Chapter 2

One Sample Shrinkage Testimators under Asymmetric Loss Function

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ONE SAMPLE SHRINKAGE TESTIMATORS UNDER ASYMMETRIC LOSS FUNCTION

2.1 Introduction

The present chapter deals with one sample shrinkage testimators under Asymmetric Loss Function (ASL) for single parameter Exponential distribution and Normal distribution.

2.1.1 Exponential Distribution

Exponential distribution plays an important part in life testing problems. For a situation where the failure rate appears to be more or less constant, the Exponential distribution would be an adequate choice.

Exponential distribution also occurs in several other contexts, such as the waiting time problems. Maguire, Pearson and Wynn (1952) studied mine accidents and showed that time intervals between accidents follow Exponential distribution.

Exponential is a very interesting continuous type distribution due to its being endowed with the Markovian character of having 'complete lack of memory'. Its importance is stressed by Epstein (1961) by saying that the Exponential distribution occupies as commanding a position in life-testing, fatigue testing and other types of destructive test situations as does the Normal distribution in other areas of statistics. It may be defined as a special case of Gamma or Weibull distribution. Situations such as sampling from the Income-distribution, waiting time for telephonic conversation or waiting time for scooter services etc. can also be modeled by Exponential distribution.

In the estimation of reliability function use of symmetric loss function may be in appropriate as has been recognized by Canfiled (1970). Overestimate of reliability function or average failure time is usually much more serious than underestimate of reliability function or mean failure time. Also, an underestimate of the failure rate results in more serious consequences than an overestimate of the failure rate. For example, in the recent disaster of the space shuttle (Ref: Basu and Ebrahimi (1991)) the management underestimated the failure rate and therefore overestimated the reliability of solid-fuel rocket booster.

2.1.2 Normal Distribution

The Normal (or Gaussian) distribution is often used as a first approximation to describe real-valued random variables that tend to cluster around a single mean value. Normal distribution is commonly encountered in practice, and is used throughout statistics, natural sciences as a simple model for complex phenomenon.

The Normal distribution plays an important role in both the application and inferential statistics. In modeling applications, the normal curve is an excellent approximation to the frequency distributions of observations taken on a variety of variables and as a limiting form of various other distributions. Many psychological measurements and physical phenomena can be approximated well by the Normal distribution. In addition, there are many applications of the Normal distribution in engineering. One application deals with analysis of items which exhibit failure due to wear, such as mechanical devices. Other applications are, the analysis of the variation of component dimensions in manufacturing, modeling global irradiation data, and the intensity of laser light, and so on. Indeed the wide application and occurrence of the Normal distribution in life testing and reliability problems are a wonder. In the context of reliability problems and life testing, a number of failure time data have been examined and it was shown that the Normal distribution give quite a good fit for the most cases.

In the estimation of a parameter sometimes there exists in certain situations some prior information about the parameter which one would like to utilize in order to get a better estimator (say in the sense of efficiency). This prior information could be either in the form of an initial guessed value or an interval in which the parameter lies (Thompson 1968 a, b) or a relation between the parameter e.g. Coefficient of Variation, Kurtosis (Khan 1968, Searles 1964). In all these cases no apriori distribution of the parameter is assumed.

According to Thomson sometimes there is a natural origin say θ_0 of the parameter θ and one would like the MVUE $\hat{\theta}$ of the parameter θ to move it close to θ_0. This leads to shrinkage estimator of θ which performs better (in the sense of smaller mean square error) than $\hat{\theta}$ in the neighbourhood of θ_0. Searles (1974), Pandey and Singh (1977) and others have proposed such estimators utilizing guess value(s) of the parameter coupled with sample observations. In proposing shrinkage estimators the available prior information is always used along with the sample observations. However, if we do not want to use it, indiscriminately, we may decide to use it or not on the evidence of a test of significance. This gives us what is known a preliminary test estimator, pre-test estimator or a testimator. The pre-test estimator or a testimator has two components viz. : (i) when the outcome of the test of significance results in acceptance of the hypothesis $H_0: \theta = \theta_0$, then we use θ_0 along with sample observations which leads to a shrinkage testimator and (ii) the minimum variance unbiased estimator or the minimum mean square error estimator, when the hypothesis is rejected.

Mathematically, a Testimator of the parameter θ is defined as follows

$$\hat{\theta}_{ST} = \begin{cases} k\theta_0 + (1-k)\hat{\theta} \;\; or \;\; \hat{\theta}_s\,, & if\ H_0\ is\ accepted \\ \hat{\theta} & , if\ H_0\ is\ rejected \end{cases} \qquad \text{\_\_\_\_\_(2.1.1)}$$

$\hat{\theta}_{ST}$ is the shrinkage estimator of θ with shrinkage factor k ($0 \leq k \leq 1$) and $\hat{\theta}$ is the best estimator of θ.

In the present chapter we have considered shrinkage testimators for (i) Scale parameter of an Exponential distribution, (ii) variance of a Normal distribution, and studied their risk properties. In all these cases it has been assumed that we are given an initial estimate (or guess) of the parameter and a single random sample of size n from the underlying populations. The salient feature of the proposed testimators is that the arbitrariness in the choice of the shrinkage factors has been removed by making it dependent on the test statistics.

In section 2.2 we have proposed four different testimators for the parameter θ (mean life time) of the Exponential distribution and we have studied the risk properties of all the four shrinkage testimators under Asymmetric Loss Function. Section 2.3 deals with the derivation of the risk(s) of these four estimators. Section 2.4 deals with the relative risk(s) of these four estimators. Section 2.5 concludes with the comparison of UMVUE and the proposed shrinkage testimators in terms of their relative risks. Suggestions for the choice of shrinkage factor, level of significance and degrees of asymmetry have been made.

In section 2.6 we have proposed the two different shrinkage testimators for the variance of a Normal distribution and we have studied the risk properties of these two shrinkage testimators under Asymmetric Loss Function. Section 2.7 deals with the derivation of the risk(s) of these two estimators. Section 2.8 deals with the relative risk(s) of these two estimators. Section 2.9 concludes with the comparison of UMVUE and the proposed shrinkage testimatiors in terms of their relative risks. Further in the same section a suggestion for the choice of shrinkage factor, level of significance, degrees of asymmetry have been made.

ASYMMETRIC LOSS FUNCTIONS

The loss function $L(\hat{\theta}, \theta)$ provides a measure of financial consequences arising from a wrong estimate of the unknown quantity θ. As in many real life situations, particularly in insurance claims, estimating any health statistics parameter the over-estimation and under-estimation are having different impacts. So giving 'equal' importance to these as the squared error loss function (SELF) does, may not be appropriate. Several authors such as canfield (1970), zellner (1986), Basu and Ebrahimi (1991), Srivastava (1996), Srivastava and Tanna (2001), Srivstava and Shah (2010) and others have demonstrated the superiority of the Asymmetric Loss Functions, over squared error loss functions in several contexts.

A useful Asymmetric Loss Function known as LINEX loss function was introduced by Varian (1975), extended by Zellner (1986) is given by

$$L(\Delta) = b[e^{a\Delta} - a\Delta - 1] \ , a \neq 0, \ b > 0 \ \text{ where } \Delta = \left(\frac{\hat{\theta}}{\theta} - 1\right) \qquad \_\_\_(2.1.2)$$

The sign and magnitude of 'a' represents the direction and degree of asymmetry respectively. Positive values of 'a' are suggested for situations where overestimation is more serious than the under estimation, while negative values of 'a' are recommended in reverse situations. 'b' is constant of proportionality. $L(\Delta)$ rises exponentially when $\Delta < 0$ and almost linearly when $\Delta > 0$. Hence, the loss function defined by (2.1.2) is known as LINEAR EXPONENTIAL (LINEX) loss function. 'b' is the factor of proportionality.

2.2 Shrinkage Testimator(s) for Scale Parameter of an Exponential Distribution.

Let X: $(x_1, x_2 \ldots\ldots, x_n)$ have the distribution

$$f(x;\theta) = \begin{cases} \frac{1}{\theta}\exp(-x/\theta), & x \geq 0,\ \theta > 0 \\ 0 & ,\ otherwise \end{cases}, \qquad \text{\_\_\_\_\_(2.2.1)}$$

It is assumed that the prior knowledge about θ is available in the form of an initial estimate θ_0. We are interested in considering an estimator of θ possibly using the information about θ and the sample observations $x_1, x_2 \ldots\ldots, x_n$ from (2.2.1). We then propose a testimator of θ which can be described as follows:

1. Compute the sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, which is the best estimator of θ in the absence of any information about θ. Actually it is UMVUE.
2. Test the hypothesis $H_0 : \theta = \theta_0$ against the two sided alternative $H_1 : \theta \neq \theta_0$ at level α using the test statistic $\frac{2n\bar{x}}{\theta_0}$ which follows χ^2 – distribution with 2n degrees of freedom.
3. If H_0 is accepted, i.e., $\chi_1^2 < \frac{2n\bar{x}}{\theta_0} < \chi_2^2$, where χ_1^2 and χ_2^2 are the lower and upper points of χ^2 – distribution with 2n degrees of freedom at a given level of significance, use the conventional shrinkage estimator $\hat{\theta}_{ST}$ with shrinkage factor k ; otherwise, ignore θ_0 and use $\bar{x}$, when the hypothesis H_0 is rejected.

The shrinkage testimator $\hat{\theta}_{ST1}$ of θ is defined as:

$$\hat{\theta}_{ST1} = \begin{cases} k\bar{x} + (1-k)\theta_0 & ,\ if\ H_0\ is\ accepted \\ \bar{x} & ,\ if\ H_0\ is\ rejected \end{cases} \qquad \text{\_\_\_\_\_(2.2.2)}$$

Estimators of this type with 'k' arbitrary ($0 \leq k \leq 1$) have been defined and studied in different contexts by Bhattacharya and Srivastava (1974), Hogg (1974), Panday and Shah (1983).

We observe that 'k' defined in (2.2.2) can take any value between '0' and '1'. We know that the test statistic for testing $H_0: \theta = \theta_0$ against the two sided alternative $H_1: \theta \neq \theta_0$ at level α is given by $\frac{2n\bar{x}}{\theta_0}$ which follows χ^2 – distribution with 2n degrees of freedom. Pandey and Srivastava (1987) and others have proposed shrinkage testimator where the arbitrariness in the choice of shrinkage factor has been removed by making it dependent on the test statistics. Waiker (1984) at el. have proposed and studied the properties of shrinkage testimator of the parameter of Exponential distribution.

Now we propose a shrinkage testimator in which the shrinkage factor depends on the test statistics.

The shrinkage testimator $\hat{\theta}_{ST2}$ of θ is defined as:

$$\hat{\theta}_{ST2} = \begin{cases} \left(\frac{2n\bar{x}}{\theta_0 x^2}\right)\bar{x} + \left(1 - \frac{2n\bar{x}}{\theta_0 x^2}\right)\theta_0 & , \text{ if } H_0 \text{ is accepted} \\ \bar{x} & , \text{ if } H_0 \text{ is rejected} \end{cases} \qquad \text{\_\_\_\_\_(2.2.3)}$$

where $k = \frac{2n\bar{x}}{\theta_0 \chi^2}$, $x^2 = (x_2^2 - x_1^2)$. Properties of these estimators $\hat{\theta}_{ST1}$ & $\hat{\theta}_{ST2}$ have been studied by Srivastava and Shah (2010) using Asymmetric Loss Function.

In all these studies it has been shown that shrinkage testimators perform better than the conventional estimator, if k is near zero, n is small, θ_0 (the guess) is in the vicinity of θ. This motivated workers to select a shrinkage factor which approaches to zero rapidly and an obvious choice was to take the **square** of the shrinkage factor.

Thus the shrinkage testimator $\hat{\theta}_{ST3}$ of θ is defined as:

$$\hat{\theta}_{ST3} = \begin{cases} \left(\frac{2n\bar{x}}{\theta_0 x^2}\right)^2 \bar{x} + \left(1 - \left(\frac{2n\bar{x}}{\theta_0 x^2}\right)^2\right)\theta_0 & , \text{ if } H_0 \text{ is accepted} \\ \bar{x} & , \text{ if } H_0 \text{ is rejected} \end{cases} \qquad \_\_\_\_(2.2.4)$$

Where $k = \left(\frac{2n\bar{x}}{\theta_0 x^2}\right)^2$, $x^2 = (x_2^2 - x_1^2)$

It may be noted that different choices of 'k' have been taken by several authors keeping in mind that it should lie between '0' and '1'. But these limits are not attained unless $\chi_1^2 = 0$ or $\chi_2^2 = \infty$. So, we propose another estimator of θ as $\hat{\theta}_{ST4}$ given by

$$\hat{\theta}_{ST4} = \begin{cases} \left(\frac{2n\bar{x}}{\theta_0 x^2} - \frac{x_1^2}{x^2}\right)\bar{x} + \left(1 + \frac{x_1^2}{x^2} - \frac{2n\bar{x}}{\theta_0 x^2}\right)\theta_0 & , \text{ if } H_0 \text{ is accepted} \\ \bar{x} & , \text{ if } H_0 \text{ is rejected} \end{cases}$$

$$\_\_\_\_\_(2.2.5)$$

where $k = \frac{2n\bar{x}}{\theta_0 x^2} - \frac{x_1^2}{x^2}$, $x^2 = (x_2^2 - x_1^2)$ with this choice of 'k' the limits '0' and '1' can actually be attained.

Pandey, Srivastava and Malik (1989) considered another choice of shrinkage factor which lies exactly between 0 and 1.

We have considered all the four different choices of the shrinkage factor(s) and proposed four different estimators.

2.3 Risk of Testimators

In this section we derive the risk of all the four testimators which are defined in the previous section.

2.3.1 Risk of $\widehat{\theta}_{ST1}$

The risk of $\hat{\theta}_{ST_1}$ under L(Δ) is defined by

$$R(\hat{\theta}_{ST_1}) = E[\hat{\theta}_{ST_1} \mid L(\Delta)]$$

$$= E\left[k\bar{x} + (1-k)\theta_0 \middle/ \chi_1^2 < \frac{2n\bar{x}}{\theta_0} < \chi_2^2\right] \cdot p\left[\chi_1^2 < \frac{2n\bar{x}}{\theta_0} < \chi_2^2\right]$$
$$+ E\left[\bar{x} \,\middle|\, \frac{2n\bar{x}}{\theta_0} < \chi_1^2 \cup \frac{2n\bar{x}}{\theta_0} > \chi_2^2\right] \cdot p\left[\frac{2n\bar{x}}{\theta_0} < \chi_1^2 \cup \frac{2n\bar{x}}{\theta_0} > \chi_2^2\right]$$

______(2.3.1.1)

$$= e^{-a} \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} e^{a\left[\frac{k(\bar{x}-\theta_0)+\theta_0}{\theta}\right]} f(\bar{x})\, d\bar{x} - a \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} \left[\frac{k(\bar{x}-\theta_0)+\theta_0}{\theta} - 1\right] f(\bar{x})\, d\bar{x} - \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\, d\bar{x}$$

$$+ e^{-a} \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} e^{a\left(\bar{x}/\theta\right)} f(\bar{x})\, d\bar{x} - a \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} \left(\frac{\bar{x}}{\theta} - 1\right) f(\bar{x})\, d\bar{x} - \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\, d\bar{x}$$

______(2.3.1.2)

Where $f(\overline{x}) = \dfrac{1}{\Gamma n}\left(\dfrac{n}{\theta}\right)^n (\bar{x})^{n-1} e^{\frac{-n\bar{x}}{\theta}}\, d\overline{x}$

Straight forward integration of (2.3.1.2) gives

$$R(\hat{\theta}_{ST_1}) = \left\{ I\left(\frac{\chi_1^2\phi}{2}, n\right) - I\left(\frac{\chi_2^2\phi}{2}, n\right) + 1 \right\} \left\{ \frac{e^{-a}}{\left(1 - a/n\right)^n} - 1 \right\} +$$
$$\left[a \left\{ I\left(\frac{\chi_2^2\phi}{2}, n+1\right) - I\left(\frac{\chi_1^2\phi}{2}, n+1\right) \right\} \right] (1-k) +$$
$$\left\{ I\left(\frac{\chi_2^2\phi}{2}, n\right) - I\left(\frac{\chi_1^2\phi}{2}, n\right) \right\} \left\{ \frac{e^{-a} e^{a\phi(1-k)}}{\left(1 - ak/n\right)^n} - ak\phi - a\phi - 1 \right\}$$
______(2.3.1.3)

Where $I(x;p) = (1/\Gamma p) \int_0^x e^{-x} x^{p-1}\, dx$ refers to the standard incomplete gamma function and $\emptyset = \frac{\theta_0}{\theta}$

2.3.2 Risk of $\widehat{\theta}_{ST2}$

The risk of $\hat{\theta}_{ST_2}$ under L(Δ) given by

$$R(\hat{\theta}_{ST_2}) = E[\hat{\theta}_{ST_2} \mid L(\Delta)]$$

$$= E\left[\frac{2n\bar{x}}{\theta_0 \chi^2} (\bar{x} - \theta_0) + \theta_0 \Big/ \chi_1^2 < \frac{2n\bar{x}}{\theta_0} < \chi_2^2 \right] \cdot p\left[\chi_1^2 < \frac{2n\bar{x}}{\theta_0} < \chi_2^2 \right]$$
$$+ E\left[\bar{x} \,\middle|\, \frac{2n\bar{x}}{\theta_0} < \chi_1^2 \cup \frac{2n\bar{x}}{\theta_0} > \chi_2^2 \right] \cdot p\left[\frac{2n\bar{x}}{\theta_0} < \chi_1^2 \cup \frac{2n\bar{x}}{\theta_0} > \chi_2^2 \right]$$
______(2.3.2.1)

$$= e^{-a} \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} e^{a\left[\frac{\left(\frac{2n\bar{x}}{\theta_0\chi^2}\right)^2 (\bar{x} - \theta_0) + \theta_0}{\theta} \right]} f(\bar{x})\, d\bar{x} \; - \; a \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} \left[\frac{\left(\frac{2n\bar{x}}{\theta_0\chi^2}\right)^2 (\bar{x} - \theta_0) + \theta_0}{\theta} - 1 \right] f(\bar{x})\, d\bar{x}$$

$$- \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\, d\bar{x} + e^{-a} \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} e^{a\left(\bar{x}/\theta\right)} f(\bar{x})\, d\bar{x} - a \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} \left(\tfrac{\bar{x}}{\theta}-1\right) f(\bar{x})\, d\bar{x} - \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\, d\bar{x}$$

_______(2.3.2.2)

Where $f(\overline{x}) = \dfrac{1}{\Gamma n}\left(\dfrac{n}{\theta}\right)^n (\bar{x})^{n-1} e^{\frac{-n\bar{x}}{\theta}}\, d\bar{x}$

A straight forward integration of (2.3.2.2) gives:

$$R(\hat{\theta}_{ST_2}) = I^* - \frac{2a(n+1)}{\phi\,(\chi^2)^2}\left\{I\left(\frac{\chi_2^2\phi}{2}, n+2\right) - I\left(\frac{\chi_1^2\phi}{2}, n+2\right)\right\} + \left(\frac{2n}{x^2}+1\right)$$
$$\left[a\left\{I\left(\frac{\chi_2^2\phi}{2}, n+1\right) - I\left(\frac{\chi_1^2\phi}{2}, n+1\right)\right\}\right] + \left\{\frac{e^{-a}}{\left(1-a/n\right)^n} - 1\right\}$$
$$\left\{I\left(\frac{\chi_1^2\phi}{2}, n\right) - I\left(\frac{\chi_2^2\phi}{2}, n\right) + 1\right\} - \left\{I\left(\frac{\chi_2^2\phi}{2}, n\right) - I\left(\frac{\chi_1^2\phi}{2}, n\right)\right\}(a\phi+1)$$

_______(2.3.2.3)

Where $I^* = e^{a(\phi-1)} \displaystyle\int_{\frac{\chi_1^2\phi}{2}}^{\frac{\chi_2^2\phi}{2}} e^{a\left[\frac{2t^2}{n\phi\chi^2} - \frac{2t}{\chi^2}\right]} \frac{1}{\Gamma n} e^{-t} t^{n-1}\, dt$

and I(x; p) as defined previously.

2.3.3 <u>Risk of</u> $\hat{\theta}_{ST3}$

The risk of $\hat{\theta}_{ST3}$ under L(Δ) defined by

$$R(\hat{\theta}_{ST_3}) = E[\hat{\theta}_{ST_3} \mid L(\Delta)]$$

$$= E\left[\left(\frac{2n\bar{x}}{\theta_0\chi^2}\right)^2\bar{x}+\left[1-\left(\frac{2n\bar{x}}{\theta_0\chi^2}\right)^2\right]\theta_0 \Big/ \chi_1^2<\frac{2n\bar{x}}{\theta_0}<\chi_2^2\right]\cdot p\left[\chi_1^2<\frac{2n\bar{x}}{\theta_0}<\chi_2^2\right]$$

$$+E\left[\bar{x}\,\middle|\,\frac{2n\bar{x}}{\theta_0}<\chi_1^2 \cup \frac{2n\bar{x}}{\theta_0}>\chi_2^2\right]\cdot p\left[\frac{2n\bar{x}}{\theta_0}<\chi_1^2 \cup \frac{2n\bar{x}}{\theta_0}>\chi_2^2\right]$$

_______(2.3.3.1)

$$= \quad e^{-a}\int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} e^{a\left[\frac{\left(\frac{2n\bar{x}}{\theta_0\chi^2}\right)^2(\bar{x}-\theta_0)+\theta_0}{\theta}\right]} f(\bar{x})\,d\bar{x}- \ a\int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}}\left[\frac{\left(\frac{2n\bar{x}}{\theta_0\chi^2}\right)^2(\bar{x}-\theta_0)+\theta_0}{\theta}-1\right] f(\bar{x})\,d\bar{x}$$

$$-\int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\,d\bar{x}+e^{-a}\int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} e^{a\left(\bar{x}/\theta\right)} f(\bar{x})\,d\bar{x} \ - a\int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}}\left(\tfrac{\bar{x}}{\theta}-1\right)f(\bar{x})\,d\bar{x} - \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\,d\bar{x}$$

_____(2.3.3.2)

Where $\quad f(\bar{x})=\frac{1}{\Gamma n}\left(\frac{n}{\theta}\right)^n(\bar{x})^{n-1}e^{\frac{-n\bar{x}}{\theta}}$

Straight forward integration of (2.3.3.2) gives

$$R(\hat{\theta}_{ST3}) = I_1^* - I_2^* + \left\{I\left(\frac{\chi_1^2\phi}{2},n\right)-I\left(\frac{\chi_2^2\phi}{2},n\right)+1\right\}\left\{\frac{e^{-a}}{\left(1-a/n\right)^n}-1\right\}$$

$$-\ a\left\{I\left(\frac{\chi_1^2\phi}{2},n+1\right)-I\left(\frac{\chi_2^2\phi}{2},n+1\right)\right\} \ - \left\{I\left(\frac{\chi_2^2\phi}{2},n\right)-I\left(\frac{\chi_1^2\phi}{2},n\right)\right\}(a+1)$$

_____(2.3.3.3)

Where $$I_1^* = e^{a\phi - a} \int_{\frac{\chi_1^2\phi}{2}}^{\frac{\chi_2^2\phi}{2}} e^{a\left[\frac{4t^3}{n\phi^2(\chi^2)^2} - \frac{4t^2}{\phi(\chi^2)^2}\right]} \frac{1}{\Gamma n} e^{-t}\, t^{n-1}\, dt$$

$$I_2^* = a \int_{\frac{\chi_1^2\phi}{2}}^{\frac{\chi_2^2\phi}{2}} \left\{\left[\left(\frac{2t}{\phi\chi^2}\right)^2\left(\frac{t}{n} - \phi\right)\right] + \phi - 1\right\} \frac{1}{\Gamma n} e^{-t}\, t^{n-1}\, dt$$

and I(x;p) refers to incomplete gamma function defined previously.

2.3.4 <u>Risk of</u> $\widehat{\theta}_{ST4}$

The risk of $\hat{\theta}_{ST4}$ under L(Δ) given by

$$R(\hat{\theta}_{ST_4}) = E[\hat{\theta}_{ST_4} \mid L(\Delta)]$$

$$= E\left[\left(\frac{2n\bar{x}}{\theta_0\chi^2} - \frac{\chi_1^2}{\chi^2}\right)\bar{x} + \left[1 + \frac{\chi_1^2}{\chi^2} - \frac{2n\bar{x}}{\theta_0\chi^2}\right]\theta_0 \Big/ \chi_1^2 < \frac{2n\bar{x}}{\theta_0} < \chi_2^2\right] \cdot p\left[\chi_1^2 < \frac{2n\bar{x}}{\theta_0} < \chi_2^2\right]$$
$$+ E\left[\bar{x} \,\middle|\, \frac{2n\bar{x}}{\theta_0} < \chi_1^2 \cup \frac{2n\bar{x}}{\theta_0} > \chi_2^2\right] \cdot p\left[\frac{2n\bar{x}}{\theta_0} < \chi_1^2 \cup \frac{2n\bar{x}}{\theta_0} > \chi_2^2\right]$$

______(2.3.4.1)

$$= e^{-a} \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} e^{a\left[\frac{\left(\frac{2n\bar{x}}{\theta_0\chi^2} - \frac{\chi_1^2}{\chi^2}\right)(\bar{x} - \theta_0) + \theta_0}{\theta}\right]} f(\bar{x})\, d\bar{x} - a \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} \left[\frac{\left(\frac{2n\bar{x}}{\theta_0\chi^2} - \frac{\chi_1^2}{\chi^2}\right)(\bar{x} - \theta_0) + \theta_0}{\theta} - 1\right] f(\bar{x})\, d\bar{x}$$

$$- \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\, d\bar{x} + e^{-a} \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} e^{a\left(\bar{x}/\theta\right)} f(\bar{x})\, d\bar{x} - a \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} \left(\frac{\bar{x}}{\theta} - 1\right) f(\bar{x})\, d\bar{x} - \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\, d\bar{x}$$

______(2.3.4.2)

Where $f(\bar{x})=\frac{1}{\Gamma n}\left(\frac{n}{\theta}\right)^{n}(\bar{x})^{n-1}e^{\frac{-n\bar{x}}{\theta}}$

A straight forward integration of (2.3.4.2) gives:

$$R(\hat{\theta}_{ST_2})=I_1^*-I_2^*+\left\{I\left(\frac{\chi_1^2\phi}{2},n\right)-I\left(\frac{\chi_2^2\phi}{2},n\right)+1\right\}\left\{\frac{e^{-a}}{\left(1-\frac{a}{n}\right)^n}-1\right\}-$$

$$a\left\{I\left(\frac{\chi_1^2\phi}{2},n+1\right)-I\left(\frac{\chi_2^2\phi}{2},n+1\right)\right\}-\left\{I\left(\frac{\chi_2^2\phi}{2},n\right)-I\left(\frac{\chi_1^2\phi}{2},n\right)\right\}(a+1)$$

______(2.3.4.3)

Where
$$I_1^*=e^{a(\phi-1)}\int_{\frac{\chi_1^2\phi}{2}}^{\frac{\chi_2^2\phi}{2}}e^{a\left[\frac{2t^2}{n\phi\chi^2}-\frac{2t}{\chi^2}-\frac{\chi_1^2t}{\chi^2n}+\frac{\chi_1^2\phi}{\chi^2}\right]}\frac{1}{\Gamma n}e^{-t}t^{n-1}\,dt$$

$$I_2^*=a(\phi-1)\int_{\frac{\chi_1^2\phi}{2}}^{\frac{\chi_2^2\phi}{2}}a\left[\frac{2t^2}{n\phi\chi^2}-\frac{2t}{\chi^2}-\frac{\chi_1^2t}{\chi^2n}+\frac{\chi_1^2\phi}{\chi^2}\right]\frac{1}{\Gamma n}e^{-t}t^{n-1}\,dt$$

and I(x;p) refers to incomplete gamma function defined previously.

2.4 Relative Risks of $\hat{\theta}_{ST_i}$

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator $\bar{x}$ in this case, which is also the UMVUE. For this purpose, we obtain the risk of $\bar{x}$ under L(Δ) as:

$R_E(\bar{x})=E[\bar{x}|L(\Delta)]$

$$=e^{-a}\int_0^{\infty}e^{a\left(\bar{x}/\theta\right)}f(\bar{x})\,d\bar{x}-a\int_0^{\infty}\left(\frac{\bar{x}}{\theta}-1\right)f(\bar{x})d\bar{x}-\int_0^{\infty}f(\bar{x})d\bar{x}$$

________(2.4.1)

A straight forward integration of (2.4.1) gives

$$R_E(\bar{x}) = \frac{e^{-a}}{(1-a/n)^n} - 1 \qquad \text{\_\_\_\_\_\_\_(2.4.2)}$$

Now, we define the Relative Risk of $\hat{\theta}_{ST_i}$ $i=1...4$ with respect to $\bar{x}$ under L(Δ) as follows

$$RR_1 = \frac{R_E(\bar{x})}{R(\hat{\theta}_{ST_1})} \qquad \text{\_\_\_\_\_\_\_(2.4.3)}$$

Using (2.4.2) and (2.3.1.3) the expression for RR_1 is given by (2.4.3). It is observed that RR_1 is a function of ϕ, n, α, k and 'a'.

Again, we define the Relative Risk of $\hat{\theta}_{ST_2}$ by

$$RR_2 = \frac{R_E(\bar{x})}{R(\hat{\theta}_{ST_2})} \qquad \text{\_\_\_\_\_\_\_(2.4.4)}$$

The expression for RR_2 is given by (2.4.4) which can be obtained by using equations (2.4.2) and (2.3.2.3).

Now, we define the Relative Risk of $\hat{\theta}_{ST_3}$ as follows

$$RR_3 = \frac{R_E(\bar{x})}{R(\hat{\theta}_{ST_3})} \qquad \text{\_\_\_\_\_\_\_(2.4.5)}$$

Using (2.4.2) and (2.3.3.3) the expression for RR_3 is given by (2.4.5).

Finally, we define the Relative Risk of $\hat{\theta}_{ST_4}$ as follows

$$RR_4 = \frac{R_E(\bar{x})}{R(\hat{\theta}_{ST_4})} \qquad \text{\_\_\_\_\_\_\_(2.4.6)}$$

Same way the RR_4 is given by (2.4.6) which can be obtained by using equations (2.4.2) and (2.3.4.3)

It is observed that RR_2, RR_3 and RR_4 are functions of ϕ, n, α, and 'a'.

2.5 Recommendations for $\hat{\theta}_{ST_i}$

In this section we provide the comparison of UMVUE and the proposed shrinkage testimators in terms of their relative risks. Recommendations regarding the applications of proposed testimators are provided in terms of the range of 'k' and '$\emptyset$'. The objective of present investigation is also to make recommendations for the degrees of asymmetry and level of significance. The following sections provide these separately for all the proposed testimators.

2.5.1 Recommendations for $\hat{\theta}_{ST_1}$

We observe that the expression for RR_1 is a function of 'k', '$\emptyset$', 'a', 'n' and 'α'. To study the behaviour of RR_1, we have taken these values as k = 0.2 (0.2)...0.8 , $\emptyset$ = 0.2 (0.2)...1.6, α = 1%, 5%, 10%, n = 5, 8, 10 and a = ±1, ±2, ±3 ,'a' is the prime important factor and decides about the seriousness of over/under estimation in the real life situation. It is observed that $\hat{\theta}_{ST_1}$ performs better than the conventional estimator for almost the whole range of k. The performance is best at k = 0.2, n = 8, for a = -1, however as 'k' increases to k = 0.4, there is a sudden change and the performance improves at a = 1 (positive) and the same trend remains for a = 2 and 3 but the range of $\emptyset$ changes. It may be stated that for smaller weights a negative value of 'a' is suggested however for higher weights positive value particularly a = 3 should be used. We have taken α = 5% and α = 10% also, it is observed that the $\hat{\theta}_{ST_1}$ still performs better for these values of α^s, but the magnitude of relative risk is maximum at α = 1% out of the three values of α, so α = 1% is the **recommended** level of significance. As regards the choice of degree of asymmetry 'a' no fixed pattern is observed for various values of 'k' i.e. for some values of 'k', positive 'a' and for some values of 'k' (particularly lower), negative values of 'a' are recommended (say a = -1 for k = 0.2). Looking at the different values of 'a' for different choice of 'k' it seems more logical to remove the arbitrariness in the choice of 'k'. $\hat{\theta}_{ST_2}$ removes this arbitrariness and our conclusions for $\hat{\theta}_{ST_2}$ are as follows:

2.5.2 Recommendations for $\hat{\theta}_{ST_2}$

There will be too many tables for varying '$\emptyset$', 'α', and 'a' all the tables are not presented here. However our recommendations based on all these computations are summarized as follows:

- For small n = 5 and for different levels of significance considered here $\hat{\theta}_{ST_2}$ performs better than the usual estimator in the whole range of $\emptyset$. However, its performance is best for a = ±3, (still better for a = 3) and α = 1%. Hence it is **recommended** to use the proposed estimator for the positive values of 'a' and small values of 'n'. Similar results hold for n = 8 and 10 however the magnitude of RR_2 is maximum for **n = 8.**
- For α = 5% and for n = 5, 8, 10 and for $0.2 \leq \emptyset \leq 1.6$, the magnitude of relative risk is still higher, i.e. usual estimator has more risk under $L(\Delta)$ compared to $\hat{\theta}_{ST_1}$. Again, $\hat{\theta}_{ST_2}$ performs better for positive values of "a", The higher magnitude of relative risk values implies better risk control in this situation, for the proposed testimator $\hat{\theta}_{ST_2}$ compared to $\bar{x}$.
- For α = 10%, rest of the findings are same, i.e., values of n considered here, range of $\emptyset$ $(0.2 \leq \emptyset \leq 1.6)$ and a = ±1, ±2, ±3. But comparing the values of

relative risks for varying α^{s} (the level of significance) ; It is observed that the magnitude of these values is maximum for $\alpha = 1\%$ and a = 1 for all the values as "n" considered here and for $0.2 \le \emptyset \le 1.6$

So, it is recommended to use $\hat{\theta}_{ST_2}$ for n = 8, a = 3, $0.2 \le \emptyset \le 1.6$ and $\alpha = 1\%$ However, it performs well for other values of 'n' and 'a' also, considered here, but for the above values its performance is at its best.

2.5.3 Recommendations for $\hat{\theta}_{ST_3}$

For various values of n = 5, 8, 10 by fixing $\alpha = 1\%$ and also varying the degree of asymmetry 'a' = ±1, ±2, ±3, it is observed that the magnitude of relative risk of $\hat{\theta}_{ST_3}$ is higher for all these values of n^s and a^s for the whole range of $\emptyset$. However, it is still higher for the positive values of 'a' in particular a=3. It is suggested therefore, to use this estimator for a=3, α = 1%.

Next we change the level of significance to $\alpha = 5\%$ for the same set of values of other parameters, again $\hat{\theta}_{ST_3}$ performs better than the conventional estimator in the whole range of '$\emptyset$' and for different values of 'n' and 'a'. However the magnitude of relative risk is higher in case of $\alpha = 1\%$ compared to $\alpha = 5\%$.

While taking $\alpha = 10\%$ and observing the behavior of relative risk, it is found that $\hat{\theta}_{ST_3}$ performs better for positive values of 'a' in particular for a = 2.

In all the above situations it is observed that the magnitude of relative risk decreases as 'α' increases and shows higher values of it for 'positive' values of a.

So, we recommend to use $\hat{\theta}_{ST_3}$ for all values of 'n' and α^s considered here. In particular its performance is at its best for $\alpha = 1\%$, $a = 3$ and $n = 8, 10$.

2.5.4 <u>Recommendations for</u> $\hat{\theta}_{ST_4}$

The testimator $\hat{\theta}_{ST_4}$ behaves nicely compared to the conventional estimator in the sense of having 'smaller' risk for different values of 'n', 'α', and 'a'. In fact $\hat{\theta}_{ST_4}$ has lower risk for almost whole range of $\emptyset = 0.2(0.2)\ 1.6$. As 'n' increases the magnitude of relative risk decreases and it is lowest for $n = 10$, $a = -1$, for $\alpha = 1\%$. However, for smaller values of n i.e. $n = 5$ and $n = 8$, $\hat{\theta}_{ST_4}$ has better control over risk values and in particular for $n = 5$ and $a = 3$ the magnitude of relative risk is highest.

For the other value of $\alpha = 5\%$ and different values of $n = 5, 8$, and 10, $\hat{\theta}_{ST_4}$ performs better for higher positive values of 'a' compared to the negative values of 'a'. Particularly for $a = -3, -2$ there is not much difference in the performances however, the trend starts changing from $a = 1$ and the highest magnitude of it is observed at $a = 3$ for the values of n, in particular for $n = 5$, the gain is maximum, which remains true for $n = 8$ and to some extent for $n = 10$ for the whole range of $\emptyset$.

Finally taking $\alpha = 10\%$ for various values of 'n' and 'a' again the performance of $\hat{\theta}_{ST_4}$ is better compared to the conventional estimator, in particular for $n = 5$ and $a = 2$, $a = 3$, still the magnitude of relative risk is higher for $a = 3$. For $n = 10$ and for the negative values of 'a' the performance of its relative risk is not so good as compared to conventional estimator.

CONCLUSIONS:

1. It is concluded that both $\hat{\theta}_{ST_3}$ and $\hat{\theta}_{ST_4}$ perform better than the UMVUE for almost the whole range of ∅ = 0.2 (0.2) 1.6, various values of n = 5, 8, 10 and different 'positive' values of 'a'. The performance is not so good for the negative values of 'a'.
2. Comparing the values of relative risk(s) of $\hat{\theta}_{ST_3}$ and $\hat{\theta}_{ST_4}$, it is observed that the magnitude of relative risk is higher for $\hat{\theta}_{ST_3}$, so the choice of weights (Shrinkage factor) suggested is to take the '**square**' of the shrinkage factor making it 'dependent' on test statistics.
3. It is observed that using the Asymmetric Loss Function the effective range of ∅ for which $\hat{\theta}_{ST_3}$ or $\hat{\theta}_{ST_4}$ perform better than the usual estimator **increases** considerably as compared to the **same in case of squared error loss function** .
4. In particular both the testimators $\hat{\theta}_{ST_3}$ and $\hat{\theta}_{ST_4}$ perform better for a = 3, α = 1% and n = 5. Positive value of 'a' indicate that it should be used in those situations where over-estimation is more serious than underestimation, which remains true in case of insurance and re-insurance problems.

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 2.5.1.1 **Relative Risk of** $\hat{\theta}_{ST_1}$ $\alpha = 1\%$, n = 5, a = 3

$\emptyset$	k = 0.2	k = 0.4	k = 0.6	k = 0.8
0.20	1.14	1.139	1.132	1.104
0.40	2.35	2.363	2.231	1.796
0.60	4.994	4.847	4.182	2.747
0.80	5.484	5.384	5.558	2.985
1.00	7.01	6.884	6.813	3.743
1.20	5.007	5.805	5.08	2.414
1.40	3.547	4.792	3.578	2.097
1.60	1.872	2.516	2.499	1.813

Table : 2.5.1.2 **Relative Risk of** $\hat{\theta}_{ST_1}$ $\alpha = 1\%$, n = 8, a = 3

$\emptyset$	k = 0.2	k = 0.4	k = 0.6	k = 0.8
0.20	1.035	1.135	1.032	1.023
0.40	2.726	2.752	2.67	1.416
0.60	4.516	4.742	3.672	2.093
0.80	5.648	6.379	4.874	2.191
1.00	7.952	7.332	5.137	4.971
1.20	5.452	5.507	2.877	2.708
1.40	2.473	2.887	1.857	1.453
1.60	1.593	1.924	1.206	1.224

Table : 2.5.1.3 **Relative Risk of** $\hat{\theta}_{ST_1}$ $\alpha = 1\%$, n = 8, k = 0.2

$\emptyset$	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.20	0.777	0.824	0.826	1.163	1.059	1.035
0.40	0.31	0.393	0.417	2.525	2.409	1.726
0.60	0.41	1.548	1.611	3.185	3.17	3.516
0.80	1.327	2.836	2.147	5.678	4.517	4.648
1.00	7.861	6.177	5.585	6.878	6.385	5.952
1.20	5.257	4.641	3.796	3.946	3.958	3.452
1.40	1.796	1.663	1.831	1.928	1.139	1.473
1.60	0.833	0.835	0.96	0.434	0.493	0.593

Table : 2.5.2.1 **Relative Risk of** $\hat{\theta}_{ST_2}$ $\alpha = 1\%$, n = 5

Ø	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.20	0.511	0.596	0.61	1.03	1.263	1.153
0.40	1.337	1.47	1.535	2.529	2.397	2.689
0.60	1.568	2.813	2.967	3.227	3.697	4.451
0.80	2.383	3.917	4.357	4.635	4.988	5.131
1.00	3.418	4.376	5.560	5.088	6.05	6.953
1.20	2.863	4.073	3.837	3.298	3.687	5.94
1.40	1.987	2.031	2.384	2.218	2.537	2.421
1.60	1.031	1.102	1.331	1.654	1.824	1.361

Table : 2.5.2.2 **Relative Risk of** $\hat{\theta}_{ST_2}$ $\alpha = 1\%$, n = 8

Ø	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.20	0.74	0.796	0.799	1.217	1.075	1.042
0.40	1.292	1.405	1.457	2.289	2.49	2.173
0.60	2.394	2.585	2.698	3.074	4.374	4.346
0.80	3.051	3.489	4.791	4.246	5.618	6.676
1.00	4.398	5.162	6.000	5.284	6.945	7.321
1.20	2.362	4.888	4.246	3.853	3.166	4.534
1.40	1.903	2.694	2.854	2.829	2.997	2.216
1.60	0.918	1.875	1.994	1.414	1.476	1.566

Table : 2.5.2.3 **Relative Risk of** $\hat{\theta}_{ST_2}$ $\alpha = 5\%$, n = 5

Ø	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.20	0.64	0.707	0.712	1.387	1.125	1.074
0.40	1.346	1.468	1.526	2.568	2.848	2.814
0.60	1.426	1.639	2.783	3.349	3.178	3.858
0.80	2.858	2.33	3.713	3.842	4.059	4.918
1.00	3.468	4.358	5.224	4.634	5.705	5.714
1.20	2.648	2.185	3.285	2.718	3.42	3.977
1.40	1.996	2.058	2.006	1.887	2.302	2.193
1.60	1.66	1.643	1.721	0.532	1.757	1.269

Table : 2.5.2.4 **Relative Risk of** $\hat{\theta}_{ST_2}$ $\alpha = 5\%$, n = 8

Ø	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.20	0.865	0.895	0.894	1.082	1.03	1.017
0.40	0.346	1.457	1.501	2.889	2.213	1.523
0.60	1.317	1.48	2.583	3.834	3.592	3.959
0.80	1.633	2.006	3.286	4.481	4.075	4.202
1.00	2.505	3.319	4.006	5.561	6.554	6.761
1.20	2.44	2.757	3.861	3.184	3.692	4.207
1.40	1.856	2.484	2.739	2.567	2.806	2.052
1.60	1.256	1.607	1.456	1.334	1.445	1.555

Table : 2.5.4.1 **Relative Risk of** $\hat{\theta}_{ST_4}$ $\alpha = 1\%$, n = 5

Ø	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.20	0.348	0.455	0.502	1.145	1.487	1.185
0.40	0.165	0.246	0.303	1.584	1.805	1.547
0.60	1.233	1.346	1.435	1.921	2.051	2.491
0.80	1.644	1.918	2.155	2.002	2.679	3.985
1.00	2.908	3.724	4.625	3.111	4.019	5.748
1.20	1.483	1.096	3.966	2.455	3.018	2.151
1.40	0.305	0.77	2.581	1.251	1.562	1.256
1.60	0.235	0.687	1.741	0.159	0.334	0.74

Table : 2.5.4.2 **Relative Risk of** $\hat{\theta}_{ST_4}$ $\alpha = 1\%$, n = 8

Ø	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.20	0.603	0.701	0.732	1.579	1.15	1.065
0.40	1.15	1.225	1.273	2.467	1.875	2.166
0.60	1.161	1.244	1.304	2.486	3.193	3.056
0.80	1.426	2.625	2.779	3.201	4.255	3.827
1.00	2.404	3.893	4.775	3.874	4.511	5.151
1.20	1.331	1.749	1.328	2.294	3.602	3.998
1.40	1.203	1.52	1.073	1.152	2.297	1.488
1.60	0.155	0.454	1.15	0.095	1.17	0.258

2.6 Shrinkage Testimator for the Variance of a Normal Distribution

Shrinkage testimators for the mean μ of a Normal distribution N(μ, σ^2) when σ^2 is known or unknown, have been proposed by Waiker, Schuurman and Raghunandan (1984). In this section we have proposed single sample shrinkage testimator(s) for the variance of a Normal distribution. Let X be normally distributed with mean μ and variance σ^2, both being unknown. It is assumed that the prior knowledge about σ^2 is available in the form of an initial estimate σ_0^2. Using the sample observations $x_1, x_2 ...., x_n$ and possibly the given information we wish to construct a shrinkage testimator for the variance. The procedure described as follows:

1. First test with a sample of size n, the null hypothesis $H_0 : \sigma^2 = \sigma_0^2$ against the alternative $H_1 : \sigma^2 \neq \sigma_0^2$ using the test statistics $\frac{vs^2}{\sigma_0^2}$, where v = (n -1) and $s^2 = \frac{1}{n-1}\sum(x_i - \bar{x})^2$. The test statistics is distributed as χ^2 with v degrees of freedom.
2. If H_0 is accepted at α level of significance i.e. $x_1^2 < \frac{vs^2}{\sigma_0^2} < x_2^2$ where x_1^2 and x_2^2 are the lower and upper points of the uniformly most powerful unbiased (UMPU) test of H_0, use the conventional shrinkage estimator with shrinkage factor $k = \frac{vs^2}{\sigma_0^2 x^2}$, which is inversely proportional to χ^2 and it depends on the test statistic, so the arbitrariness in the choice of shrinkage factor has been removed by making it dependent on the test statistic.
3. If H_0 is rejected, use s^2, the Uniformly Minimum Variance Unbiased Estimator (UMVUE) as the estimator of σ^2.

Now, the proposed shrinkage testimator $\hat{\sigma}^2_{ST1}$ of σ^2 is

$$\hat{\sigma}^2_{ST1} = \begin{cases} k_1 s^2 + (1-k_1)\sigma_0^2 & , \ if\ H_0\ is\ accepted \\ s^2 & , \ otherwise \end{cases}$$

Where $k_1 = \frac{\nu s^2}{{\sigma_0}^2 x^2}$

Estimators of this type with an arbitrary k ($0 \le k \le 1$) have been proposed by Pandey and Singh (1976,77), Srivastava (1976) and others. In these studies it has been shown that the shrinkage testimators work well if k is near zero, n is small and $|\sigma^2 - \sigma_0^2|$ is also small. Hence, we should select the shrinkage factor which approaches to zero rapidly. We have, therefore, define another shrinkage Testimator $\hat{\sigma}^2_{ST2}$ of σ^2 by taking square of the shrinkage factor $k_2 = k_1^2$.

$$\hat{\sigma}^2{}_{ST_2} = \begin{cases} \left(\frac{\nu s^2}{{\sigma_0}^2 x^2}\right)^2 s^2 + \left[1 - \left(\frac{\nu s^2}{{\sigma_0}^2 x^2}\right)^2\right] {\sigma_0}^2 & ; \ if\ \ H_0\ \ is\ accepted \\ s^2 & ; \quad otherwise \end{cases}$$

2.7 Risk of Testimators

In this section we derive the risk of these two testimators which are defined in the previous section.

2.7.1 Risk of $\hat{\sigma}^2_{ST1}$

The risk of $\hat{\sigma}^2{}_{ST1}$ under L(Δ) is defined by

$$R(\hat{\sigma}^2{}_{ST_1}) = E[\hat{\sigma}^2{}_{ST_1} | L(\Delta)]$$

$$= E\left[k_1 s^2 + (1-k_1)\sigma_0^2 \Big/ \chi_1^2 < \frac{\upsilon s^2}{\sigma_0^2} < \chi_2^2\right] \cdot p\left[\chi_1^2 < \frac{\upsilon s^2}{\sigma_0^2} < \chi_2^2\right]$$

$$+ E\left[s^2 \,\middle|\, \frac{\upsilon s^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon s^2}{\sigma_0^2} > \chi_2^2\right] \cdot p\left[\frac{\upsilon s^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon s^2}{\sigma_0^2} > \chi_2^2\right]$$

_____(2.7.1.1)

$$= e^{-a}\int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} e^{a\left[\frac{\frac{\upsilon s^2}{\sigma_0^2\chi^2}\left(s^2-\sigma_0^2\right)+\sigma_0^2}{\sigma^2}\right]} f(s^2)\,ds^2 - a\int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}}\left[\frac{\frac{\upsilon s^2}{\sigma_0^2\chi^2}\left(s^2-\sigma_0^2\right)+\sigma_0^2}{\sigma^2}-1\right] f(s^2)\,ds^2$$

$$-\int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} f(s^2)\,ds^2 + e^{-a}\int_{0}^{\frac{\chi_1^2\sigma_0^2}{\upsilon}} e^{a\left(s^2/\sigma^2\right)} f(s^2)\,ds^2 + e^{-a}\int_{\frac{\chi_2^2\sigma_0^2}{\upsilon}}^{\infty} e^{a\left(s^2/\sigma^2\right)} f(s^2)\,ds^2$$

$$-a\int_{0}^{\frac{\chi_1^2\sigma_0^2}{\upsilon}}\left(\frac{s^2}{\sigma^2}-1\right) f(s^2)\,ds^2 - a\int_{\frac{\chi_2^2\sigma_0^2}{\upsilon}}^{\infty}\left(\frac{s^2}{\sigma^2}-1\right) f(s^2)\,ds^2 - \int_{0}^{\frac{\chi_1^2\sigma_0^2}{\upsilon}} f(s^2)\,ds^2 - \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon}}^{\infty} f(s^2)\,ds^2$$

______(2.7.1.2)

Where $f(s^2) = \dfrac{1}{2^{\upsilon/2}\,\Gamma\left(\upsilon/2\right)}\left(s^2\right)^{\frac{\upsilon}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon s^2}{\sigma^2}\right)} ds^2$

Straight forward integration of (2.7.1.2) gives

$$R(\hat{\sigma}_{ST1}^2) = \left(\frac{\sigma^2}{\nu}\right)^{\nu/2}\left[\begin{array}{c} I^* - \frac{2a}{\lambda\chi^2}\left(\frac{\nu}{2}+1\right)\left\{I\left(\chi_2^2\lambda,\ \frac{\nu}{2}+2\right) - I\left(\chi_1^2\lambda,\ \frac{\nu}{2}+2\right)\right\} \\ +\ a\left\{I\left(\chi_2^2\lambda,\ \frac{\nu}{2}+1\right) - I\left(\chi_1^2\lambda,\ \frac{\nu}{2}+1\right)\right\}\left(\frac{\nu}{\chi^2}+1\right) \\ -a\lambda\left\{I\left(\chi_2^2\lambda,\ \frac{\nu}{2}\right) - I\left(\chi_1^2\lambda,\ \frac{\nu}{2}\right)\right\} \\ +\ \frac{e^{-a}}{2^{\nu/2}\left(\frac{1}{2}-\frac{a}{\nu}\right)^{\frac{\nu}{2}}}\left[1 - I\left(\chi_2^2\lambda,\ \frac{\nu}{2}\right) - I\left(\chi_1^2\lambda,\ \frac{\nu}{2}\right)\right] + 1 \end{array}\right]$$

______(2.7.1.3)

Where $I(x;p) = (1/\Gamma p) \int_0^x e^{-x} x^{p-1}\, dx$ refers to the standard incomplete gamma function, $\lambda = \frac{\sigma_0^2}{\sigma^2}$, and

$$I^* = \frac{e^{a(\lambda-1)}}{2^{v/2}\, \Gamma\left(\frac{v}{2}\right)} \int_{x_1^2 \lambda}^{x_2^2 \lambda} e^{\left[\frac{at^2}{\lambda v x^2} - \frac{at}{x^2}\right]} e^{-\frac{1}{2}t}\, t^{\frac{v}{2}-1}\, \mathrm{dt}$$

2.7.2 Risk of $\hat{\sigma}^2_{ST2}$

Again, we obtain the risk of $\hat{\sigma}^2{}_{ST_2}$ under L(Δ) with respect to s^2, given by

$$R(\hat{\sigma}^2{}_{ST_2}) = E[\hat{\sigma}^2{}_{ST_2} \mid L(\Delta)]$$

$$= E\left[\left(\frac{\upsilon s^2}{\sigma_0^2 \chi^2}\right)^2 s^2 + \left(1 - \left(\frac{\upsilon s^2}{\sigma_0^2 \chi^2}\right)^2\right)\sigma_0^2 \Big/ \chi_1^2 < \frac{\upsilon s^2}{\sigma_0^2} < \chi_2^2\right] \cdot p\left[\chi_1^2 < \frac{\upsilon s^2}{\sigma_0^2} < \chi_2^2\right]$$
$$+ E\left[s^2 \;\middle|\; \frac{\upsilon s^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon s^2}{\sigma_0^2} > \chi_2^2\right] \cdot p\left[\frac{\upsilon s^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon s^2}{\sigma_0^2} > \chi_2^2\right]$$

______(2.7.2.1)

$$= e^{-a} \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} e^{a\left[\frac{\left(\frac{\upsilon s^2}{\sigma_0^2 \chi^2}\right)^2 (s^2 - \sigma_0^2) + \sigma_0^2}{\sigma^2}\right]} f(s^2)\, ds^2 - a \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} \left[\frac{\left(\frac{\upsilon s^2}{\sigma_0^2 \chi^2}\right)^2 (s^2 - \sigma_0^2) + \sigma_0^2}{\sigma^2} - 1\right] f(s^2)\, ds^2$$

$$- \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} f(s^2)\, ds^2 + e^{-a} \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon}} e^{a\left(s^2/\sigma^2\right)} f(s^2)\, ds^2 + e^{-a} \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon}}^{\infty} e^{a\left(s^2/\sigma^2\right)} f(s^2)\, ds^2$$

$$- a \int_{0}^{\frac{\chi_1^2\sigma_0^2}{\upsilon}} \left(\frac{s^2}{\sigma^2}-1\right) f(s^2)\,ds^2 - a \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon}}^{\infty} \left(\frac{s^2}{\sigma^2}-1\right) f(s^2)\,ds^2 - \int_{0}^{\frac{\chi_1^2\sigma_0^2}{\upsilon}} f(s^2)\,ds^2 - \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon}}^{\infty} f(s^2)$$

______(2.7.2.2)

Where $f(s^2) = \dfrac{1}{2^{\upsilon/2}\,\Gamma\left(\upsilon/2\right)} \left(s^2\right)^{\frac{\upsilon}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon s^2}{\sigma^2}\right)} ds^2$

Straight forward integration of (2.7.2.2) gives

$$R\left(\hat{\sigma}^2_{ST2}\right) = \left(\frac{\sigma^2}{\nu}\right)^{\nu/2} \left[\begin{array}{c} I^* - \frac{4a}{\lambda^2 (x^2)^2}\left(\frac{\nu}{2}+1\right)\left(\frac{\nu}{2}+2\right) \\ \left\{I\left(\chi_2^2\lambda,\, \frac{\nu}{2}+3\right) - I\left(\chi_1^2\lambda,\, \frac{\nu}{2}+3\right)\right\} \\ + \frac{4a}{\lambda (x^2)^2}\left(\frac{\nu}{2}\right)\left(\frac{\nu}{2}+1\right) \\ \left\{I\left(\chi_2^2\lambda,\, \frac{\nu}{2}+2\right) - I\left(\chi_1^2\lambda,\, \frac{\nu}{2}+2\right)\right\} \\ - \ a\left\{I\left(\chi_1^2\lambda,\, \frac{\nu}{2}+1\right) - I\left(\chi_2^2\lambda,\, \frac{\nu}{2}+1\right)\right\} \\ - a\lambda\left\{I\left(\chi_2^2\lambda,\, \frac{\nu}{2}\right) - I\left(\chi_1^2\lambda,\, \frac{\nu}{2}\right)\right\} \\ + \frac{e^{-a}}{2^{\nu/2}\left(\frac{1}{2}-\frac{a}{\nu}\right)^{\frac{\nu}{2}}}\left[1 - I\left(\chi_2^2\lambda,\, \frac{\nu}{2}\right) - I\left(\chi_1^2\lambda,\, \frac{\nu}{2}\right)\right] + 1 \end{array}\right]$$

______(2.7.2.3)

Where $I^* = \dfrac{e^{a(\lambda-1)}}{2^{\nu/2}\,\Gamma\left(\frac{\nu}{2}\right)} \displaystyle\int_{x_1^2\lambda}^{x_2^2\lambda} e^{\left[\frac{at^3}{\lambda^2 \nu (x^2)^2} - \frac{at^2}{\lambda (x^2)^2}\right]} e^{-\frac{1}{2}t}\, t^{\frac{\nu}{2}-1}\, dt$

Where $I(x;p) = (1/\Gamma p) \displaystyle\int_0^x e^{-x} x^{p-1}\, dx$ refers to the standard incomplete gamma function and λ is same as defined earlier.

2.8 Relative Risk of $\hat{\sigma}^2_{STi}$

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator s^2 in this case. For this purpose, we obtain the risk of s^2 under $L_E\left(\hat{\sigma}^2, \sigma^2\right)$ as:

$$R_E(s^2) = E[s^2 \mid L(\hat{\sigma}^2, \sigma^2)]$$

$$= e^{-a} \int_0^{\infty} e^{a\left[\frac{s^2}{\sigma^2}\right]} f(s^2)\, ds^2 \; - a \int_0^{\infty} \left[\frac{s^2}{\sigma^2} - 1\right] f(s^2)\, ds^2 \; - \int_0^{\infty} f(s^2)\, ds^2 \qquad \text{\_\_\_\_\_\_(2.8.1)}$$

Where $f(s^2) = \dfrac{1}{2^{\nu/2}\,\Gamma\left(\nu/2\right)} \left(s^2\right)^{\frac{\nu}{2}-1} e^{\left(-\frac{1}{2}\frac{\nu s^2}{\sigma^2}\right)}$

A straightforward integration of (2.8.1) gives

$$R_E(s^2) = \left(\frac{\sigma^2}{\nu}\right)^{\nu/2} \left[\frac{e^{-a}}{2^{\nu/2}\left(\frac{1}{2} - \frac{a}{\nu}\right)^{\nu/2}} - 1 \right] \qquad \text{\_\_\_\_\_\_(2.8.2)}$$

Now, we define the Relative Risk of $\hat{\sigma}^2{}_{ST_i}$, $i = 1,2$ with respect to s^2 under $L\left(\hat{\sigma}^2, \sigma^2\right)$ as follows:

$$RR_1 = \frac{R_E(s^2)}{R(\hat{\sigma}^2{}_{ST1})} \qquad \text{\_\_\_\_\_\_(2.8.3)}$$

Using (2.8.2) and (2.7.1.3) the expression for RR_1 given in (2.8.3) can be obtained; it is observed that RR_1 is a function of 'λ', ' ν ', 'α' and 'a'.

Finally, we define the Relative Risk of $\hat{\sigma}^2{}_{ST_2}$ by

$$RR_2 = \frac{R_E(s^2)}{R(\hat{\sigma}^2{}_{ST2})} \qquad \text{\_\_\_\_\_\_(2.8.4)}$$

The expression for RR_2 is given by (2.8.4) can be obtained by using (2.8.2) and (2.7.2.3). Again we observed that RR_2 is a function of 'λ', ' v ', 'α' and 'a'.

2.9 Recommendations for $\hat{\sigma}^2_{STi}$

In this section we wish to compare the performance of $\hat{\sigma}^2{}_{ST_1}$ and $\hat{\sigma}^2{}_{ST_2}$ with respect to the best available (unbiased) estimator of σ^2.

2.9.1 Recommendations for $\hat{\sigma}^2_{ST1}$

It is observed that the expressions of RR_1 and RR_2 are the functions of v, α, λ and the degrees of asymmetry " a ". For the comparison of the proposed testimators with the best available estimator we have considered $\alpha = 1\%,\ 5\%\ and\ 10\%$, $v = 5,\ 8, 10\ and\ 12\ and\ a = \ -2.0,\ -1.0,\ 1.0,\ 1.5,\ and\ 1.75$ and $\lambda = 0.2\ (0.2)\ 2.0$. There will be several tables and graphs for RR values for both the testimators. We have assembled some of graphs at the end of the chapter. However our recommendations based on all these computations are as follows:

(i) $\hat{\sigma}^2{}_{ST_1}$ Performs better than $\hat{\sigma}^2$ for a considerably large range of λ for different degrees of asymmetry. For a= -2 the range is $0.6 \le \lambda \le 1.8$, which changes slightly for a = -1 and becomes $0.6 \le \lambda \le 1.6$. For the positive values of 'a' we have observed a different pattern for RR_1 as when different values $0.8 \le \lambda \le 1.4$, the performance of $\hat{\sigma}^2{}_{ST_1}$ is better than $\hat{\sigma}^2$. Similar pattern is observed for the other two positive values of 'a' i.e. a = 1.5 and a = 1.75. However the values of RR_1 are smaller in magnitude. Further the magnitude of RR_1 is higher when 'a' is negative as compared to those values of when 'a' is positive.

(ii) For higher values of α i.e. 5% and 10% a similar kind of behaviour of RR values is observed but the range of 'λ' changes, it is $0.6 \leq \lambda \leq 2.0$ for a = -2 and $\alpha = 5\%$ and this becomes $0.8 \leq \lambda \leq 2.0$ for a = -2 and $\alpha = 10\%$. Similarly for other values of negative 'a' when 'a' is positive the range of 'λ' is $0.8 \leq \lambda \leq 1.8$ for a = 1.75.

(iii) It is observed that for some negative values of 'a' as well as for some positive values of 'a' the magnitude of RR_1 is greater than unity which indicates that $\hat{\sigma}^2_{ST_1}$ performs better than usual estimator.

(iv) As the value of 'ν' increases there is a decrease in the RR_1 values for different values of levels of significance and degrees of asymmetries. However the best performance of $\hat{\sigma}^2_{ST_1}$ is observed at $\alpha = 1\%$ for a = -2 and $\alpha = 1\%$ for a = 1.75

(v) It is recommended therefore to consider a smaller level of significance (preferably $\alpha = 1\%$) and smaller sample size ν = '5' or '8' for positive / negative values of 'a' in particular a = 1.75 and a = -2.0.

2.9.2 Recommendations for $\hat{\sigma}^2_{ST2}$

Next we have considered another testimator $\hat{\sigma}^2_{ST_2}$ which is obtained by squaring the shrinkage factor, we have evaluated the expression RR_2 for the same set of values as considered for RR_1 and our recommendations are as follows:

(i) $\hat{\sigma}^2_{ST_2}$ performs better than the usual estimator $\hat{\sigma}^2$ for different range of λ i.e. for a = -2, it is $0.6 \leq \lambda \leq 1.8$, however for a = -1 it becomes $0.6 \leq$

$\lambda \leq 1.6$ i.e. almost the same whole range as observed for $\hat{\sigma}^2_{ST_1}$ but the magnitude of RR_2 values are **HIGHER** than the magnitude of RR_1 values indicating a 'better' control over the risk of $\hat{\sigma}^2_{ST_2}$ as compared to $\hat{\sigma}^2_{ST_1}$. This is observed when $\alpha = 1\%$, $\nu = 5$ and a = -1.0 also when a = +1.75.

(ii) A Similar kind of pattern for the performance of $\hat{\sigma}^2_{ST_2}$ is observed for $\alpha = 5\%$ *and* $\alpha = 10\%$ for the range of 'λ' as mentioned above.

(iii) It is observed that the values of RR_2 are more than unity for some positive and negative values of 'a'. So, it is conclude that in both the situations i.e. over/under estimation the proposed testimators behaves nicely.

(iv) The maximum values of RR_2 are observed for $\alpha = 1\%$, a = -2.0 and $\nu = 5$. Similarly for a = +1.75, $\alpha = 1\%$ and $\nu = 5$.

(v) The general behaviour observed is that of 'decreasing' values of RR for higher values of 'ν' and 'α'.

(vi) So, it is recommended to consider smaller level of significance along with a smaller sample size with proper choice of 'a'.

CONCLUSION:

We have proposed shrinkage testimator(s) for the variance of Normal distribution and we recommend that: A shrinkage testimator $\hat{\sigma}^2_{ST_2}$ (i.e. 'square' of shrinkage factor) should be considered for $\alpha = 1\%$, $\nu = 5$ *or* 8 *and* $a = 1.75$ (for situations where overestimation is more serious) and $a = -2.0$ (for situations where under estimation is more serious).

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 2.9.1.1 **Relative Risk of $\hat{\sigma}^2_{ST1}$** $\alpha = 1\%$, $\nu = 5$

λ	a = -2	a = -1	a = 1	a = 1.5	a = 1.75
0.20	0.699	0.597	0.497	0.176	0.718
0.40	0.76	0.794	0.642	1.071	1.229
0.60	1.311	1.116	1.196	1.934	2.047
0.80	2.606	1.939	2.693	2.92	3.307
1.00	4.647	3.256	2.817	3.448	4.375
1.20	4.64	3.092	1.725	2.436	3.071
1.40	2.935	2.288	1.187	1.763	2.246
1.60	1.802	1.459	0.801	1.233	1.609
1.80	1.206	0.974	0.541	0.84	1.121
2.00	0.878	0.695	0.373	0.569	0.768

Table : 2.9.1.2 **Relative Risk of $\hat{\sigma}^2_{ST1}$** $\alpha = 1\%$, $\nu = 8$

λ	a = -2	a = -1	a = 1	a = 1.5	a = 1.75
0.20	0.863	0.455	0.244	0.685	0.891
0.40	1.642	0.983	0.651	1.599	1.165
0.60	2.516	1.455	1.301	1.794	2.005
0.80	3.579	1.878	1.806	2.133	2.44
1.00	4.156	2.446	2.562	3.726	4.164
1.20	3.693	2.01	1.757	2.755	3.333
1.40	2.777	1.503	1.476	1.751	2.887
1.60	1.738	1.062	1.008	1.482	1.574
1.80	1.191	0.782	0.208	0.315	0.371
2.00	0.895	0.611	0.146	0.211	0.245

Table : 2.9.1.3 **Relative Risk of $\hat{\sigma}^2_{ST1}$** $\alpha = 5\%$, $\nu = 5$

λ	a = -2	a = -1	a = 1	a = 1.5	a = 1.75
0.20	0.802	0.617	0.663	0.362	0.672
0.40	1.018	0.876	0.981	1.162	1.278
0.60	1.714	1.167	1.923	2.117	2.03
0.80	2.833	1.598	2.442	2.548	3.336
1.00	3.641	1.909	2.665	3.117	3.632
1.20	3.279	1.886	1.879	2.254	2.977
1.40	2.465	1.627	1.188	1.762	2.116
1.60	1.829	1.334	0.828	1.049	1.771
1.80	1.416	1.094	0.523	0.568	1.031
2.00	0.92	0.666	0.249	0.429	0.583

Table : 2.9.1.4 **Relative Risk of $\hat{\sigma}^2_{ST1}$** $\alpha = 5\%$, $\nu = 8$

λ	**a = -2**	**a = -1**	**a = 1**	**a = 1.5**	**a = 1.75**
0.20	0.768	0.781	0.315	0.475	0.603
0.40	1.003	1.076	0.971	0.916	0.713
0.60	1.801	1.583	1.387	1.59	1.374
0.80	2.621	1.963	1.638	2.464	2.563
1.00	3.265	2.377	2.219	2.764	3.103
1.20	2.221	1.844	1.843	2.105	2.34
1.40	1.909	1.631	1.474	1.611	1.744
1.60	1.601	1.193	0.937	1.231	1.51
1.80	1.374	0.766	0.711	0.831	0.974
2.00	1.009	0.684	0.393	0.438	0.582

Table : 2.9.2.1 **Relative Risk of $\hat{\sigma}^2_{ST2}$** $\alpha = 1\%$, $\nu = 5$

λ	**a = -2**	**a = -1**	**a = 1**	**a = 1.5**	**a = 1.75**
0.20	0.426	0.393	0.343	0.789	0.996
0.40	0.499	1.193	0.458	1.225	1.689
0.60	1.059	1.907	1.494	1.874	2.768
0.80	2.782	2.614	2.389	2.751	3.961
1.00	6.727	4.124	3.826	4.747	6.296
1.20	5.728	3.306	2.129	4.055	4.127
1.40	2.833	2.432	1.28	3.227	2.709
1.60	1.592	1.309	0.781	2.05	1.72
1.80	1.04	0.836	0.495	1.265	1.081
2.00	0.754	0.589	0.328	0.785	0.686

Table : 2.9.2.2 **Relative Risk of $\hat{\sigma}^2_{ST2}$** $\alpha = 5\%$, $\nu = 5$

λ	**a = -2**	**a = -1**	**a = 1**	**a = 1.5**	**a = 1.75**
0.20	0.509	0.552	0.441	0.992	0.688
0.40	0.989	0.855	1.326	1.69	1.913
0.60	1.405	1.514	1.921	2.054	2.874
0.80	2.87	2.081	2.784	2.371	4.183
1.00	3.975	2.833	2.973	3.354	5.017
1.20	3.193	2.302	1.917	2.649	4.216
1.40	2.464	1.825	1.581	1.806	3.446
1.60	1.732	1.574	1.105	1.232	2.005
1.80	1.168	1.195	0.295	0.52	1.715
2.00	1.048	0.936	0.221	0.375	0.514