CHAPTER II :

ON NECESSARY BEST ESTIMATOR

2.0 SUMMARY

In this chapter we consider the concept of NBE of various orders as introduced by Prabhu-Ajgaonkar [12] and defined in definition 1.2.5. We simplify the proof of Prabhu-Ajgaonkar for NBE of order 1 and also correct his assertion regarding NBE of order 2.

2.1 NBE OF ORDER 1

While introducing the concept of NBE, Prabhu-Ajgaonkar [12], proved that a NBE of order 1 exists and it coincides with the Horvitz-Thompson estimator given by

$T(s,\underset{\sim}{Y}) = \sum_{i \in s} [Y_i/\pi_i]$ where $\pi_i = \sum_{s \ni i} p(s)$ is the inclusion probability for unit i.

His original proof involves complicated notation. We give below a simpler proof for the same.

Let $t(s,\underset{\sim}{Y})$ be a linear estimator of the population total for the finite population $\mathcal{U} = \{1,2,\ldots, N\}$. Then $t(s,\underset{\sim}{Y})$ has the form (1.2.6). Also $t(s,\underset{\sim}{Y})$ is unbiased for Y if, and only if

$$\sum_{s \ni i} b(s,i)\, p(s) = 1, \quad i=1,2,\ldots,N. \qquad \ldots(2.1.1)$$

Further the variance of $t(s,\underset{\sim}{Y})$ is given by (1.2.10).

Then NBE of order 1 can be obtained by choosing the coefficients $b(s,i)$ in (1.2.6) by minimising (1.2.10), subject to conditions (2.1.1), at those values of $\underset{\sim}{Y}$ for which exactly one Y_i is non-zero. For this we consider

$$\phi = \sum_{i=1}^{N} \sum_{s \ni i} b^2(s,i)\, Y_i^2\, p(s) + \sum_{i \neq j}^{N} \sum_{s \ni \{i,j\}} b(s,i).\, b(s,j) Y_i Y_j\, p(s) - 2\lambda_i \sum_{s \ni i} b(s,i)\, p(s). \qquad \ldots(2.1.2)$$

Here the λ_i are Lagrange's multipliers. Differentiating (2.1.2) with respect to $b(s,i)$ and equating the derivative to zero, we get

$$b(s,i)\, Y_i^2\, p(s) = \lambda_i p(s). \qquad \ldots(2.1.3)$$

Since $p(s) > 0$,

$$b(s,i) = \lambda_i / Y_i^2. \qquad \ldots(2.1.4)$$

Thus it is clear from (2.1.4) that, for $s \in S$, the coefficient $b(s,i)$ depends only upon i and not on the sample.

Now from (2.1.1) one can see that $b(s,i) = \frac{1}{\pi_i}$. This completes the proof.

2.2 NBE OF ORDER 2

We now prove here that NBE of Order 2 and higher do not exist for a non-unicluster design. This contradicts a result of Prabhu-Ajgaonkar [12].

To get NBE of Order 2, the coefficients $b(s,i)$ in (1.2.6) have to be obtained by minimising (1.2.10), subject to conditions (2.1.1), at those $\underset{\sim}{Y}$ for which exactly two Y_i's are non zero. We show that this procedure leads to a $t(s,\underset{\sim}{Y})$ for which $b(s,i)$ depends on i and not on s. We next show that this condition leads to a contradiction.

Choose s_1, s_2 in S such that $s_1 \neq s_2$ and a unit $i \in s_1 \cap s_2$. This is possible because the design is non-unicluster. We may assume that some unit $j \in s_1$ but $j \notin s_2$. Let $\underset{\sim}{Y}$ be such that only Y_i and Y_j are non-zero. To minimise (1.2.10) subject to conditions (2.1.1) we consider

$$\emptyset = \text{Var}\left[t(s,\underset{\sim}{Y})\right] - 2\lambda_i \sum_{s \ni i} b(s,i)\ p(s) - 2\lambda_j \sum_{s \ni j} b(s,j)p(s) \qquad \ldots(2.2.1)$$

where λ_i, λ_j are Lagrange's multipliers.

Differentiating (2.2.1) with respect to $b(s_1,i)$ and $b(s_2,i)$ and equating the results to zero, we get

$$Y_i^2 b(s_1,i)\ p(s_1) + Y_i Y_j b(s_1,j)p(s_1) = \lambda_i p(s_1) \qquad \ldots(2.2.2)$$

and $$Y_i^2\ b(s_2,i)\ p(s_2) = \lambda_i p(s_2). \qquad \ldots(2.2.3)$$

Since $p(s_1)$ and $p(s_2)$ are positive, we get

$$b(s_1,i)\ Y_i^2 + b(s_1,j)\ Y_i Y_j = \lambda_i \qquad \ldots(2.2.4)$$

and $$b(s_2,i)\ Y_i^2 = \lambda_i\ . \qquad \ldots(2.2.5)$$

From (2.2.4) and (2.2.5), we get

$$\left[b(s_1,i)-b(s_2,i)\right] Y_i^2 + b(s_1,j)\ Y_i Y_j = 0. \qquad \ldots(2.2.6)$$

Since (2.2.6) holds for all Y_i, Y_j, we must have

$$b(s_1,i) = b(s_2,i) \text{ and } b(s_1,j) = 0.$$

Thus on the one hand $b(s,i)$ depends only on i; but on other hand $b(s_1,j) = 0$. This shows that $b(s,j) = 0$ for all $s \ni j$.

This contradicts the unbiasedness of $t(s,\underset{\sim}{Y})$. This contradiction proves that a NBE of Order 2 does not exist.

Example : Suppose that we take simple random samples of size 2 from a population of size 3. Then the samples with positive probabilities are $s_1 = \{1,2\}$; $s_2 = \{2,3\}$ and $s_3 = \{1,3\}$.

Let T_1 be the HT estimator. Then

$$E(T_1^2) = \frac{3}{4}\{ (Y_1+Y_2)^2 + (Y_2+Y_3)^2 + (Y_3+Y_1)^2\}$$

$$= \frac{3}{2}\{Y_1^2 + Y_2^2 + Y_3^2 + Y_1Y_2 + Y_2Y_3 + Y_3Y_1\}$$

Consider the estimator T_2 for which

$$T_2(s_1,\underset{\sim}{Y}) = T_2(s_3,\underset{\sim}{Y}) = \frac{3}{2} Y_1 \text{ and}$$

$$T_2(s_2,\underset{\sim}{Y}) = 3(Y_2+Y_3).$$

Then

$$E(T_2^2) = \frac{3}{2} Y_1^2 + 3(Y_2+Y_3)^2$$

$$= \frac{3}{2} Y_1^2 + 3Y_2^2 + 3Y_3^2 + 6Y_2Y_3.$$

Hence

$$\text{Var}(T_2) - \text{Var}(T_1) = \underset{\sim}{Y}' A \underset{\sim}{Y}$$

where

$$A = \begin{bmatrix} 0 & -3/4 & -3/4 \\ -3/4 & 3/2 & 9/4 \\ -3/4 & 9/4 & 3/2 \end{bmatrix}.$$

The leading 2 x 2 principal minor of A is negative. Thus T_1 is not a NBE of Order 2.