Chapter 3

Double Stage Shrinkage Testimators under Asymmetric Loss Function

Chapter – 3

DOUBLE STAGE SHRINKAGE TESTIMATORS UNDER ASYMMETRIC LOSS FUNCTION

3.1 Introduction

In this chapter we have extended our studies of chapter 2 in the sense that now instead of drawing only one sample form the population, the experimenter may possibly drawn one or two samples. Estimation of the mean from double sample in the presence of a priori information was first considered by Katti (1962) and later by many others. Katti's method consisted in constructing a region R using the a priori information available in the form of a guess value say θ_0 of the parameter θ and the observations $x_1, x_2 \ldots\ldots, x_n$ from the first sample. If the estimator constructed or proposed belonged to R; there was no need to draw a second sample of size n_2. However, if it did not lie in R; a second sample of size n_2 was drawn and the proposed estimator used observations from both samples. Shah (1964) used this method in estimating variance of a Normal distribution when a guess of the population variance is given. He also proposed a pre-test estimator of the variance. The procedure adopted by Shah has something in common with the two stage procedure due to Stein (1945). Arnold and Al-Bayyati (1970) modified the estimator proposed by Katti using the shrinkage technique and studied the properties of the estimator. Waiker and Katti (1971) have also studied two stage estimation of the mean. Pandey (1979) considered estimation of variance of a normal population using a priori information.

Waiker et al (1984) have suggested and studied a two stage shrinkage testimator of the mean of a normal population when the variance of the population may be known or unknown. Their approach is different from that of Katti and others in the sense that (i) no region R is constructed in the sample space (ii) the shrinkage factor $\boldsymbol{k}$ is no longer arbitrary but is a function of the test statistic used in testing the hypothesis regarding the given a priori information. In both techniques $\boldsymbol{k}$ being arbitrary or not, no assumption is made regarding the distribution of the parameter θ on (the parameter space). At the most one may take it a singular distribution with entire mass concentrated at a single point $\theta = \theta_0$.

Similar studies for estimating the scale parameter θ in one parameter Exponential distribution with p.d.f.

$$f(x;\theta) = \begin{cases} \frac{1}{\theta}\exp(-x/\theta), & x \geq 0, \theta > 0 \\ 0 & , otherwise \end{cases} \qquad \text{\_\_\_\_\_} (3.1.1)$$

have been made. Using the priori information available in the form of an initial estimate say θ_0 of the parameter θ. Shah (1975) considered estimation of θ in censored sampling. Ojha and Srivastava (1980) have studied a pre-test double stage shrunken estimators of θ using complete samples. The object of the present chapter is to propose and study shrinkage testimators for scale parameters of (3.1.1).

Recently Srivastava and Tanna (2007) have studied the risk properties of a Double stage shrinkage testimator under General Entropy Loss Function. Further Srivastava and Tanna (2012) have studied the risk properties of such estimators under Asymmetric Loss Function.

DOUBLE STAGE ESTIMATION:

The first stage sample is used to test H_0 and if H_0 is not rejected, it is suggested to use the prior knowledge being supported by a test, in estimating θ. However, if H_0 is rejected, we do not use the prior knowledge and obtain a second sample size $n_2 = (n - n_1)$ to make up for the loss of the prior knowledge and estimate θ using both the samples.

In section 3.2 we have proposed the three different shrinkage testimators for scale parameter of an Exponential Distribution and we have studied the risk properties of these three shrinkage testimators under Asymmetric Loss Function. Section 3.3 deals with the derivation of the risk(s) of these three estimators. Section 3.4 deals with the relative risk(s) of these three estimators. Section 3.5 concludes with the comparison of UMVUE and the proposed shrinkage testimators in terms of their relative risks. Further in the same section a suggestion for the shrinkage factor is made.

In section 3.6 we have proposed the two different shrinkage testimators for the variance of a Normal Distribution and we have studied the risk properties of these two shrinkage testimators under Asymmetric Loss Function. Section 3.7 deals with the derivation of the risk(s) of these two estimators. Section 3.8 deals with the relative risk(s) of these two estimators. Section 3.9 concludes with the comparison of UMVUE and the proposed shrinkage testimators regarding their choice in terms of their relative efficiency. Further in the same section a suggestion for the choice of shrinkage factor is made.

3.2 Shrinkage Testimator(s) for Scale Parameter of an Exponential Distribution.

Let x_{11}, x_{12}, ______, x_{1n1} be the first stage sample of size n_1 from and exponential population is given by (3.1.1). Let θ_0 be the guess estimate of the mean θ. Compute the sample mean $\bar{x}_1 = \frac{1}{n} \sum_{i=1}^{n1} x_{1i}$ and test the preliminary hypothesis H_0 : $\theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$, using the test statistic $T = \frac{2n_1\bar{x}_1}{\theta_0}$ which follows $x^2_{2n_1}$. It is to be noted that H_0 is accepted if $x_1^2 \leq \frac{2n_1\bar{x}_1}{\theta_0} \leq x_2^2$ and H_0 is rejected, otherwise. Then take $n_2 = n - n_1$ additional observations x_{21}, x_{22}, ______, x_{2n2} and use the pooled estimator $\bar{x}_p$ as the estimator of the mean where $\bar{x}_p = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1+n_2}$

x_1^2 and x_2^2 being given by

$$P\left[x^2_{2n_1} \geq x_2^2\right] + P\left[x^2_{2n_1} \leq x_1^2\right] = \alpha \qquad \text{\_\_\_\_\_\_\_\_(3.2.1)}$$

where α is the pre-assigned level of significance.

When $\theta = \theta_0$, the probability of avoiding the second sample is (1- α) and the expected sample size is given by

$$n^* = E[n|\,\theta = \theta_0]$$

$$= n_1\, P\left[x_1^2 < \frac{2n_1\bar{x}_1}{\theta_0} < x_2^2\right] + (n_1 + n_2)\, P\left[\frac{2n_1\bar{x}_1}{\theta_0} < x_1^2 \cup \frac{2n_1\bar{x}_1}{\theta_0} > x_2^2\right]$$

or, $n^* = n_1\,(1 + u\alpha)$ where $u = {}^{n_2}\!/_{n_1}$.

When $\theta \neq \theta_0$, the probability of avoiding the second sample is

$P = \frac{1}{2^{n_1}\,\Gamma n_1}\left(\frac{2n_1}{\theta_0}\right)^{n_1} (\bar{x}_1)^{n_1-1}\, e^{\left(\frac{-n_1\bar{x}_1}{\theta_0}\right)} d\bar{x}_1$ and the expected sample size is

$$n^{**} = n_1 + n_2\left[1 - P\left\{\lambda x_1^2 < \frac{2n_1\bar{x}_1}{\theta_0} < \lambda x_2^2\right\}\right]$$

Now we propose a shrinkage testimator $\hat{\theta}_{DST1}$ of θ defined as:

$$\hat{\theta}_{DST1} = \begin{cases} k_1\bar{x}_1 + (1-k_1)\theta_0 & ; \ if \ \ x_1^2 \le \dfrac{2n_1\bar{x}_1}{\theta_0} \le x_2^2 \\ \bar{x}_p & ; \quad otherwise \end{cases}$$

______(3.2.2)

Where $\bar{x}_p = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1+n_2}$ and $\bar{x}_i = \frac{1}{n_i}\sum_{j=1}^{ni} x_{ij} \quad ; \ i = 1,2$

and k_1 being dependent on test statistic is given by $k_1 = \frac{2n_1\bar{x}_1}{\theta_0\, x^2}$

where $x^2 = (x_2^2 - x_1^2)$

Now, taking the 'square' of k_1 (i.e. $k_2 = k_1^2$), another testimator is defined as

$$\hat{\theta}_{DST2} = \begin{cases} \left(\frac{2n_1\bar{x}_1}{\theta_0 x^2}\right)^2 \bar{x}_1 + \left[1 - \left(\frac{2n_1\bar{x}_1}{\theta_0 x^2}\right)^2\right]\theta_0 & ; if \ H_0 \ \ is \ accepted \\ \bar{x}_p & ; \ otherwise \end{cases}$$ ________(3.2.3)

Finally, taking k_3 , the third testimator can be proposed as

$$\hat{\theta}_{DST3} = \begin{cases} k_3\,\bar{x}_1 + (1-k_3)\theta_0 & ; \ if \ \ H_0 \ is \ accepted \\ \bar{x}_p & ; \quad otherwise \end{cases}$$ ________(3.2.4)

Where $k_3 = \frac{2n_1\bar{x}_1}{\theta_0 x^2} - \frac{x_1^2}{x^2}$ and $x^2 = (x_2^2 - x_1^2)$

In this case 'k_3' exactly lies between '0' and '1'.

3.3 Risk of Testimators

In this section we derive the risk of all the three testimators which are defined in the previous section.

3.3.1 Risk of $\widehat{\theta}_{DST1}$

The risk of $\hat{\theta}_{DST_1}$ under L(Δ) is given by

$$R(\hat{\theta}_{DST_1}) = E[\hat{\theta}_{DST_1} | L(\Delta)]$$

$$= E\left[k_1\,\bar{x}_1 + (1-k_1)\theta_0 \Big/ \chi_1^2 < \frac{2n_1\bar{x}_1}{\theta_0} < \chi_2^2\right]\cdot p\left[\chi_1^2 < \frac{2n_1\bar{x}_1}{\theta_0} < \chi_2^2\right]$$
$$+ E\left[\bar{x}_p \;\middle|\; \frac{2n_1\bar{x}_1}{\theta_0} < \chi_1^2 \cup \frac{2n_1\bar{x}_1}{\theta_0} > \chi_2^2\right]\cdot p\left[\frac{2n_1\bar{x}_1}{\theta_0} < \chi_1^2 \cup \frac{2n_1\bar{x}_1}{\theta_0} > \chi_2^2\right]$$

_______(3.3.1.1)

$$= e^{-a}\int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} e^{a\left[\frac{\frac{2n_1\bar{x}_1}{\theta_0}(\bar{x}_1-\theta_0)+\theta_0}{\theta}\right]} f(\bar{x}_1)\,d\bar{x}_1$$

$$- a\int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} \left[\frac{\frac{2n_1\bar{x}_1}{\theta_0}(\bar{x}_1-\theta_0)+\theta_0}{\theta} - 1\right] f(\bar{x}_1)\,d\bar{x}_1$$

$$- \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} f(\bar{x}_1)\,d\bar{x}_1 \;+ e^{-a}\int_0^{\frac{\chi_1^2\theta_0}{2n_1}}\int_0^{\infty} e^{a\left(\bar{x}_p/\theta\right)} f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2$$

$$- a\int_0^{\frac{\chi_1^2\theta_0}{2n_1}}\int_0^{\infty}\left(\frac{\bar{x}_p}{\theta}-1\right) f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2 - \int_0^{\frac{\chi_1^2\theta_0}{2n_1}}\int_0^{\infty} f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2$$

$$+ e^{-a}\int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty}\int_0^{\infty} e^{a\left(\bar{x}_p/\theta\right)} f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2$$

$$- a\int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty}\int_0^{\infty}\left(\frac{\bar{x}_p}{\theta}-1\right) f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2 - \int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty}\int_0^{\infty} f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2$$

_______(3.3.1.2)

Where $f(\bar{x}_1) = \frac{1}{\Gamma n_1}\left(\frac{n_1}{\theta}\right)^{n_1} (\bar{x}_1)^{n_1-1} e^{\frac{-n_1\bar{x}_1}{\theta}}$

and $f(\bar{x}_2) = \frac{1}{\Gamma n_2}\left(\frac{n_2}{\theta}\right)^{n_2} (\bar{x}_2)^{n_2-1} e^{\frac{-n_2\bar{x}_2}{\theta}}$

Straight forward integration of (3.3.1.2) gives

$$
\begin{aligned}
R(\hat{\theta}_{DST_1}) = I^* - \frac{2a(n_1+1)}{\phi x^2}\left\{I\left(\frac{x_2^2\phi}{2}, n_1+2\right) - I\left(\frac{x_1^2\phi}{2}, n_1+2\right)\right\} + \\
\left\{\frac{2an_1}{x^2} + \frac{an_1}{n_1+n_2}\right\}\left\{I\left(\frac{x_2^2\phi}{2}, n_1+1\right) - I\left(\frac{x_1^2\phi}{2}, n_1+1\right)\right\} - \\
\left\{I\left(\frac{x_2^2\phi}{2}, n_1\right) - I\left(\frac{x_1^2\phi}{2}, n_1\right)\right\}(a\phi - a + 1) + \left\{I\left(\frac{x_2^2\phi}{2}, n_1\right) - I\left(\frac{x_1^2\phi}{2}, n_1\right)\right\} \\
\left\{1 - \frac{an_1}{n_1+n_2} - \frac{e^{-a}}{\left(1-\frac{a}{n_1+n_2}\right)^{n_1+n_2}}\right\} + \left(\frac{e^{-a}}{\left(1-\frac{a}{n_1+n_2}\right)^{n_1+n_2}} - 1\right)
\end{aligned}
$$

_______(3.3.1.3)

Where $I^* = e^{a\phi - a}\int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} e^{\left[\frac{2at^2}{n_1\phi x^2} - \frac{2at}{x^2}\right]}\frac{1}{\Gamma n_1} e^{-t}\, t^{n_1-1}\, dt$; $\emptyset = \frac{\theta_0}{\theta}$ and

$I(x;p) = (1/\Gamma p)\int_0^x e^{-x} x^{p-1}\, dx$ refers to the standard incomplete gamma function

3.3.2 <u>Risk of</u> $\hat{\theta}_{DST2}$

Again, we obtain the risk of $\hat{\theta}_{DST_2}$ under L(Δ) with respect to $\bar{x}_1$, given by

$R(\hat{\theta}_{DST_2}) = E[\hat{\theta}_{DST_2} \mid L(\Delta)]$

$$= E\left[\left(\frac{2n_1\bar{x}_1}{\theta_0\,\chi^2}\right)^2(\bar{x}_1-\theta_0)+\theta_0 \Big/ \chi_1^2 < \frac{2n_1\bar{x}_1}{\theta_0} < \chi_2^2 \right]\cdot p\left[\chi_1^2 < \frac{2n_1\bar{x}_1}{\theta_0} < \chi_2^2\right]$$

$$+E\left[\bar{x}_p \,\middle|\, \frac{2n_1\bar{x}_1}{\theta_0} < \chi_1^2 \cup \frac{2n_1\bar{x}_1}{\theta_0} > \chi_2^2\right]\cdot p\left[\frac{2n_1\bar{x}_1}{\theta_0} < \chi_1^2 \cup \frac{2n_1\bar{x}_1}{\theta_0} > \chi_2^2\right]$$

(3.3.2.1)

$$= e^{-a}\int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} e^{a\left[\frac{\left(\frac{2n_1\bar{x}_1}{\theta_0\,\chi^2}\right)^2(\bar{x}_1-\theta_0)+\theta_0}{\theta}\right]} f(\bar{x}_1)\,d\bar{x}_1$$

$$-a\int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} \left[\frac{\left(\frac{2n_1\bar{x}_1}{\theta_0\,\chi^2}\right)^2(\bar{x}_1-\theta_0)+\theta_0}{\theta}-1\right] f(\bar{x}_1)\,d\bar{x}_1$$

$$-\int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} f(\bar{x}_1)\,d\bar{x}_1 \;+ e^{-a}\int_0^{\frac{\chi_1^2\theta_0}{2n_1}}\int_0^{\infty} e^{a\left(\bar{x}_p/\theta\right)} f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2$$

$$-a\int_0^{\frac{\chi_1^2\theta_0}{2n_1}}\int_0^{\infty}\left(\frac{\bar{x}_p}{\theta}-1\right)f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2 - \int_0^{\frac{\chi_1^2\theta_0}{2n_1}}\int_0^{\infty} f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2$$

$$+ e^{-a}\int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty}\int_0^{\infty} e^{a\left(\bar{x}_p/\theta\right)} f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2$$

$$-a\int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty}\int_0^{\infty}\left(\frac{\bar{x}_p}{\theta}-1\right)f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2 - \int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty}\int_0^{\infty} f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2$$

(3.3.2.2)

Where $f(\bar{x}_1)=\frac{1}{\Gamma n_1}\left(\frac{n_1}{\theta}\right)^{n_1}(\bar{x}_1)^{n_1-1}e^{\frac{-n_1\bar{x}_1}{\theta}}$

and $f(\bar{x}_2)=\frac{1}{\Gamma n_2}\left(\frac{n_2}{\theta}\right)^{n_2}(\bar{x}_2)^{n_2-1}e^{\frac{-n_2\bar{x}_2}{\theta}}$

A straight forward integration of (3.3.2.2) gives:

$$R(\hat{\theta}_{DST_2})=I_1^{*}-\frac{4a(n_1+1)(n_1+2)}{\phi^2(\chi^2)^2}\left\{I\left(\frac{\chi_2^2\phi}{2},n_1+3\right)-I\left(\frac{\chi_1^2\phi}{2},n_1+3\right)\right\}+$$
$$\frac{4an_1(n_1+1)}{\phi(\chi^2)^2}\left\{I\left(\frac{\chi_2^2\phi}{2},n_1+2\right)-I\left(\frac{\chi_1^2\phi}{2},n_1+2\right)\right\}-\left\{I\left(\frac{\chi_2^2\phi}{2},n_1\right)-I\left(\frac{\chi_1^2\phi}{2},n_1\right)\right\}$$
$$(a\phi-a+1)+\frac{an_1}{n_1+n_2}\left\{I\left(\frac{\chi_2^2\phi}{2},n_1+1\right)-I\left(\frac{\chi_1^2\phi}{2},n_1+1\right)\right\}+$$
$$\left\{I\left(\frac{x_2^2\phi}{2},n_1\right)-I\left(\frac{x_1^2\phi}{2},n_1\right)\right\}\left\{1-\frac{an_1}{n_1+n_2}-\frac{e^{-a}}{\left(1-\frac{a}{n_1+n_2}\right)^{n_1+n_2}}\right\}+\left(\frac{e^{-a}}{\left(1-\frac{a}{n_1+n_2}\right)^{n_1+n_2}}-1\right)$$

_______(3.3.2.3)

Where $$I_1^{*}=e^{a(\phi-1)}\int_{\frac{\chi_1^2\phi}{2}}^{\frac{\chi_2^2\phi}{2}}e^{\left[\frac{4at^3}{n_1\phi^2(\chi^2)^2}-\frac{4at^2}{\phi(\chi^2)^2}\right]}\frac{1}{\Gamma n_1}e^{-t}t^{n_1-1}\,dt$$

3.3.3 <u>Risk of</u> $\hat{\theta}_{DST3}$

Finally, we obtain the risk of $\hat{\theta}_{DST_3}$ under L(Δ) with respect to $\bar{x}_1$, given by

$$R(\hat{\theta}_{DST_3})=E[\,\hat{\theta}_{DST_3}\,|\,L(\Delta)]$$

$$= E\left[k_3\,\bar{x}_1 + (1-k_3)\theta_0 \Big/ \chi_1^2 < \frac{2n_1\bar{x}_1}{\theta_0} < \chi_2^2\right] \cdot p\left[\chi_1^2 < \frac{2n_1\bar{x}_1}{\theta_0} < \chi_2^2\right]$$

$$+ E\left[\bar{x}_p \;\middle|\; \frac{2n_1\bar{x}_1}{\theta_0} < \chi_1^2 \cup \frac{2n_1\bar{x}_1}{\theta_0} > \chi_2^2\right] \cdot p\left[\frac{2n_1\bar{x}_1}{\theta_0} < \chi_1^2 \cup \frac{2n_1\bar{x}_1}{\theta_0} > \chi_2^2\right]$$

____(3.3.3.1)

$$= e^{-a} \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} e^{a\left[\frac{\left(\frac{2n_1\bar{x}_1}{\theta_0 x^2} - \frac{x_1^2}{x^2}\right)(\bar{x}_1-\theta_0)+\theta_0}{\theta}\right]} f(\bar{x}_1)\, d\bar{x}_1$$

$$- a \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} \left[\frac{\left(\frac{2n_1\bar{x}_1}{\theta_0 x^2} - \frac{x_1^2}{x^2}\right)(\bar{x}_1-\theta_0)+\theta_0}{\theta} - 1\right] f(\bar{x}_1)\, d\bar{x}_1$$

$$- \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} f(\bar{x}_1)\, d\bar{x}_1 \; + \; e^{-a} \int_0^{\frac{\chi_1^2\theta_0}{2n_1}} \int_0^{\infty} e^{a\left(\bar{x}_p/\theta\right)} f(\bar{x}_1) f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2$$

$$- a \int_0^{\frac{\chi_1^2\theta_0}{2n_1}} \int_0^{\infty} \left(\tfrac{\bar{x}_p}{\theta} - 1\right) f(\bar{x}_1) f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2 - \int_0^{\frac{\chi_1^2\theta_0}{2n_1}} \int_0^{\infty} f(\bar{x}_1) f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2$$

$$+ e^{-a} \int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty} \int_0^{\infty} e^{a\left(\bar{x}_p/\theta\right)} f(\bar{x}_1) f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2$$

$$- a \int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty} \int_0^{\infty} \left(\tfrac{\bar{x}_p}{\theta} - 1\right) f(\bar{x}_1) f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2 - \int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty} \int_0^{\infty} f(\bar{x}_1) f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2$$

________(3.3.3.2)

Where $f(\bar{x}_1) = \frac{1}{\Gamma n_1}\left(\frac{n_1}{\theta}\right)^{n_1} (\bar{x}_1)^{n_1-1} e^{\frac{-n_1\bar{x}_1}{\theta}}$

and $f(\bar{x}_2) = \frac{1}{\Gamma n_2}\left(\frac{n_2}{\theta}\right)^{n_2} (\bar{x}_2)^{n_2-1} e^{\frac{-n_2\bar{x}_2}{\theta}}$

A straight forward integration of (3.3.3.2) gives:

$$R(\hat{\theta}_{DST_3}) = I_2^* - \frac{2a(n_1+1)}{\phi\chi^2}\left\{I\left(\frac{\chi_2^2\phi}{2}, n_1+2\right) - I\left(\frac{\chi_1^2\phi}{2}, n_1+2\right)\right\} + \left\{\frac{2a\,n_1}{\chi^2} + \frac{\chi_1^2 a}{\chi^2} + \frac{an_1}{n_1+n_2}\right\}$$
$$\left\{I\left(\frac{\chi_2^2\phi}{2}, n_1+1\right) - I\left(\frac{\chi_1^2\phi}{2}, n_1+1\right)\right\} - \frac{\chi_1^2\phi a}{\chi^2}\left\{I\left(\frac{\chi_2^2\phi}{2}, n_1\right) - I\left(\frac{\chi_1^2\phi}{2}, n_1\right)\right\}$$
$$-\left\{I\left(\frac{\chi_2^2\phi}{2}, n_1\right) - I\left(\frac{\chi_1^2\phi}{2}, n_1\right)\right\}(a\phi - a - 1) + \left\{I\left(\frac{\chi_2^2\phi}{2}, n_1\right) - I\left(\frac{\chi_1^2\phi}{2}, n_1\right)\right\}$$
$$\left\{1 - \frac{an_1}{n_1+n_2} - \frac{e^{-a}}{\left(1 - \frac{a}{n_1+n_2}\right)^{n_1+n_2}}\right\} + \left(\frac{e^{-a}}{\left(1 - \frac{a}{n_1+n_2}\right)^{n_1+n_2}} - 1\right)$$

_____(3.3.3.3)

Where $$I_2^* = e^{a\phi - a}\int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} e^{\left[\frac{2at^2}{n_1\phi x^2} - \frac{2at}{x^2} - \frac{x_1^2 ta}{x^2 n_1} + \frac{x_1^2 a\phi}{x^2}\right]} \frac{1}{\Gamma n_1} e^{-t}\, t^{n_1-1}\, dt$$

3.4 Relative Risks of $\hat{\theta}_{DST_i}$

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator $\bar{x}_1$ in this case. For this purpose, we obtain the risk of $\bar{x}_1$ under L(Δ) as:

$$R_E(\bar{x}_1) = E[\bar{x}_1 | L(\Delta)]$$

$$= e^{-a}\int_0^{\infty}\int_0^{\infty} e^{a\left(\bar{x}_1/\theta\right)} f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2$$

$$-a\int_0^{\infty}\int_0^{\infty}\left(\tfrac{\bar{x}_1}{\theta}-1\right) f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2 - \int_0^{\infty}\int_0^{\infty} f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2 \qquad \text{\_\_\_\_\_\_\_(3.4.1)}$$

A straightforward integration of (3.4.1) gives

$$R_E(\bar{x}_1) = \frac{e^{-a}}{(1-a/n_1)^{n_1}} - 1 \qquad \text{\_\_\_\_\_\_\_(3.4.2)}$$

Now, we define the Relative Risk of $\hat{\theta}_{DST1}$ with respect to $\bar{x}_1$ under L(Δ) as follows –

$$RR_1 = \frac{R_E(\bar{x}_1)}{R(\hat{\theta}_{DST_1})} \qquad \text{\_\_\_\_\_\_\_(3.4.3)}$$

Using (3.4.2) and (3.3.1.3) the expression for RR_1 given in (3.4.3) can be obtained;

Similarly, we define the Relative Risk of $\hat{\theta}_{DST_2}$ with respect to $\bar{x}_1$ under L(Δ) as follows

$$RR_2 = \frac{R_E(\bar{x}_1)}{R(\hat{\theta}_{DST_2})} \qquad \text{\_\_\_\_\_\_\_(3.4.4)}$$

The expression for RR_2 given in (3.4.4) which can be obtained by using (3.4.2) and (3.3.2.3).

Finally, we define the Relative Risk of $\hat{\theta}_{DST3}$ with respect to $\bar{x}_1$ under L(Δ) as follows

$$RR_3 = \frac{R_E(\bar{x}_1)}{R(\hat{\theta}_{DST3})} \qquad \_\_\_\_\_(3.4.5)$$

Using (3.4.2) and (3.3.3.3) the expression for RR_3 given in (3.4.5) can be obtained.

Now, it is observed that RR_1, RR_2 and RR_3 are functions of 'ϕ', 'n_1', 'n_2', 'α' and 'a'. In order to study the behaviour of Relative Risk(s), we have taken a set of values of $(n_1, n_2) = (4,4), (4,6), (4,8), (4,10)$ and $(4,12)$, $\alpha^S = 1\%, 5\%$ and 10%, $\emptyset = 0.6\ (0.2)\ 1.8$ and $a = \pm 1$ to ± 3. The recommendations regarding the applications of proposed testimators are provided as follows:

The values of n^* and n^{**} are defined in section 3.2. For some values of (n_1, n_2) these values are obtained as follows:

Table -1 shows the values of n^* for $\phi = 1.0$ and $n_1 = 4$, $n_2 = 8$ and table - 2 shows the values of n^{**} for $\phi = 0.8$ and $n_1 = 4$, $n_2 = 10$

Table -1 $\phi = 1.0$

(n_1, n_2)	$\alpha = 1\%$	$\alpha = 5\%$
(4, 8)	4.08	4.40

Table -2 $\phi = 0.8$

(n_1, n_2)	$\alpha = 1\%$	$\alpha = 5\%$
(4, 8)	4.17	4.62
(4,10)	4.21	4.78

Similarly the other values of n^* and n^{**} can be computed for other values of (n_1, n_2) considered here.

3.5 Recommendations for $\hat{\theta}_{DST_i}$

In this section we wish to compare the performance of $\hat{\theta}_{DST_1}$, $\hat{\theta}_{DST_2}$ and $\hat{\theta}_{DST_3}$ with respect to the best available (unbiased) estimator of $\bar{x}_1$.

3.5.1 Recommendations for $\hat{\theta}_{DST_1}$

1. For various set of values of (n_1, n_2), keeping $\alpha = 1\%$ and allowing the variations in all the values of 'a', it is observed that the proposed testimator $\hat{\theta}_{DST1}$ performs better than $\bar{x}_1$ for $0.6 \leq \emptyset \leq 1.4$ considered here, except for few higher values i.e. $\emptyset = 1.8$. The magnitude of RR is higher for all the values of 'a' however maximum gain is achieved at a=3 and a= -3. Similar pattern is observed for other values of α^s i.e. 5% and 10% but the magnitude of Relative Risk is higher at α` = 1%. It is also observed that for a = -3 and $(n_1, n_2) = (4,8)$, $\hat{\theta}_{DST1}$ performs better.

2. In the next comparison stage we have fixed a=3, and have allowed the variation for values of α^s such as α = 1%, 5% and 10%. Maximum gain in risk is observed at $\emptyset = 1.0$ (though it is true for the whole range of $\emptyset$) again at α = 1%, relative risk values are higher than those at 5% and 10% so a lower level of significance i.e. α = 1% is recommended for better performance of the proposed testimator.

3. We have kept 'a' = 3.0 and have allowed the variation in α for $n_1 = 4$, $n_2 = 12$. It is seen that the Relative Risk values are much higher than unity, indicating superiority of the proposed testimator under Asymmetric Loss Function. A

value of $\alpha = 1\%$ shows maximum relative risk value implying that it is the most preferred value.

4. Again, for $n_1 = 4$, $n_2 = 10$, $\emptyset = 1.2$ for different values of α^s , the table of $\hat{\theta}_{DST\,1}$, indicates that, it dominates the usual estimator for the whole range of $\emptyset$, with best performance at $\alpha = 1\%$ and $a = 3$.

5. It has also been observed that the relative risk increases as $\emptyset$ increases from 0.6 to 1.0 reaches its maximum at $\emptyset = 1.0$ and then it decreases. The relative risk increases as n_2 increases for fixed value of n_1, and is maximum at (4, 8).

6. Thus, our recommendation for the use of $\hat{\theta}_{DST1}$ is to take $n_1 = 4$ and $n_2 = 8$ i.e. $n_2 = 2\,n_1$ and small values of α^s.

3.5.2 Recommendations for $\hat{\theta}_{DST_2}$ and $\hat{\theta}_{DST_3}$

We have considered two other choices of the weight functions viz. square of 'k' and making the values of 'k' to lie exactly between '0' and '1' and with these choices of shrinkage factors we have proposed $\hat{\theta}_{DST2}$ and $\hat{\theta}_{DST3}$, so it is natural to suggest which 'k' should be taken. This can be achieved by making a comparative study of the relative risks of values for all the three choices.

However a comparison of the values of relative risks for $\hat{\theta}_{DST1}$, $\hat{\theta}_{DST2}$ and $\hat{\theta}_{DST3}$ reveals that

(i) $\hat{\theta}_{DST2}$ is better than the usual estimator for $0.6 \leq \emptyset \leq 1.8$ however if n_1 is small similar pattern is observed for $\hat{\theta}_{DST3}$. However the magnitude of relative risk is smaller in case of $\hat{\theta}_{DST1}$ and $\hat{\theta}_{DST3}$ in comparison to $\hat{\theta}_{DST2}$.

So, we conclude that $\hat{\theta}_{DST2}$ is preferred in comparison to $\hat{\theta}_{DST1}$ and / or $\hat{\theta}_{DST3}$.

(ii) Our focus is also on recommending the degree(s) of asymmetry. A careful study of the table of Relative Risks, reveals following choices:

For $\hat{\theta}_{DST1}$, it is recommended that a = 3 and a = -3 for almost all the choices of n_1 and n_2

For $\hat{\theta}_{DST2}$, it is recommended to take a = -3 and a = 3 for several choices of n_1 and n_2

For $\hat{\theta}_{DST3}$, it is recommended to choose a = -3 and a = 3 and $\alpha = 1\%$. The performance of $\hat{\theta}_{DST3}$ is better than $\bar{x}_1$ in almost the whole range of $\emptyset$ ($0.6 \leq \emptyset \leq 1.4$)

CONCLUSION

To conclude it is recommended to use 'square' of the weight function (Shrinkage factor) with high positive / negative values of degrees of asymmetry along with lower level(s) of significance viz 1% and 5%. However 1% is preferable as the magnitude of relative risk is higher in this case showing better control over risk of the proposed estimator.

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 3.5.1.1 **Relative Risk of** $\hat{\theta}_{DST_1}$ $\alpha = 1\%, n_1 = 4, n_2 = 4$

Ø	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.60	1.028	1.286	1.435	1.063	2.592	3.138
0.80	2.103	2.656	3.197	2.061	3.394	4.895
1.00	3.852	4.902	6.384	3.835	4.388	6.405
1.20	3.508	4.076	5.113	2.64	3.159	5.44
1.40	1.893	2.129	2.629	1.535	2.009	4.921
1.60	1.036	1.195	1.508	0.855	1.162	3.1
1.80	0.639	0.765	0.995	0.5	0.664	1.849

Table : 3.5.1.2 **Relative Risk of** $\hat{\theta}_{DST_1}$ $\alpha = 1\%, n_1 = 4, n_2 = 8$

Ø	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.60	1.255	1.467	1.583	1.885	2.328	2.959
0.80	2.392	2.892	3.419	3.06	3.122	4.804
1.00	4.015	5.052	6.564	3.769	4.379	6.689
1.20	3.432	4.039	5.095	2.691	3.188	5.48
1.40	1.839	2.101	2.611	1.571	2.031	3.952
1.60	1.011	1.181	1.499	0.872	1.173	2.119
1.80	0.625	0.758	0.989	0.508	0.669	1.858

Table : 3.5.1.3 **Relative Risk of** $\hat{\theta}_{DST_1}$ $\alpha = 1\%, n_1 = 4, n_2 = 10$

Ø	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.60	1.339	1.528	1.631	1.109	3.77	3.586
0.80	2.49	2.967	3.488	4.83	6.044	6.737
1.00	4.064	5.096	6.617	3.75	4.375	9.438
1.20	3.411	4.029	5.09	2.706	3.195	7.49
1.40	1.825	2.094	2.605	1.581	2.037	4.96
1.60	1.004	1.177	1.496	0.877	1.176	3.123
1.80	0.622	0.755	0.987	0.511	0.671	1.861

Table : 3.5.1.4 **Relative Risk of** $\hat{\theta}_{DST_1}$ $\alpha = 5\%$, $n_1 = 4$, $n_2 = 8$

Ø	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.60	1.073	1.348	1.498	1.741	1.821	2.276
0.80	2.022	2.668	3.206	4.55	5.587	4.22
1.00	3.75	4.78	6.103	5.637	6.942	7.115
1.20	3.899	4.253	5.206	2.619	3.632	5.939
1.40	2.301	2.4	2.89	1.451	2.068	4.08
1.60	1.302	1.405	1.73	0.859	1.218	3.564
1.80	0.82	0.929	1.179	0.541	0.74	2.132

Table : 3.5.1.5 **Relative Risk of** $\hat{\theta}_{DST_1}$ $\alpha = 10\%$, $n_1 = 4$, $n_2 = 8$

Ø	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.60	1.005	1.283	1.439	2.066	2.578	2.588
0.80	1.833	2.523	3.065	3.567	3.055	3.929
1.00	3.771	4.873	6.119	6.985	7.386	7.903
1.20	4.793	4.72	5.529	2.753	4.258	5.462
1.40	2.972	2.739	3.167	1.417	2.208	4.227
1.60	1.668	1.635	1.947	0.858	1.291	3.059
1.80	1.053	1.106	1.365	0.567	0.81	2.431

Table : 3.5.2.1 **Relative Risk of** $\hat{\theta}_{DST_2}$ $\alpha = 1\%$, $n_1 = 4$, $n_2 = 8$

Ø	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.60	1.224	1.41	1.508	1.531	1.661	2.664
0.80	3.037	3.746	4.437	3.93	3.367	3.359
1.00	6.154	7.193	8.043	6.574	7.717	8.331
1.20	4.046	4.508	5.469	3.635	4.902	5.35
1.40	1.61	1.817	2.243	1.512	2.192	3.766
1.60	0.82	0.969	1.241	0.724	1.02	2.237
1.80	0.501	0.62	0.822	0.399	0.522	1.578

3.6 Shrinkage Testimator for the Variance of a Normal Distribution

Let X be normally distributed with mean μ and variance σ^2, both unknown. It is assumed that the prior knowledge about σ^2 is available in the form of an initial estimate σ_0^2. We are interested in constructing an estimator of σ^2 using the sample observations and possibly the guess value σ_0^2. We define a double stage shrinkage testimator of σ^2 as follows:

1. Take a random sample x_{1i} $(i = 1,2,\_\_, n_1)$ of size n_1 from $N(\mu, \sigma^2)$ and compute $\bar{x}_1 = \frac{1}{n_1}\sum x_{1i}$, $s_1^2 = \frac{1}{n_1-1}\sum(x_{1i} - \bar{x}_1)^2$.
2. Test the hypothesis $H_0 : \sigma^2 = \sigma_0^2$ against the alternative $H_1 : \sigma^2 \neq \sigma_0^2$ at level α using the test statistic $\frac{\nu_1 s_1^2}{\sigma_0^2}$, which is distributed as χ^2 with $\nu_1 = (n_1 - 1)$ degrees of freedom.
3. If H_0 is accepted at α level of significance i.e. $x_1^2 < \frac{\nu_1 s_1^2}{\sigma_0^2} < x_2^2$,where x_1^2 and x_2^2 refer to lower and upper critical points of the unbiased portioning of the test statistic at a given level of significance α, take $k_1 s_1^2 + (1 - k_1)\sigma_0^2$ as the shrinkage estimator of σ^2 with shrinkage factor k_1 dependent on the test statistic.
4. If H_0 is rejected, take a second sample x_{2j} $(j = 1,2,\_\_, n_2)$ of size $n_2 = (n - n_1)$ compute $\bar{x}_2 = \frac{1}{n_2}\sum x_{2j}$, $s_2^2 = \frac{1}{n_2-1}\sum(x_{2j} - \bar{x}_2)^2$ and take $(\nu_1 s_1^2 + \nu_2 s_2^2)/(\nu_1 + \nu_2)$ where $\nu_2 = (n_2 - 1)$ as the estimator of σ^2.

To summarize, we define the double- stage shrinkage Testimator $\hat{\sigma}_{DST1}^2$ of σ^2 as follows:

$$\hat{\sigma}^2_{DST1} = \begin{cases} k_1 s_1^2 + (1-k_1)\sigma_0^2 , & if \;\; H_0 \;\; is\; accepted \\ s_p^2 = \dfrac{(v_1 s_1^2 + v_2 s_2^2)}{(v_1+v_2)}, & if\; H_0 \;\; is \;\; rejected \end{cases}$$

Where $k_1 = \frac{v_1 s_1^2}{\sigma_0{}^2\,\chi^2}$

Estimators of this type with k arbitrary and lying between 0 and 1 have been proposed by Katti (1962), Shah(1964), Arnold and Al-Bayyati (1970), Waiker and Katti (1971), Pandey (1979) and k being dependent on the test statistics by Waiker Schuurman and Raghunandan (1984).

We define another double stage shrinkage Testimator $\hat{\sigma}^2_{DST2}$ of σ^2 by taking square of the shrinkage factor as $k_2 = k_1^2 = \left(\frac{v_1 s_1^2}{\sigma_0{}^2\,\chi^2}\right)^2$ which tends to zero more rapidly than k_1 as follows

$$\hat{\sigma}^2_{DST2} = \begin{cases} \left(\dfrac{v_1 s_1^2}{\sigma_0{}^2\,\chi^2}\right)^2 s_1^2 + \left(1 - \left(\dfrac{v_1 s_1^2}{\sigma_0{}^2\,\chi^2}\right)^2\right)\sigma_0^2 & , if \;\; H_0 \;\; is\; accepted \\ s_p^2 & , \;\; if\; H_0 \;\; is \;\; rejected \end{cases}$$

3.7 <u>Risk of Testimators</u>

In this section we derive the risk of two proposed testimators which are defined in the previous section.

3.7.1 <u>Risk of $\hat{\sigma}^2_{DST1}$</u>

The risk of $\hat{\sigma}^2{}_{DST1}$ under L(Δ) is defined by

$$R(\hat{\sigma}^2{}_{DST_1}) = E[\,\hat{\sigma}^2{}_{DST_1} \,|\, L(\Delta)]$$

$$= E\left[k_1 s_1{}^2 + (1-k_1)\sigma_0^2 \Big/ \chi_1^2 < \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} < \chi_2^2\right] \cdot p\left[\chi_1^2 < \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} < \chi_2^2\right]$$

$$+ E\left[s_p{}^2 \left| \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} > \chi_2^2\right.\right] \cdot p\left[\frac{\upsilon_1 s_1{}^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} > \chi_2^2\right]$$

_____ (3.7.1.1)

$$= \quad e^{-a} \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}} e^{a\left[\frac{\frac{\upsilon_1 s_1^2}{\sigma_0^2\chi^2}\left(s_1^2-\sigma_0^2\right)+\sigma_0^2}{\sigma^2}\right]} f(s_1^2)\,d s_1^2$$

$$- \; a \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}} \left[\frac{\frac{\upsilon_1 s_1^2}{\sigma_0^2\chi^2}\left(s_1^2-\sigma_0^2\right)+\sigma_0^2}{\sigma^2}-1\right] f(s_1^2)\,d s_1^2$$

$$- \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}} f(s_1^2)\,ds_1^2 \; + \; e^{-a} \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}} \int_0^{\infty} e^{a\left({s_p^2}/{\sigma^2}\right)} f(s_1^2)\,f(s_2^2)ds_1^2ds_2^2$$

$$+ \; e^{-a} \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}}^{\infty} \int_0^{\infty} e^{a\left({s_p^2}/{\sigma^2}\right)} f(s_1^2)\,f(s_2^2)ds_1^2ds_2^2$$

$$- \; a \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}} \int_0^{\infty} \left(\frac{s_p^2}{\sigma^2}-1\right) f(s_1^2)\,f(s_2^2)ds_1^2ds_2^2$$

$$- \; a \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}}^{\infty} \int_0^{\infty} \left(\frac{s_p^2}{\sigma^2}-1\right) f(s_1^2)\,f(s_2^2)ds_1^2ds_2^2$$

$$- \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}} \int_0^{\infty} f(s_1^2)\,f(s_2^2)ds_1^2ds_2^2 - \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}}^{\infty} \int_0^{\infty} f(s_1^2)\,f(s_2^2)ds_1^2ds_2^2$$

______(3.7.1.2)

Where $f(s_1^2) = \dfrac{1}{2^{\upsilon_1/2}\,\Gamma\left(\upsilon_1/2\right)} \left(s_1^2\right)^{\frac{\upsilon_1}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_1 s_1^2}{\sigma^2}\right)} ds_1^2$

$$f(s_2^2) = \frac{1}{2^{\upsilon_2/2}\,\Gamma\left(\upsilon_2/2\right)} \left(s_2^2\right)^{\frac{\upsilon_2}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_2 s_2^2}{\sigma^2}\right)} ds_2^2$$

Straight forward integration of (3.7.1.2) gives

$$R(\hat{\sigma}^2_{DST1}) = \left(\frac{\sigma^2}{\mathcal{V}_1}\right)^{\mathcal{V}_1/2} \left(\frac{\sigma^2}{\mathcal{V}_2}\right)^{\mathcal{V}_2/2}$$

$$R(\hat{\sigma}^2_{DST1}) = \left(\frac{\sigma^2}{\mathcal{V}_1}\right)^{\mathcal{V}_1/2} \left(\frac{\sigma^2}{\mathcal{V}_2}\right)^{\mathcal{V}_2/2} \begin{bmatrix} I_1^* - \frac{2a}{\lambda\chi^2}\left(\frac{\mathcal{V}_1}{2}+1\right) \\ \left\{I\left(\chi_2^2\lambda,\frac{\mathcal{V}_1}{2}+2\right) - I\left(\chi_1^2\lambda,\frac{\mathcal{V}_1}{2}+2\right)\right\} \\ + \frac{a\mathcal{V}_1}{\chi^2}\left\{I\left(\chi_2^2\lambda,\frac{\mathcal{V}_1}{2}+1\right) - I\left(\chi_1^2\lambda,\frac{\mathcal{V}_1}{2}+1\right)\right\} \\ -(a\lambda - a + 1)\left\{I\left(\chi_2^2\lambda,\frac{\mathcal{V}_1}{2}\right) - I\left(\chi_1^2\lambda,\frac{\mathcal{V}_1}{2}\right)\right\} \\ -\frac{a\mathcal{V}_1}{\mathcal{V}_1+\mathcal{V}_2}\left\{I\left(\chi_1^2\lambda,\frac{\mathcal{V}_1}{2}+1\right) - I\left(\chi_2^2\lambda,\frac{\mathcal{V}_1}{2}+1\right)\right\} \\ + \frac{a\mathcal{V}_1}{\mathcal{V}_1+\mathcal{V}_2}\left\{I\left(\chi_1^2\lambda,\frac{\mathcal{V}_1}{2}\right) - I\left(\chi_2^2\lambda,\frac{\mathcal{V}_1}{2}\right)\right\} \\ -\left\{I\left(\chi_1^2\lambda,\frac{\mathcal{V}_1}{2}\right) - I\left(\chi_2^2\lambda,\frac{\mathcal{V}_1}{2}\right)+1\right\} + I_2^* \end{bmatrix}$$

_____ (3.7.1.3)

Where $I(x;p) = (1/\Gamma p)\int_0^x e^{-x}\,x^{p-1}\,dx$ refers to the standard incomplete gamma function, $\lambda = \frac{\sigma_0^2}{\sigma^2}$ and

$$I_1^* = \frac{e^{a(\lambda-1)}}{2^{\mathcal{V}_1/2}\,\Gamma\left(\frac{\mathcal{V}_1}{2}\right)} \int_{x_1^2\lambda}^{x_2^2\lambda} e^{\left[\frac{a t_1^2}{\lambda \mathcal{V}_1 x^2} - \frac{a t_1}{x^2}\right]} e^{-\frac{1}{2}t_1} (t_1)^{\frac{\mathcal{V}_1}{2}-1}\, dt_1$$

$$I_2^* = \frac{e^{-a}}{2^{\left(\frac{v_1}{2}+\frac{v_2}{2}\right)}\left(\frac{1}{2}-\frac{a}{v_1+v_2}\right)^{\left(\frac{v_1}{2}+\frac{v_2}{2}\right)}}\left[I\left(\chi_1^2\lambda,\ \frac{v_1}{2}\right) - I\left(\chi_2^2\lambda,\ \frac{v_1}{2}\right) + 1\right]$$

3.7.2 Risk of $\hat{\sigma}^2_{DST2}$

Again, we obtain the risk of $\hat{\sigma}^2{}_{DST_2}$ under L(Δ) with respect to $s_p{}^2$, given by

$$R(\hat{\sigma}^2{}_{DST_2}) = E[\hat{\sigma}^2{}_{DST_2} \mid L(\Delta)]$$

$$= E\left[\left(\frac{\upsilon_1 s_1{}^2}{\sigma_0^2\chi^2}\right)^2 s_1{}^2 + \left(1-\left(\frac{\upsilon_1 s_1{}^2}{\sigma_0^2\chi^2}\right)^2\right)\sigma_0^2 \Big/ \chi_1^2 < \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} < \chi_2^2\right]\cdot p\left[\chi_1^2 < \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} < \chi_2^2\right]$$

$$+ E\left[s_p{}^2 \;\Big|\; \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} > \chi_2^2\right]\cdot p\left[\frac{\upsilon_1 s_1{}^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} > \chi_2^2\right]$$

______(3.7.2.1)

$$= \quad e^{-a}\int_{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}} e^{a\left[\frac{\left(\frac{\upsilon_1 s_1{}^2}{\sigma_0^2\chi^2}\right)^2\left(s_1{}^2-\sigma_0^2\right)+\sigma_0^2}{\sigma^2}\right]} f(s_1{}^2)\, ds_1{}^2$$

$$- \quad a\int_{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}} \left[\frac{\left(\frac{\upsilon_1 s_1{}^2}{\sigma_0^2\chi^2}\right)^2\left(s_1{}^2-\sigma_0^2\right)+\sigma_0^2}{\sigma^2} - 1\right] f(s_1{}^2)\, ds_1{}^2$$

$$- \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}} f(s_1{}^2)\,ds_1{}^2 + e^{-a}\int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}\int_0^{\infty} e^{a\left({s_p{}^2}/{\sigma^2}\right)} f(s_1{}^2)f(s_2{}^2)\,ds_1{}^2 ds_2{}^2$$

$$+\ e^{-a}\int_{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}}^{\infty}\int_{0}^{\infty} e^{a\left(s_p^2/\sigma^2\right)} f(s_1^2)f(s_2^2)ds_1^2ds_2^2 - a\int_{0}^{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}\int_{0}^{\infty}\left(\frac{s_p^2}{\sigma^2}-1\right)f(s_1^2)f(s_2^2)ds_1^2ds_2^2$$

$$-a\int_{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}}^{\infty}\int_{0}^{\infty}\left(\frac{s_p^2}{\sigma^2}-1\right)f(s_1^2)f(s_2^2)ds_1^2ds_2^2 - \int_{0}^{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}\int_{0}^{\infty} f(s_1^2)f(s_2^2)ds_1^2ds_2^2$$

$$-\int_{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}}^{\infty}\int_{0}^{\infty} f(s_1^2)f(s_2^2)ds_1^2ds_2^2$$

______(3.7.2.2)

Where $f(s_1^2)=\dfrac{1}{2^{\upsilon_1/2}\,\Gamma\left(\upsilon_1/2\right)}\left(s_1^2\right)^{\frac{\upsilon_1}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_1 s_1^2}{\sigma^2}\right)} ds_1^2$

$$f(s_2^2)=\frac{1}{2^{\upsilon_2/2}\,\Gamma\left(\upsilon_2/2\right)}\left(s_2^2\right)^{\frac{\upsilon_2}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_2 s_2^2}{\sigma^2}\right)} ds_2^2$$

Straight forward integration of (3.7.2.3) gives

$$\left(\hat{\sigma}^2_{DST2}\right)=\left(\frac{\sigma^2}{V_1}\right)^{V_1/2}\left(\frac{\sigma^2}{V_2}\right)^{V_2/2}\left[\begin{array}{c} I_1^* - \frac{4a}{\lambda^2\,(\chi^2)^2}\left(\frac{V_1}{2}+1\right)\left(\frac{V_1}{2}+2\right) \\ \left\{I\left(\chi_2^2\lambda,\ \frac{V_1}{2}+3\right)-I\left(\chi_1^2\lambda,\ \frac{V_1}{2}+3\right)\right\} \\ +\ \frac{4a}{\lambda(\chi^2)^2}\left(\frac{V_1}{2}\right)\left(\frac{V_1}{2}+1\right) \\ \left\{I\left(\chi_2^2\lambda,\ \frac{V_1}{2}+2\right)-I\left(\chi_1^2\lambda,\ \frac{V_1}{2}+2\right)\right\} \\ -(a\lambda-a+1)\left\{I\left(\chi_2^2\lambda,\ \frac{V_1}{2}\right)-\ I\left(\chi_1^2\lambda,\ \frac{V_1}{2}\right)\right\} \\ -\frac{aV_1}{V_1+V_2}\left\{I\left(\chi_1^2\lambda\ ,\ \frac{V_1}{2}+1\right)-\ I\left(\chi_2^2\lambda\ ,\ \frac{V_1}{2}+1\right)\right\} \\ +\ \frac{aV_1}{V_1+V_2}\left\{I\left(\chi_1^2\lambda\ ,\ \frac{V_1}{2}\right)-\ I\left(\chi_2^2\lambda\ ,\ \frac{V_1}{2}\right)\right\} \\ -\left\{I\left(\chi_1^2\lambda\ ,\ \frac{V_1}{2}\right)-\ I\left(\chi_2^2\lambda\ ,\ \frac{V_1}{2}\right)+1\right\}+\ I_2^* \end{array}\right]$$

_____(3.7.2.3)

Where $I(x;p) = (1/\Gamma p)\int_0^x e^{-x}x^{p-1}\,dx$ refers to the standard incomplete gamma function and

$$I_1^* = \frac{e^{a(\lambda-1)}}{2^{v_1/2}\,\Gamma\left(\frac{v_1}{2}\right)}\int_{x_1^2\lambda}^{x_2^2\lambda} e^{\left[\frac{at_1^3}{\lambda^2 v_1 (x^2)^2}-\frac{at_1^2}{\lambda (x^2)^2}\right]}\, e^{-\frac{1}{2}t_1}\,(t_1)^{\frac{v_1}{2}-1}\,dt_1$$

$$I_2^* = \frac{e^{-a}}{2^{\left(\frac{v_1}{2}+\frac{v_2}{2}\right)}\left(\frac{1}{2}-\frac{a}{v_1+v_2}\right)^{\left(\frac{v_1}{2}+\frac{v_2}{2}\right)}}\left[I\left(\chi_1^2\lambda,\frac{v_1}{2}\right)-I\left(\chi_2^2\lambda,\frac{v_1}{2}\right)+1\right]$$

3.8 Relative Risk of $\hat{\sigma}^2_{DSTi}$

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator $s_p{}^2$ in this case. For this purpose, we obtain the risk of $s_p{}^2$ under $L_E\left(\hat{\sigma}^2,\sigma^2\right)$ as:

$$R_E(s_p^2) = E[s_p^2 \mid L(\hat{\sigma}^2,\sigma^2)]$$

$$= e^{-a}\int_0^\infty\int_0^\infty e^{a\left[\frac{s_p{}^2}{\sigma^2}\right]} f(s_1{}^2)f(s_2{}^2)\,ds_1{}^2 ds_2{}^2$$

$$-a\int_0^\infty\int_0^\infty\left[\frac{s_p^2}{\sigma^2}-1\right] f(s_1{}^2)f(s_2{}^2)\,ds_1{}^2 ds_2{}^2 - \int_0^\infty\int_0^\infty f(s_1{}^2)f(s_2{}^2)\,ds_1{}^2 ds_2{}^2$$

______(3.8.1)

Where $f(s_1{}^2) = \dfrac{1}{2^{\upsilon_1/2}\,\Gamma\left(\upsilon_1/2\right)}\left(s_1{}^2\right)^{\frac{\upsilon_1}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_1 s_1{}^2}{\sigma^2}\right)}\,ds_1{}^2$

$$f(s_2^2)=\frac{1}{2^{\upsilon_2/2}\,\Gamma\left(\upsilon_2/2\right)}\left(s_2^2\right)^{\frac{\upsilon_2}{2}-1}e^{\left(-\frac{1}{2}\frac{\upsilon_2 s_2^2}{\sigma^2}\right)}ds_2^2$$

A straightforward integration of (3.8.1) gives

$$R_E(s_p^2)=\left(\frac{\sigma^2}{\nu_1}\right)^{\nu_1/2}\left(\frac{\sigma^2}{\nu_2}\right)^{\nu_2/2}\left[\frac{e^{-a}}{2^{\left(\nu_1/2+\nu_2/2\right)}\left(\frac{1}{2}-\frac{a}{\nu_1+\nu_2}\right)^{\nu_1/2+\nu_2/2}}-1\right]$$

_______(3.8.2)

Now, we define the Relative Risk of $\hat{\sigma}^2{}_{DST_i}$, $i=1,2$ with respect to s_p^2 under $L\left(\hat{\sigma}^2,\sigma^2\right)$ as follows:

$$RR_1=\frac{R_E(s_p^2)}{R(\hat{\sigma}^2{}_{DST1})}$$ _______(3.8.3)

Using (3.8.2) and (3.7.1.3) the expression for RR_1 given in (3.8.3) can be obtained; it is observed that RR_1 is a function of 'ν_1', 'ν_2', 'λ', 'α' and 'a'.

Finally, we define the Relative Risk of $\hat{\sigma}^2{}_{ST_2}$ by

$$RR_2=\frac{R_E(s_p^2)}{R(\hat{\sigma}^2{}_{DST2})}$$ _______(3.8.4)

The expression for RR_2 is given by (3.8.4) can be obtained by using (3.8.2) and (3.7.2.3). Again we observed that RR_2 is a function of 'ν_1', 'ν_2', 'λ', 'α' and 'a'.

3.9 Recommendations for $\hat{\sigma}^2_{DSTi}$

In this section we wish to compare the performance of $\hat{\sigma}^2{}_{DST_1}$ and $\hat{\sigma}^2{}_{DST_2}$ with respect to the best available (unbiased) estimator of σ^2 $i.e.$ $\hat{\sigma}^2$.

3.9.1 Recommendations for $\hat{\sigma}^2_{DST1}$

It is observed that the above expressions (3.8.3) and (3.8.4) are functions of $\alpha, \lambda, \nu_1, \nu_2$ and the degrees of asymmetry " a ". For the comparison purpose we have considered several values for these viz. (ν_1, ν_2) = (6,6), (6,9), (6,12), (6,15), (6,18); (8,8), (8,12), (8,16), (8,20), (8,24) and (10,10), (10,15), (10,20), (10,25), (10,30) ; α = 1%, 5% and 10%, and a = -3, -2, -1, 1, 1.25, 1.50 and λ = 0.2 (0.2) 2.0 .

In all there will be several tables for these data sets of Relative Risk (RR_1). We have presented some of the tables at the end of the chapter. However, our recommendations based on all these findings are as follows:

(i) The proposed testimator $\hat{\sigma}^2{}_{DST_1}$ performs better than the pooled estimator $s_p{}^2$ for almost all the values considered as above. However some of the best performances are outlined specifically.

(ii) $\hat{\sigma}^2{}_{DST_1}$ dominates the usual estimator when (ν_1, ν_2) = (6,6) ; α = 1% ; a = -1 for $0.2 \le \lambda \le 2.0$ and for a = +1 the range of λ is $0.2 \le \lambda \le 2.0$.

(iii) As 'ν_2' increases the RR_1 values are still greater than unity, but decrease in magnitude also the range of 'λ' changes slightly now it becomes $0.6 \le \lambda \le 1.8$ for negative values of 'a'. A similar pattern is observed when 'a' is positive for almost $0.6 \le \lambda \le 1.8$.

(iv) The performance of $\hat{\sigma}^2{}_{DST_1}$ is the best when a = +1 or a = -1 in terms of the range of λ, the magnitude of RR_1 values for the first data set i.e. (6,6). The same remains true when ν_2 increases i.e. (6,9) etc. Here we have considered these values for α = 1%.

(v) As the other quantity of interest i.e. the level of significance in addition to the degrees of asymmetry. We change 'α' to 5% and 10% it is observed that still the proposed testimator performs better for the 'ranges' mentioned as above. i.e. when 'a' is negative $0.2 \leq \lambda \leq 2.0$ and when 'a' is positive it becomes $0.2 \leq \lambda \leq 1.6$ indicating that range shrinks for overestimation case. Still the values of RR_1 are more than unity but their magnitude decreases slightly.

(vi) Now, we have considered the other values of (ν_1, ν_2) as mentioned above and it is observed that RR_1 values are still higher than unity for these different data sets, with almost the same ranges of 'λ' as above for positive as well as negative values of 'a'. Again as ν_2 increases the magnitude of RR_1 values decreases but not falling below 1.

(vii) Overall recommendations are: ν_1 should be small i.e. $\nu_1 \ngtr 10$ and $\nu_2 \leq 3\nu_1$, $\alpha = 1\%$ i.e. a smaller level of significance and for various degrees of asymmetry i.e. 'a' could be extreme negative as a = -3 or it could be considerably positive i.e. a = 1.5. The best suggested values are a = -1 or a = +1.

(viii) When these RR_1 values are compared with the Mean Square values of $\hat{\sigma}^2_{DST_1}$ proposed by Pandey and Srivastava (1987) it is observed that the magnitude of RR_1 values are **HIGHER**, the range of 'λ' increases considerably as it was (0.5 – 1.5) and now it becomes almost (0.2 – 2.0) earlier it was recommended that $\nu_2 \leq 2\nu_1$ now it becomes $\nu_2 \leq 3\nu_1$ a considerable increase in the choice of ν_2. Implying that the use of ASL not only allows to take account for various degrees of asymmetry (i.e. choose 'a' accordingly when over / under estimation is more serious) but also increases the range of 'λ', ν_2 etc.

3.9.2 Recommendations for $\hat{\sigma}^2_{DST2}$

We have also proposed $\hat{\sigma}^2{}_{DST_2}$ which is obtained by squaring the shrinkage factor. The performance of it, is compared with respect to $s_p{}^2$ for the **same** data as considered for $\hat{\sigma}^2{}_{DST_1}$. Again, similar tables of RR_2 will be generated for these data sets. Our recommendations based on all these computations are as follows:

(i) It is observed that the magnitude of RR_2 values is higher than RR_1 values. The proposed testimator performs better than the best available estimator for almost all the values considered here. The best performing data sets are mentioned briefly.

(ii) $\hat{\sigma}^2{}_{DST_2}$ dominates $s_p{}^2$ when $(\nu_1, \nu_2) =$ (6,6), $\alpha = 1\%$ for a = -1, $0.2 \leq \lambda \leq 2.0$ and for a = +1, $0.2 \leq \lambda \leq 2.0$ as obtained earlier.

(iii) As 'ν_2' increases the RR_2 values decrease in their magnitude (but still above unity). Here the range of 'λ' change shortens slightly as it is now $0.6 \leq \lambda \leq 1.8$ for negative values of 'a' however when 'a' is positive it remains unchanged i.e. $0.2 \leq \lambda \leq 2.0$.

(iv) The performance of $\hat{\sigma}^2{}_{DST_2}$ is at its best when a = ± 1. As 'ν_2' increases i.e. for the other data set (6,9), (6,12), (6,15) or (6,18) the magnitude of RR_2 decreases slightly but not below unity. Again, if we increase ν_1 i.e. (8,8), (8,12) etc. Similar behaviour of RR_2 values is observed but their magnitude change.

(v) Now taking $\alpha = 5\%$ and $\alpha = 10\%$ when the values of RR_2 are obtained again these values are 'good' in the sense of being more than unity. But there is a decrease in the magnitude of RR_2 values as 'α' increase.

(vi) We therefore recommend as, ν_1 should be small i.e. $\nu_1 \ngtr 10$ and $\nu_2 \leq 3\nu_1$, and choose $\alpha = 1\%$. However the degree of asymmetry could chosen for a fairly large range i.e. from a = -3 to a = 1.5. The best performing values are observed for $a = \pm 1$.

(vii) Comparing these RR_2 values with those obtained by Pandey and Srivastava (1987) under the MSE criterion (or the use of 'SELF') indicate that these values are 'better' than those values showing that the application of Asymmetric Loss Function yields better result also providing a choice to tailor the risk by choosing 'a' appropriately. Further the range of 'λ' increases.

CONCLUSIONS:

Two shrinkage testimators viz. $\hat{\sigma}^2_{DST_1}$ and $\hat{\sigma}^2_{DST_2}$ have been proposed for the variance of a Normal distribution. It is concluded that (i) use asymmetric loss function to study the risk properties. (ii) ν_1 should be small preferably should not exceed 10 for both the cases. (iii) $\nu_2 \leq 3\nu_1$ (iv) take $\alpha = 1\%$ and take $0.2 \leq \lambda \leq 2.0$ for negative values of 'a' and take $0.2 \leq \lambda \leq 1.8$ for positive values of 'a'. (v) take 'SQUARE' of the shrinkage factor.

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 3.9.1.1 **Relative Risk of** $\hat{\sigma}^2_{DST_1}$ $\alpha = 1\%$, $(\nu_1, \nu_2) = (6, 6)$

λ	a = -3	a = -2	a = -1	a = 1	a = 1.25	a = 1.50
0.20	1.059	1.531	1.229	1.835	1.884	1.890
0.40	1.257	1.649	2.081	2.06	1.984	1.975
0.60	1.658	2.618	3.714	3.514	3.762	3.509
0.80	3.484	4.013	5.103	5.913	4.623	3.974
1.00	4.433	5.486	6.834	7.02	5.153	4.851
1.20	4.086	5.332	6.08	6.884	4.774	3.368
1.40	3.753	4.414	5.827	4.499	3.213	2.336
1.60	2.357	3.417	4.518	2.909	2.087	1.541
1.80	1.637	2.339	3.117	1.911	1.354	0.999
2.00	1.239	1.735	2.236	1.295	0.899	0.654

Table : 3.9.1.2 **Relative Risk of** $\hat{\sigma}^2_{DST_1}$ $\alpha = 5\%$, $(\nu_1, \nu_2) = (6, 6)$

λ	a = -3	a = -2	a = -1	a = 1	a = 1.25	a = 1.50
0.20	1.379	1.831	1.813	1.49	1.692	1.057
0.40	1.972	1.912	2.436	2.592	2.54	1.536
0.60	1.339	2.021	3.939	2.855	2.711	2.676
0.80	2.271	3.074	4.42	3.909	3.568	3.334
1.00	3.593	4.462	5.081	5.001	4.111	4.67
1.20	4.153	4.172	5.051	3.563	2.842	2.267
1.40	3.549	3.736.	4.476	2.299	1.837	1.472
1.60	2.754	2.888	3.762	1.634	1.294	1.034
1.80	2.182	2.166	3.133	1.212	0.945	0.748
2.00	1.815	1.658	2.649	0.926	0.707	0.551

Table : 3.9.1.1 **Relative Risk of** $\hat{\sigma}^2_{DST_1}$ $\alpha = 1\%$, $(\nu_1, \nu_2) = (8, 8)$

λ	a = -3	a = -2	a = -1	a = 1	a = 1.25	a = 1.50
0.20	1.486	1.74	1.481	1.775	1.792	1.851
0.40	1.989	1.839	1.861	2.693	2.606	2.56
0.60	1.617	2.722	2.195	3.211	3.123	3.249
0.80	3.301	3.041	3.476	4.793	4.437	4.311
1.00	4.105	5.296	6.005	6.403	5.446	5.405
1.20	4.077	4.315	5.212	5.968	4.572	3.507
1.40	3.886	3681	4.75	3.679	2.869	2.256
1.60	2.5	2.75	3.089	2.326	1.797	1.413
1.80	1.782	2.675	2.395	1.534	1.159	0.897
2.00	1.392	2.069	2.338	1.059	0.779	0.589

Table : 3.9.2.1 **Relative Risk of** $\hat{\sigma}^2_{DST_2}$ $\alpha = 1\%$, $(\nu_1, \nu_2) = (6, 6)$

λ	a = -3	a = -2	a = -1	a = 1	a = 1.25	a = 1.50
0.20	0.647	1.117	1.897	1.214	1.164	1.156
0.40	0.652	1.997	2.725	1.452	1.347	1.299
0.60	1.381	2.054	3.202	2.629	2.496	3.603
0.80	3.826	4.273	4.73	4.669	3.67	4.078
1.00	5.225	5.732	5.933	5.684	5.396	6.077
1.20	4.882	4.758	4.747	4.077	4.857	4.248
1.40	3.444	3.075	3.385	3.59	3.369	2.531
1.60	2.035	2.952	3.022	2.702	1.951	1.464
1.80	1.397	1.976	2.755	1.68	1.181	0.871
2.00	1.06	1.463	2.735	1.103	0.752	0.541

Table : 3.9.2.2 **Relative Risk of** $\hat{\sigma}^2_{DST_2}$ $\alpha = 1\%$, $(\nu_1, \nu_2) = (8, 8)$

λ	a = -3	a = -2	a = -1	a = 1	a = 1.25	a = 1.50
0.20	0.74	1.306	1.328	1.143	1.074	1.036
0.40	0.61	1.962	2.865	1.177	1.173	1.909
0.60	1.285	2.009	4.475	3.825	3.595	3.514
0.80	3.539	4.15	5.735	4.962	4.863	3.643
1.00	6.627	7.176	7.917	6.658	5.38	4.939
1.20	4.728	5.556	5.968	4.261	4.945	3.906
1.40	3.439	3.151	4.721	3.488	2.737	2.179
1.60	2.076	2.192	3.519	2.069	1.577	1.233
1.80	1.466	1.692	3.76	1.313	0.968	0.736
2.00	1.151	1.406	2.141	0.885	0.631	0.465

Table : 3.9.2.3 **Relative Risk of** $\hat{\sigma}^2_{DST_2}$ $\alpha = 5\%$, $(\nu_1, \nu_2) = (6, 6)$

λ	a = -3	a = -2	a = -1	a = 1	a = 1.25	a = 1.50
0.20	0.848	1.521	1.017	1.577	1.549	1.582
0.40	0.957	1.87	2.343	1.947	1.967	1.927
0.60	1.987	1.302	2.97	2.942	2.777	2.698
0.80	3.798	2.522	4.836	4.478	3.225	3.658
1.00	4.434	4.561	5.914	5.711	4.446	5.626
1.20	3.403	3.652	4.607	3.579	2.887	2.326
1.40	2.456	3.585	3.919	2.183	1.751	1.413
1.60	1.887	2.773	2.884	1.487	1.171	0.936
1.80	1.555	2.272	2.015	1.065	0.818	0.641
2.00	1.357	1.964	1.388	0.79	0.589	0.45