(s_2^2\right)^{\frac{\upsilon_2}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_2 s_2^2}{\sigma^2}\right)} ds_2^2$

Straight forward integration of (5.7.2.2) gives

$$R\left(\hat{\sigma}^2_{DST2}\right) = \left(\frac{\sigma^2}{\nu_1}\right)^{\nu_1/2}\left(\frac{\sigma^2}{\nu_2}\right)^{\nu_1/2}\left[\begin{array}{c} \mathrm{I}_1 - \mathrm{I}_2 - \left\{\mathrm{I}\left(\chi_2^2\lambda,\frac{\nu_1}{2}\right) - \mathrm{I}\left(\chi_1^2\lambda,\frac{\nu_1}{2}\right)\right\} \\ -\left[\mathrm{I}\left(\chi_1^2\lambda,\frac{\nu_1}{2}\right) - \mathrm{I}\left(\chi_2^2\lambda,\frac{\nu_1}{2}\right)+1\right] \\ +\frac{\Gamma\left(\frac{\nu_1}{2}+p\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\left(\frac{\nu_1}{2}\right)^p}\left[\mathrm{I}\left(\chi_1^2\lambda,\frac{\nu_1}{2}+p\right) - \mathrm{I}\left(\chi_2^2\lambda,\frac{\nu_1}{2}+p\right)+1\right] \\ -\mathrm{I}_3 - \mathrm{I}_4 \end{array}\right]$$

_____(5.7.2.3)

Where $I(x;p) = (1/\Gamma p)\int\limits_0^x e^{-x}x^{p-1}\,dx$ refers to the standard incomplete gamma function, $\lambda = \frac{\sigma_0^2}{\sigma^2}$, and

$$I_1 = \frac{1}{2^{v_1/2}\,\Gamma\left(\frac{v_1}{2}\right)} \int_{x_1^2\lambda}^{x_2^2\lambda} \left(\frac{{t_1}^2}{v_1\lambda\chi^2} - \frac{t_1}{\chi^2} + \lambda\right)^p e^{-\left(\frac{1}{2}\right)t_1}\, t_1^{\frac{v_1}{2}-1}\, dt_1$$

$$I_2 = \frac{p}{2^{v_1/2}\,\Gamma\left(\frac{v_1}{2}\right)} \int_{x_1^2\lambda}^{x_2^2\lambda} ln\left(\frac{{t_1}^2}{v_1\lambda\chi^2} - \frac{t_1}{\chi^2} + \lambda\right) e^{-\left(\frac{1}{2}\right)t_1}\, t_1^{\frac{v_1}{2}-1}\, dt_1$$

$$I_3 = \frac{p}{2^{v_1/2}\,\Gamma\left(\frac{v_1}{2}\right)} \int\limits_0^{x_1^2\lambda} ln\left(\frac{t_1}{v_1}\right) e^{-\left(\frac{1}{2}\right)t_1}\, t_1^{\frac{v_1}{2}-1}\, dt_1$$

$$I_4 = \frac{p}{2^{v_1/2}\,\Gamma\left(\frac{v_1}{2}\right)} \int_{x_2^2\lambda}^{\infty} ln\left(\frac{t_1}{v_1}\right) e^{-\left(\frac{1}{2}\right)t_1}\, t_1^{\frac{v_1}{2}-1}\, dt_1$$

5.8 Relative Risk of $\hat{\sigma}^2_{DSTi}$

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator s_p^2 in this case. For this purpose, we obtain the risk of s_p^2 under $L_E\left(\hat{\sigma}^2, \sigma^2\right)$ as:

$$R_E({s_p}^2) = E[{s_p}^2 \,|\, L(\hat{\sigma}^2, \sigma^2)]$$

$$= \int\limits_0^\infty \int\limits_0^\infty \left[\left({s_p}^2/\sigma^2\right)^p - p\ln\left({s_p}^2/\sigma^2\right) - 1\right] f({s_1}^2)\, f({s_2}^2)\; d{s_1}^2 d{s_2}^2$$

______(5.8.1)

Where $f({s_1}^2) = \frac{1}{2^{\upsilon_1/2}\,\Gamma\left(\upsilon_1/2\right)} \left({s_1}^2\right)^{\frac{\upsilon_1}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_1 {s_1}^2}{\sigma^2}\right)} d{s_1}^2$

and $$f(s_2^2) = \frac{1}{2^{\upsilon_2/2}\,\Gamma\left(\upsilon_2/2\right)} \left(s_2^2\right)^{\frac{\upsilon_2}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_2 s_2^2}{\sigma^2}\right)} ds_2^2$$

A straightforward integration of (5.8.1) gives

$$R_E(s_p^2) = \left[\frac{\Gamma\left(\frac{\nu_1}{2}+P\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\left(\frac{\nu_1}{2}\right)^p} - p\left\{\psi\left(\frac{\nu_1}{2}\right) - \ln\left(\frac{\nu_1}{2}\right)\right\}\right] - 1 \qquad \text{\_\_\_\_\_\_(5.8.2)}$$

Where $\psi(n) = \left(d/dn\right)\ln\Gamma n$ refers to the Euler's psi function.

Now, we define the Relative Risk of $\hat{\sigma}^2_{DST_i}$, $i = 1,2$ with respect to s^2 under $L\left(\hat{\sigma}^2, \sigma^2\right)$ as follows:

$$RR_1 = \frac{R_E(\mathrm{s_p}^2)}{R(\hat{\sigma}^2_{DST1})} \qquad \text{\_\_\_\_\_\_(5.8.3)}$$

Using (5.8.2) and (5.7.1.3) the expression for RR_1 given in (5.8.3) can be obtained; it is observed that RR_1 is a function of 'λ', '(ν_1, ν_2)', 'α', 'k' and 'p'. In order to study the risk behaviour of $\hat{\sigma}^2_{DST1}$ we have considered the following values of these quantities. k = 0.2 (0.2) 1.0, λ = 0.2 (0.2) 2.0, p = -3, -2.5, -2.0, -1.5, -1.0, 1.0 and 1.5, α = 1% and 0.1%, (ν_1, ν_2) = (5,5), (5,8), (5,10), (5,12).

Finally, we define the Relative Risk of $\hat{\sigma}^2_{DST_2}$ by

$$RR_2 = \frac{R_E(\mathrm{s_p}^2)}{R(\hat{\sigma}^2_{DST2})} \qquad \text{\_\_\_\_\_\_(5.8.4)}$$

The expression for RR_2 is given by (5.8.4) can be obtained by using (5.8.2) and (5.7.2.3). Again we observed that RR_2 is a function of 'λ', '(ν_1, ν_2)', 'α' and 'p'. We have considered same values of these as in case of RR_1 not 'k'. i.e. λ = 0.2

(0.2) 2.0, p = -3, -2.5, -2.0, -1.5, -1.0, 1.0 and 1.5, α = 1% and 0.1% , (ν_1, ν_2) = (5,5), (5,8), (5,10), (5,12).

5.9 Recommendations for $\hat{\sigma}^2_{DSTi}$

In this section we wish to compare the performance of $\hat{\sigma}^2_{DST_1}$ and $\hat{\sigma}^2_{DST_2}$ with respect to the best available (unbiased) estimator of σ^2.

5.9.1 Recommendations for $\hat{\sigma}^2_{DST1}$

There will be several tables of RR_1 , some of these tables are assembled at the end of the chapter. Recommendations for the use of $\hat{\sigma}^2_{DST1}$ are as follows:

1. For (ν_1, ν_2) = (5, 5), α = 1% the following table provides the effective ranges of 'λ' for different choice of 'k' (shrinkage factor) values. Various degrees of asymmetries are also presented.

k	λ	p
0.2	$0.6 \le \lambda \le 2.0$	p = -3 to -1.5
0.4	$0.6 \le \lambda \le 2.0$	p = -3 to -1.5
0.6	$0.8 \le \lambda \le 1.6$	p = -1
0.8	$0.8 \le \lambda \le 1.4$	p = 1 & 1.5

   From the above table it is observed that the range of 'λ' decreases as 'k' increases and it remains true for extreme negative and positive values of 'p'.

2. As (ν_1, ν_2) change i.e. (5,8), (5,10) the values of RR_1 also change in their magnitude but still higher than unity. A high value of ν_2 is not recommended. In this case, also for $0.2 \le k \le 0.8$ the effective range of 'λ' varies slightly as in the above table as for p = -3 it is $0.6 \le \lambda \le 2.0$ for k = 0.2 where as for k = 0.8 it becomes $1.0 \le \lambda \le 2.0$ for p = 1.5 .

3. Next, we have considered $\alpha = 0.1\%$ as it is reported by several authors that shrinkage testimators perform better for smaller level of significance. RR_1 values obtained for this choice of 'α' are better than those obtained for earlier value of 'α' as their magnitude is higher. A higher value of relative risk indicates better performance of the proposed estimator.
4. The effective ranges of 'λ' are more or less the same obtained previously i.e. for p = -3 it is $0.6 \leq \lambda \leq 2.0$ and for p = +1 it is $0.8 \leq \lambda \leq 1.6$ However as mentioned above the numerical values are larger.
5. As (ν_1, ν_2) change to (5,8) and (5,10) the RR_1 values are better in the range of 0.6 to 2.0 for 'λ' when p is upto -1.75, however 'λ' range becomes 0.8 to 1.8 for p = -1.5 and -1.0. This range reduces further to 0.8 to 1.6 for both the positive values of 'p'.

5.9.2 Recommendation for $\hat{\sigma}^2_{DST2}$

There will be several tables of RR_2 some of these are assembled at the end of the chapter. The recommendations are as follows:

1. For α = 1% and all the negative values of 'p' i.e. -3 upto -1.5 $\hat{\sigma}^2_{DST2}$ performs better than s_p^2 for fairly large range of λ i.e. $0.6 \leq \lambda \leq 2.0$. However for p = -1 this shrinks and it becomes $0.8 \leq \lambda \leq 1.6$. For p = +1 and 1.5 the values of

   RR_2 are better i.e. greater than unity for a range of 'λ' i.e. $0.8 \leq \lambda \leq 1.6$ for p = +1 however for p = 1.5 it becomes $0.8 \leq \lambda \leq 1.4$ it reduces very slightly. So, $\hat{\sigma}^2_{DST2}$ can be considered for various degrees of positive / negative asymmetry. This behaviour is observed for $(\nu_1, \nu_2) = (5,5)$.
2. As we have considered another data sets for (ν_1, ν_2) it is observed that as ν_2 increases i.e. (5, 8), (5,10) etc. still $\hat{\sigma}^2_{DST2}$ behaves nicely for different

positive / negative values of 'p'. But it is observed that the performance is better for larger negative values of 'p' as compared to positive values of 'p'. Further, it is noted that the magnitude of RR_2 values decrease as ν_2 increases. However it does not change the effective ranges of 'λ' i.e. again for p = -3 it is $0.6 \leq \lambda \leq 2.0$ which reduces by 0.2 units as 'p' changed from -3 to -1 but even for p = -1, it is $0.6 \leq \lambda \leq 1.6$. For much higher values of ν_2 i.e. $\nu_2 = 12$ and more the performance is not very good.

3. Next we reduce 'α' further to $\alpha = 0.1\%$ then still better values of RR_2 are obtained in the sense that they are higher in magnitude as compared to those obtained for $\alpha = 1\%$.
4. The effective ranges of 'λ' are more or less same as obtained previously i.e. for p = -3 it is $0.6 \leq \lambda \leq 2.0$ and for p = +1 it becomes $0.8 \leq \lambda \leq 1.6$. Again, it performs better for both positive/ negative degrees of asymmetry for almost all the data set considered here. But the magnitude of RR_2 values are higher uniformly than those obtained at $\alpha = 1\%$.
5. It is recommended that use large negative value of 'p', smaller level of significance and a small sample (ν_1, ν_2).

CONCLUSIONS:

We have propose two double stage shrinkage testimator(s) for the variance of a Normal distribution viz. $\hat{\sigma}^2_{DST1}$ and $\hat{\sigma}^2_{DST2}$. It is observed that both the testimators dominate the usual unbiased estimator of σ^2 for various sample sizes, degrees of asymmetries, levels of significance and a wide range of 'λ'. It is found that the use of GEL is beneficial for those situations where underestimation is more harmful than overestimation or vice- versa. In particular for p = -3 and p = 1.0, $\alpha = 0.1\%$ and $(\nu_1, \nu_2) = (5,5)$ both the testimator(s) perform at their best. However for other values also the performance is satisfactory. So, it is recommended take smaller sample sizes, smaller level of significance for both positive and negative values of degrees of asymmetry. In particular $\hat{\sigma}^2_{DST2}$ may be preferred as it removes the arbitraryness in the choice of shrinkage factor. So, it can be mentioned that shrinkage testimators perform better under GELF.

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 5.9.1.1 **Relative Risk of** $\hat{\sigma}^2{}_{DST_1}$ $\alpha = 0.1\%$,(ν_1, ν_2) = (5,5), p = -3

λ	k = 0.2	k = 0.4	k = 0.6	k = 0.8
0.20	0.129	0.179	0.192	0.149
0.40	0.598	0.669	0.564	0.337
0.60	2.371	1.989	1.272	0.596
0.80	4.35	4.172	2.479	0.936
1.00	7.854	6.174	5.214	1.365
1.20	5.518	4.561	4.974	2.173
1.40	4.134	4.009	3.873	2.43
1.60	3.264	3.496	3.612	1.966
1.80	2.854	2.903	2.845	1.426
2.00	2.03	1.868	1.981	1.237

Table : 5.9.1.2 **Relative Risk of** $\hat{\sigma}^2_{DST_1}$ $\alpha = 1\%$,(ν_1, ν_2) = (5,5), p = -3

λ	k = 0.2	k = 0.4	k = 0.6	k = 0.8
0.20	0.149	0.19	0.196	0.153
0.40	0.596	0.646	0.551	0.346
0.60	2.024	1.756	1.198	0.611
0.80	5.209	3.658	2.122	0.94
1.00	6.541	4.929	3	1.301
1.20	4.996	4.615	3.434	1.65
1.40	3.588	3.75	3.372	1.93
1.60	2.712	2.998	3.049	2.104
1.80	2.176	2.464	2.69	2.176
2.00	1.823	2.085	2.367	2.167

Table : 5.9.2.1 **Relative Risk of** $\hat{\sigma}^2_{DST_2}$ $\alpha = 1\%$, (ν_1, ν_2) = (5,5)

λ	p = -3	p = -2.5	p = --2	p = -1.5	p = -1.0	P = 1	P = 1.5
0.20	0.265	0.263	0.263	0.246	0.182	0.172	0.282
0.40	0.874	0.71	0.594	0.481	0.326	0.279	0.426
0.60	2.409	1.917	1.528	1.199	0.839	0.647	0.828
0.80	4.943	4.708	4.041	3.324	2.612	1.629	1.6
1.00	5.67	6.903	7.338	7.522	5.091	4.169	2.647
1.20	4.464	5.371	5.886	6.642	6.524	5.442	2.541
1.40	3.3	3.597	3.627	3.772	3.394	2.829	1.547
1.60	2.529	2.548	2.399	2.327	3.175	1.43	0.899
1.80	2.041	1.942	1.745	1.617	1.972	0.844	0.566
2.00	1.715	1.563	1.364	1.226	1.401	0.561	0.387

Table : 5.9.2.2 **Relative Risk of** $\hat{\sigma}^2_{DST_2}$ $\alpha = 0.1\%$, (ν_1, ν_2) = (5,5)

λ	p = -3	p = -2.5	p = --2	p = -1.5	p = -1.0	P = 1	P = 1.5
0.20	0.235	0.234	0.236	0.226	0.176	0.174	0.278
0.40	0.858	0.697	0.6	0.52	0.409	0.379	0.507
0.60	2.857	2.144	1.711	1.415	1.146	0.9	0.977
0.80	3.38	5.283	3.631	4.47	3.602	2.143	1.984
1.00	6.92	6.46	6.591	6.884	6.081	4.099	3.357
1.20	5.342	5.642	5.793	4.171	6.213	3.131	2.613
1.40	4.746	4.677	4.464	3.557	3.007	1.588	1.373
1.60	3.506	3.577	2.72	2.12	1.749	0.887	0.768
1.80	2.351	2.554	1.902	1.458	1.179	0.562	0.482
2.00	1.661	1.982	1.453	1.099	0.876	0.392	0.33