Chapter 3

Symmetric Quark Mixing & Some Consequences

3.1 Symmetric Ansatz for quark mixing

3.1.1 CKM matrix with symmetric moduli

Apart from the masses, the other existing free parameters in the standard model are the three mixing angles and a CP-violating phase, which are incorporated into the quark sector of the standard model via the Cabibbo-Kobayashi-Maskawa (CKM) matrix V. All the presently available data [21] is consistent with having symmetric moduli for CKM matrix i.e.

$$|V_{ij}| = |V_{ji}|. \tag{3.1}$$

It should be noted that for three generations, the assumption that V has symmetric moduli implies a single constraint on the matrix V because the unitarity requirement alone yields

$$A = |V_{12}|^2 - |V_{21}|^2 = |V_{31}|^2 - |V_{13}|^2 = |V_{23}|^2 - |V_{32}|^2 \tag{3.2}$$

for three generations. The fact that experimentally the asymmetry parameter A is, in general, small i.e. $A < 10^{-4}$ and in particular $|V_{12}|$ and $|V_{21}|$ are quoted to be same modulo the errors and both of them lie between 0.217 and 0.223 prompted us to believe that V has symmetric moduli.

It is well known that the individual phases of V_{ij} is devoid of any physical meaning, since under rephasing of the up and down quark fields the non-physical individual phases γ_j and β_i of V_{ij} transform as

$$V_{ij} \rightarrow (V')_{ij} = V_{ij} exp(\gamma_j - \beta_i). \tag{3.3}$$

Now, we consider the question whether starting with symmetric moduli one may use the rephasing freedom of the CKM matrix to obtain also symmetric phases. In other words, whether starting from an arbitrary V, it is possible to achieve $arg(V')_{ij} = arg(V')_{ji}$ by an appropriate choice of γ_j, β_i. It was shown by Branco and Parada that in general this is not possible for arbitrary V, but it is possible for a three-generation CKM matrix with symmetric moduli. In fact, to achieve $arg(V')_{ij} = arg(V')_{ji}$, the following relations has to be satisfied

$$arg(V')_{ij} - arg(V')_{ji} = \gamma_i - \gamma_j + \beta_i - \beta_j + 2n\pi. \tag{3.4}$$

In order for the above equation to yield a solution for γ_j, β_i, the imaginary part of a rephasing invariant sextet consisting of the off-diagonal matrix elements of V, namely

$$Im(V_{12}V_{23}V_{31}V_{21}{}^*V_{13}{}^*V_{32}{}^*) = 0. \tag{3.5}$$

The above equation is a necessary condition to have symmetric phase of V for any number of generations ($N \geq 3$). Obviously, for $N > 3$ there are other conditions, analogous to the above equation, which need also be satisfied in order to obtain symmetric phases. For three generations, symmetric moduli of the CKM matrix lead, through unitarity, to the above condition. To see this, consider the orthogonality conditions for the first two rows and first two columns of the CKM martix :

$$\begin{aligned} V_{11}V_{21}^* + V_{12}V_{22}^* + V_{13}V_{23}^* &= 0, \\ V_{11}V_{12}^* + V_{21}V_{22}^* + V_{31}V_{32}^* &= 0. \end{aligned} \tag{3.6}$$

If one multiplies the first equation by V_{21} and the second by V_{21}, and assumes $|V_{ij}| = |V_{ji}|$, then one obtains, by subtracting the resulting equations,

$$V_{13}V_{23}^*V_{21} - V_{31}V_{32}^*V_{12} = 0, \tag{3.7}$$

which in turn implies the vanishing of the imaginary part of a rephasing invariant sextet consisting of the off-diagonal matrix elements of V. It can be readily verified that for more than three generations, symmetric moduli of V do not imply symmetric phases through unitarity. For example, for four generations, even if one has exact knowledge of the moduli of V, with $|V_{ij}| = |V_{ji}|$, this would not imply a symmetric V.

3.1.2 Generalised two-angle parametrization

In general, four independent parameters are required to characterize the CKM matrix for three generations But, assuming V to be symmetric implies a single constraint and, as a result, one needs three parameters to characterize the most general symmetric CKM matrix for three generations. A symmetric form for the CKM matrix, with two parameters was first proposed by Kielanowski and was generalised later by Blundell, Mann and Sarkar. It was also pointed out by Blundell et. al and Branco and Parada that Kielanowski had implicitly assumed a restriction on the free parameters of a symmetric CKM matrix.

Now we consider the generalised two-angle parametrization of the CKM matrix. The rephasing freedom of the quark fields implies that two CKM matrices V and V' are physically equivalent provided

$$V = U_1 U V' \dagger U_1, \tag{3.8}$$

where $U = diag(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})$ and $U_1 = diag(1, e^{i\psi_1}, e^{i\psi_2})$. Let λ_i and ω_i denote the eigenvalues and the eigenvectors of KM matrix V. The eigenvectors of V may be constructed in terms of three angles $(\beta_1, \beta_2, \beta_3)$ and one phase α. The eigenvalues of V satisfy its characteristic equation

$$\lambda^3 - k_1\lambda^2 + k_2\lambda - k_3 = 0 \tag{3.9}$$

where $k_1 = trV, k_2 = \frac{1}{2}[(trV)^2 - tr(V^2)]$, and $k_3 = detV$. The unitarity of V gives $k_2 = k_1^* k_3$. Let $trV = xe^{i\phi/3}$, a general complex number with real parameters and $detV = e^{i\phi}$, a phase. Then the characteristic equation becomes

$$\lambda^3 - xe^{i\phi/3}\lambda^2 + xe^{i2\phi/3}\lambda - e^{i\phi} = 0 \tag{3.10}$$

whose solutions are

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}e^{i\phi/3}(x - 1 - i\sqrt{3 + 2x - x^2}) \\
\lambda_2 &= \frac{1}{2}e^{i\phi/3}(x - 1 + i\sqrt{3 + 2x - x^2}) \\
\lambda_3 &= e^{i\phi/3}
\end{aligned} \tag{3.11}$$

for $-1 \le x \le 3$. Unitarity of V implies that the only relevant range of x is $-1 \le x \le 3$. Note that the factor $e^{i\phi/3}$ will vanish in the magnitudes of KM matrix elements, and in the rephasing invariant plaquette J, so the observables are independent of $detV$. Since $J = 0$ for $x \le -1$ and $x \ge 3$ we will examine the range $-1 < x < 3$.

The CKM matrix has three orthonormal complex eigenvectors. The normalised eigenvectors are determined upto a phase Thus we can choose one nonvanishing component of each vector to be real The two remaining arbitrary phases can be chosen in such a way that one eigenvector is real. We use the following parametrization of the three eigenvectors of KM matrix with the above properties:

$$w_1 = \begin{pmatrix} c_1 \\ s_1c_2 \\ s_1s_2 \end{pmatrix}, \; w_2 = \begin{pmatrix} -s_1c_3 \\ c_1c_2c_3 - s_2s_3e^{i\alpha} \\ c_1s_2c_3 + c_2s_3e^{i\alpha} \end{pmatrix}, \; w_3 = \begin{pmatrix} s_1s_3 \\ -c_1c_2s_3 - s_2c_3e^{i\alpha} \\ -c_1s_2s_3 + c_2c_3e^{i\alpha} \end{pmatrix} \tag{3.12}$$

where $c_i \equiv \cos(\beta_i)$ and $s_i \equiv \sin(\beta_i)$ and the KM matrix V is written as

$$V = \sum_{i=1}^{3} \lambda_i\omega_i \otimes \omega_i^\dagger \tag{3.13}$$

In the case where $\beta_3 = 0$ and α drops out, we write the magnitudes of the elements of symmetric KM matrix in terms of three parameters *i.e.* x and two angles β_1, β_2 as:

$$\begin{aligned}
|V_{11}| &= \sqrt{1 - \frac{1}{4}sin^2(2\beta_1)(3 + 2x - x^2)} \\
|V_{12}| &= \frac{1}{2}sin(2\beta_1)cos(\beta_2)\sqrt{(3 + 2x - x^2)} \\
|V_{13}| &= \frac{1}{2}sin(2\beta_1)sin(\beta_2)\sqrt{(3 + 2x - x^2)}
\end{aligned}$$

$$
\begin{aligned}
|V_{22}| &= \sqrt{1-\frac{1}{4}sin^2(2\beta_2)[3-x]-\frac{1}{4}sin^2(2\beta_1)cos^4(\beta_2)[3+2x-x^2]} \\
|V_{23}| &= \frac{1}{2}sin(2\beta_2)\sqrt{[3-x]-\frac{1}{4}sin^2(2\beta_1)[3+2x-x^2]} \\
|V_{33}| &= \sqrt{1-\frac{1}{4}sin^2(2\beta_2)[3-x]-\frac{1}{4}sin^2(2\beta_1)sin^4(\beta_2)[3+2x-x^2]}
\end{aligned}
\tag{3.14}
$$

3.1.3 Restrictions on the eigenstates and eigenvalues

Since the CKM matrix is unitary, it can be diagonalised by a unitary transformation

$$
V = WKW^{-1}; \quad K = diag(e^{i\sigma_1}, e^{i\sigma_2}, e^{i\sigma_3}) \tag{3.15}
$$

where $exp^{(i\sigma_i)}$ are the eigenvalues of the CKM matrix, corresponding to the eigenstates with components $w_{ij}(j = 1,2,3)$ The asymmetry parameter A can be expressed in terms of the eigenvalues of V and the combinations of the elements of the matrix W as follows:

$$
A = -4I[\sin(\sigma_1-\sigma_2)+\sin(\sigma_3-\sigma_1)+\sin(\sigma_2-\sigma_3)], \tag{3.16}
$$

where $I = Im(W_{11}W_{22}W^*_{12}W^*_{21})$. From the above expression for V it is obvious that the reality of W is sufficient in order to have a symmetric V. It was shown[21] that the CKM matrix is symmetric if and only if the matrix W is real, apart from irrelevant overall phases for each one of its columns. We have also reached the same conclusion (see the subsection 3.2.2). One can easily verify that if two of the eigenvalues are degenerate, then $|V|$ is necessarily symmetric and the eigenvectors can be chosen to be real Note that the asymmetry parameter A vanishes when two of the eigenvalues are degenerate and / or when the matrix W is effectively real (i.e $I = 0$). The fact that experimentally A is small provides an indication that two of the eigenvalues of V are close to being degenerate and / or W is close to be 'effectively' real i.e. $I << 1$.

3.2 Consequences of Symmetric quark mixing

3.2.1 Top Quark Mass and a Symmetric CKM matrix

We pursued[1] the investigation of symmetric quark mixing (ie a symmetric CKM matrix) in conjunction with CP-violation in the neutral kaon-system and the extent of the B^0_d–$\overline{B}^0_d$mixings to find out what constraints it put on parameters like m_t etc of SM. We used the standard parametrization[12, 13] for CKM matrix described in the introduction.

The relation $|V_{12}| = |V_{21}|$ obviously restricts one to a three dimensional hypersurface in the parameter space spanned by s_{12}, s_{23}, $q = |V_{13}|/|V_{23}|$ and δ. While $J = Im(V_{11}V_{22}V^*_{12}V^*_{21})$, the rephasing invariant measure of CP-violation, does vary with s_{23}, q and δ do not show any such variations, as their dependence on θ_{23} is very weak. Taking s_{12} and s_{23} as phenomenological

[1]This section is based on the work reported in ref [22]

inputs from (2.69) and (2 74) leaves us with a a curve in the q–δ plane for fixed values of s_{12} and s_{23} For the situation described in ref [23] the curve shrinks to a point. By determining whether this curve lies within the region in the q–δ plane allowed by the ϵ_K and B–$\overline{B}$ mixing data we are therefore able to obtain limits on the mass m_t of the t-quark as a consequence of the symmetric CKM ansatz since these latter quantities depend upon m_t.

The K^0–$\overline{K}^0$ system indirect CP-violating measure ϵ_K in the CKM picture is expressed as [24]

$$|\epsilon_K| = C \cdot B_K \cdot s_{23}^2 q \sin(\delta) \left[(\eta_3 f_3(y_t) - \eta_1)\, y_c s_{12} + \eta_2 y_t f_2(y_t) s_{23}^2 (s_{12} - q\cos(\delta)) \right] \tag{3.17}$$

where

$$\begin{aligned} C &\equiv \frac{(G_F f_K M_W)^2 M_K}{6\pi^2\sqrt{2}(\Delta M_K)} \\ f_2(y_t) &= 1 - \frac{3}{4}\frac{y_t(1+y_t)}{(1-y_t)^2}\left[1 + \frac{2y_t}{1-y_t^2}\ln(y_t)\right] \\ f_3(y_t) &= \ln\left(\frac{y_t}{y_c}\right) - \frac{3}{4}\frac{y_t}{1-y_t}\left[1 + \frac{y_t}{1-y_t}\ln(y_t)\right] \end{aligned} \tag{3.18}$$

and $y_i \equiv m_i^2/M_W^2$ ($i = c, t$). The parameters η_i are QCD corrections [25]

$$\eta_1 = 0.7 \;, \qquad \eta_2 = 0.6 \;, \qquad \eta_3 = 0.4 \;. \tag{3.19}$$

The experimental result $|\epsilon_K| = 2.3 \times 10^{-3}$ gives a parabola in the q–δ plane for given B_K, s_{23} and m_t. The Bag factor B_K is very poorly determined and various theoretical estimates only find the bounds $1/3 \leq B_K \leq 1$ The expression for the B_d^0–$\overline{B}_d^0$ mixing parameters $x_d = \Delta M/\Gamma$ is on the other hand,

$$x_d = \tau_b \frac{G_F^2}{6\pi^2}\eta M_B \left(B_B f_B^2\right) M_W^2 y_t f_2(y_t)|V_{tb}V_{td}^*|^2 \tag{3.20}$$

where $M_B = 5.28GeV$, $B_B f_b^2 = (0.15 \pm 0.05GeV)^2$ and the QCD correction $\eta = 0.85$. Experimentally $|V_{tb}| \approx 1$ to a high degree of accuracy and

$$|V_{td}|^2 = s_{23}^2(s_{12}^2 + q^2 - 2s_{12}q\cos\delta) \quad . \tag{3.21}$$

The ARGUS result[27]

$$x_d = 0.73 \pm 0.18 \tag{3.22}$$

thus gives another curve in the q–δ plane for given s_{23} and m_t.

It is straightforward to see that the symmetric ansatz implies a strong lower bound on m_t. Eq. (3.20) shows that $x_d \approx m_t^2|V_{31}|^2$ which by the symmetric ansatz is $m_t^2|V_{13}|^2$. However eqs. (2.74,2 75) impose a severe upper limit on $|V_{13}|$, in turn yielding a strong lower bound on m_t.

In our numerical analysis we hold B_K and m_t fixed and consider the total variation of all other parameters, taken in quadrature Thus we get two interesting bands in the q–δ plane

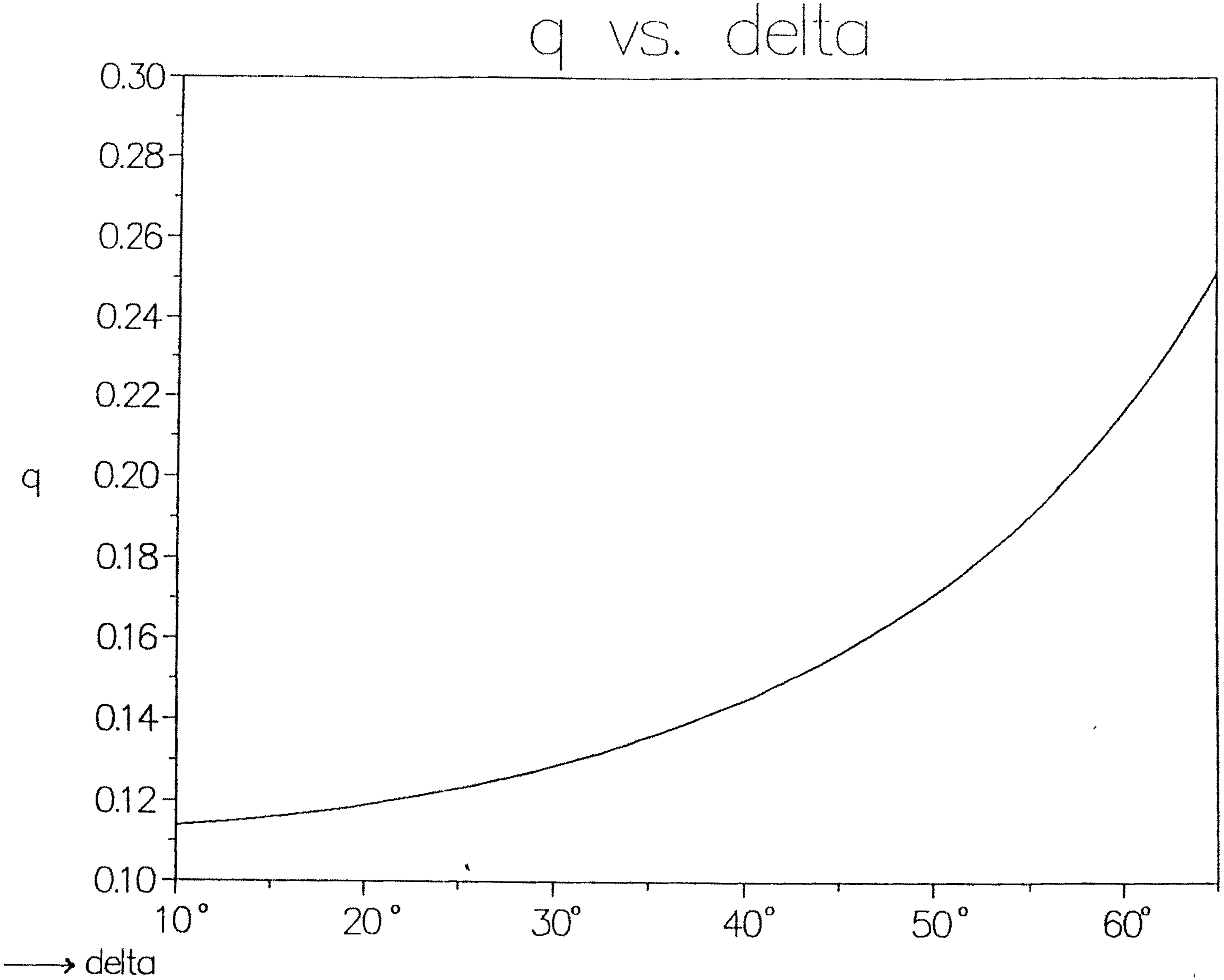

Figure 3.1: The symmetry curve for q vs. δ. Note that the existing data implies that $8.0° \leq \delta \leq 32.0°$

coming from ϵ_K and x_d. If this zone does contain the curve obtained from the symmetrical ansatz, then the assumptions are obviously valid for the given choice of m_t and B_K.

A plot of the curve in the q–δ plane for the symmetric ansatz (henceforth called the symmetric curve) is given in fig.(3.1)

We find a very narrow curve considering all the variations of s_{12} and s_{23}. We next superimpose on the symmetric curve in the q–δ plane curves parametrizing the regions allowed by the experiments with B–$\overline{B}$ mixing and the measurement of ϵ_K [28].

We find that when the top quark mass is lighter than 180 GeV, the symmetric curve does not intersect with the ARGUS measurement of x_d, implying that the top quark must be at least this heavy if the symmetric ansatz is correct. Imposing the K–$\overline{K}$ mixing result we find that for $B_K = 1/3$, the symmetric ansatz implies $m_t > 275$ GeV (although for $B_K = 2/3$ and 1 the lower limit of 180 GeV is unaltered). Alternatively, for given values of m_t, when the symmetric curve overlaps with the measurements of x_d and ϵ_K we find that the symmetric ansatz allows only a restricted range of values for q and δ, *i.e.*, the CP-violating phase is not completely arbitrary. The value of δ lies between 8° and 31°, while q is restricted to lie between .113 and .13. We have shown the allowed regions of q and δ for different m_t values in figs.(3.2, and 3.3) for two different values of B_K, namely, $B_K = 2/3$ and 1.

The experimental constraints imply that x must lie between -0.882 and .02. We also show the allowed regions of the parameter x for different values of m_t in fig.(3.4). From the allowed region of x for different m_t, we can immediately conclude that $x = 0$ is allowed for m_t about 185 GeV, in accord with an earlier result of Rosner [30].

We find that if the CKM matrix is symmetric then the top quark mass has to be heavier than 180 GeV, to be consistent with the experiments on B–$\overline{B}$ mixing and the measurement of ϵ_K; if the bag constant $B_K = 1/3$ then $m_t > 275$ GeV. The parameters q and δ are constrained to be in the range

$$.130 \geq q \geq .113 \qquad 8.0^\circ \leq \delta \leq 31.1^\circ \tag{3.23}$$

for the symmetric CKM matrix over the allowed range of the top quark mass.

3.2.2 Symmetric CKM matrix and Quark Mass matrices

The importance of studying the mass matrices lies in the fact that the structure of the quark and lepton mass matrices determines the flavour dynamics of the standard electroweak theory. However, the elements of these matrices cannot be predicted within the standard model as quark and lepton masses are the free parameters within the model. Furthermore, there exits an infinite number of mass matrices, related to each other by unitarity transformations, which yield the same physics. We have tried[2] to find out the constraints imposed on the form of the mass matrices due to the symmetric CKM matrix. In the basis, where the up-quark fields are mass

[2]This section is based on the work reported in ref [29]

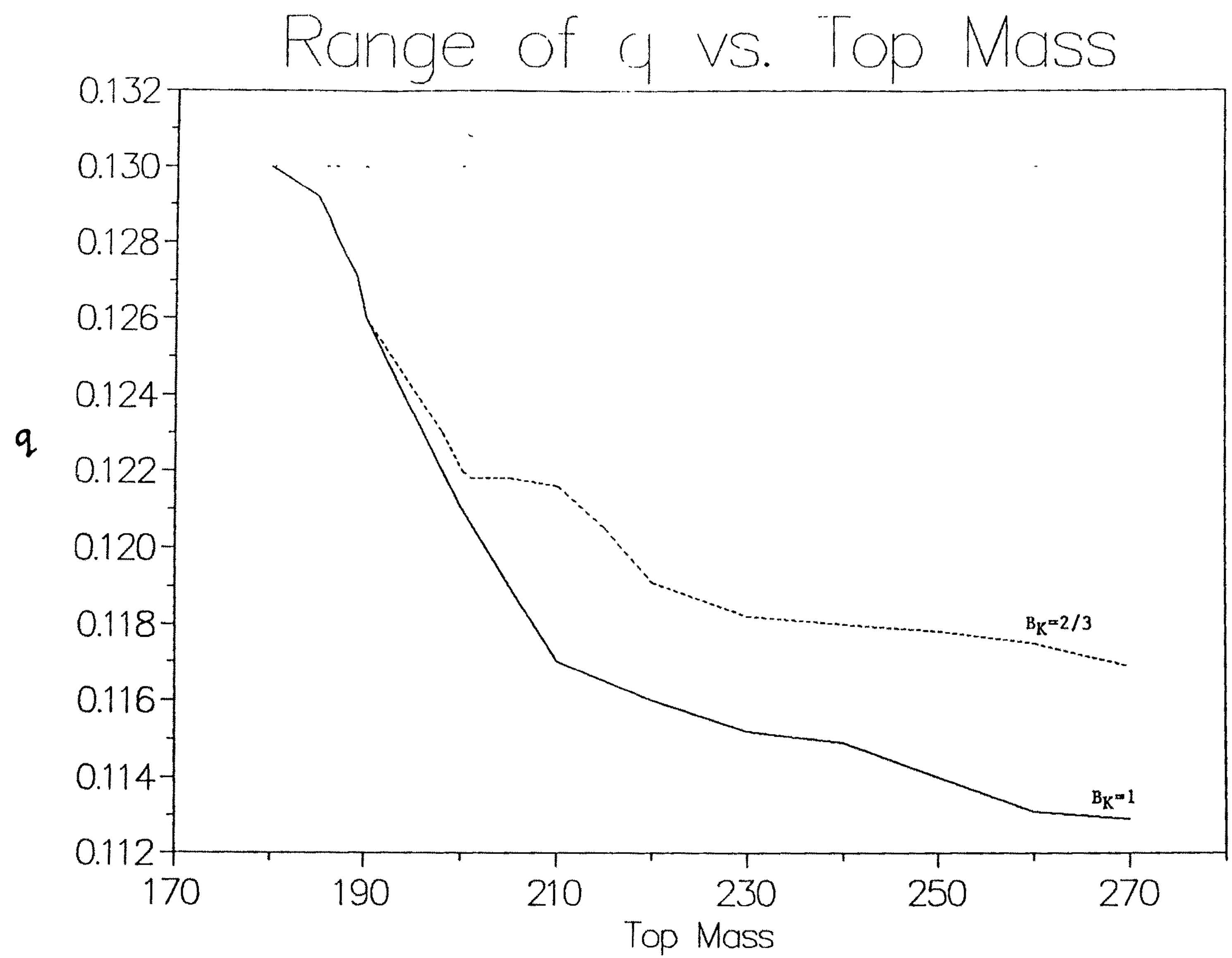

Figure 3.2: Allowed region of q as a function of m_t. The dotted line is the upper limit on q, valid for all B_K.

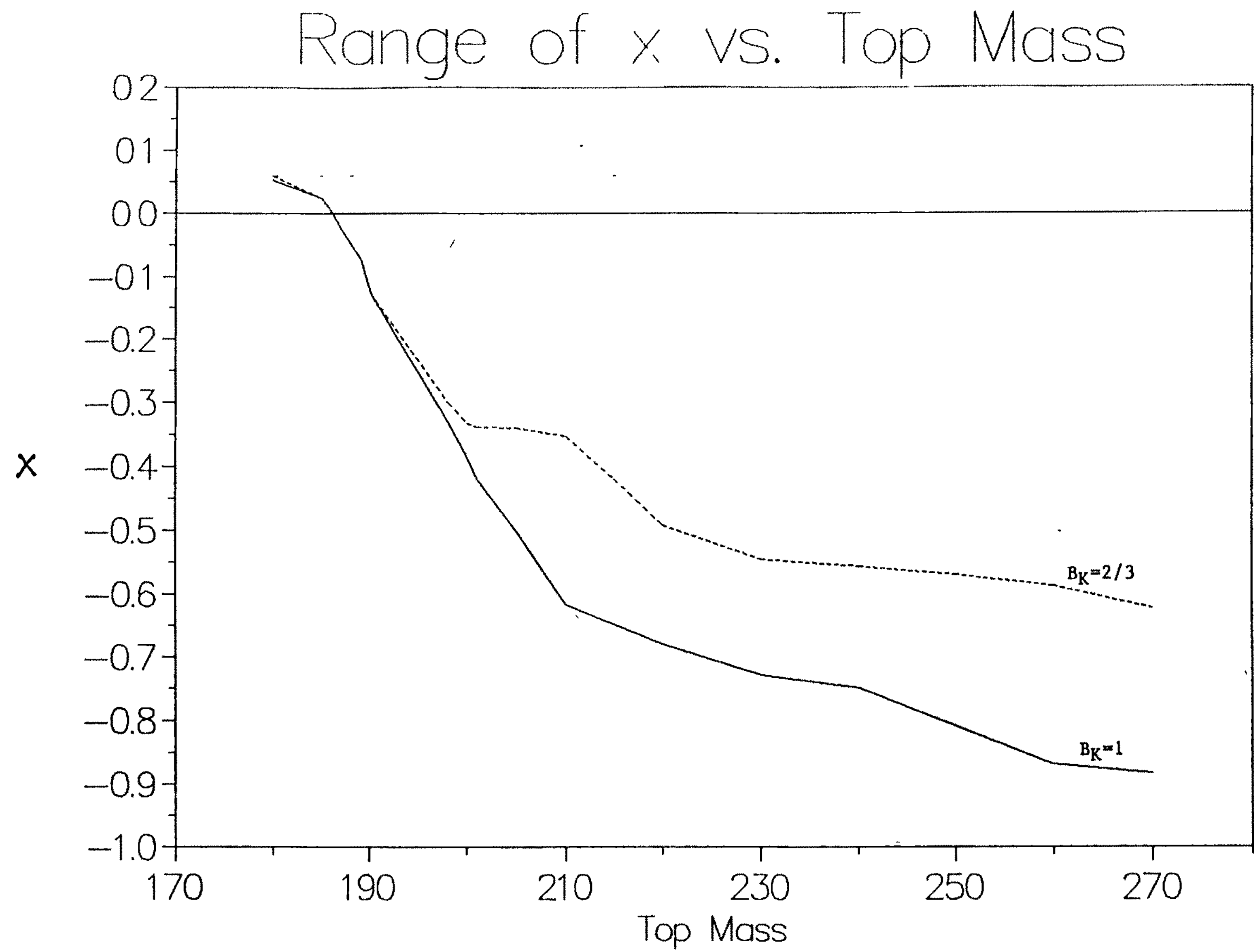

Figure 3.3: Allowed region of δ as a function of m_t. The dotted line is the upper limit on δ, valid for all B_K.

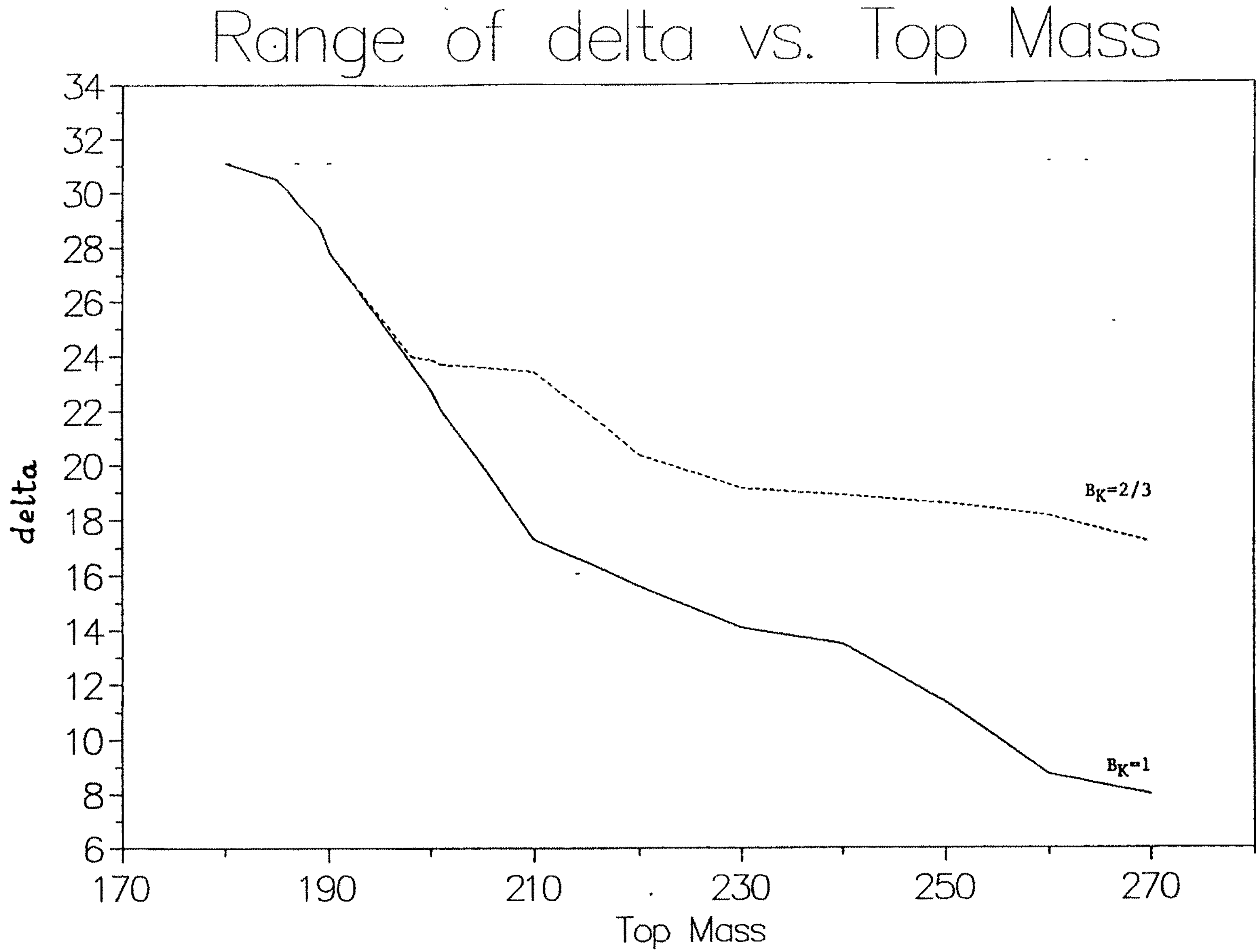

Figure 3.4: Allowed region of x as a function of m_t. The dotted line is the upper limit on x, valid for all B_K.

eigenstates, $M^{(u)}$ is diagonal i.e.

$$\mathcal{M}_u = diag(m_u, m_c, m_t) \tag{3.24}$$

In general the matrix $M^{(d)}$ is not hermitian, but we assume $M^{(d)}$ to be hermitian and write the most general hermitian $M^{(d)}$ is given by

$$M^{(d)} = h\mathcal{M}_u + A, \tag{3.25}$$

where

$$A = \begin{pmatrix} 0 & R_1 e^{i\rho_1} & R_2 e^{i\rho_2} \\ R_1 e^{-i\rho_1} & f & R_3 e^{i\rho_3} \\ R_2 e^{-i\rho_2} & R_3 e^{-i\rho_3} & d \end{pmatrix}. \tag{3.26}$$

Thus the mass matrices are a ten parameter family determined by m_u, m_c, $m_t, h, f, d, R_{1,2,3}$ and the invariant phase $(\rho_1 + \rho_3 - \rho_2)$ [32]. Taking the trace of both the sides of equation, we obtain the constant h in terms of parameters of mass matrices as

$$h = \frac{(m_d + m_s + m_b) - f - d}{(m_u + m_c + m_t)}. \tag{3.27}$$

Since the identity of the quarks is defined in the basis where the mass matrix is diagonal, the flavour projection operators[20], denoted by P_α and P'_j $(\alpha, j = 1, 2, ..., n)$ are introduced to keep track of the identity of quarks in any arbitrary basis, where the mass matrices are arbitrary, by projecting out the appropriate flavour. They are given by

$$\begin{aligned} P_\alpha(S) &= v_\alpha(S)/v, \\ P'_j(S') &= v'_j(S')/v', \end{aligned} \tag{3.28}$$

where the hermitian matrices $S(= M^{(u)}M^{(u)\dagger})$ and $S'(= M^{(d)}M^{(d)\dagger})$ has non-negative eigenvalues

$$\begin{aligned} (x_1, x_2, ..., x_n) &= ({m_u}^2, {m_c}^2, ...), \\ (x'_1, x'_2, ..., x'_n) &= ({m_d}^2, {m_s}^2, ...), \end{aligned} \tag{3.29}$$

respectively and v is a Vandermonde-type determinant given by

$$v = v(x_1, x_2, ..., x_n) = \Pi_{\beta,\alpha}(x_\beta - x_\alpha); \quad \beta > \alpha. \tag{3.30}$$

The quantity v' is the primed version of v, whereas the quantity v_α is obtained from the v by replacing x_α by the matrix S and all other $x_\beta, \beta \neq \alpha$ by $x_\beta I$ where I is the unit matrix. Thus v_α is a $n \times n$ matrix. For example, for $n = 3$ we have

$$v = v(x_1, x_2, x_3) = (x_3 - x_1)(x_3 - x_2)(x_2 - x_1), \tag{3.31}$$

and

$$v_1(S) = (x_3 - x_2)(x_3 - S)(x_2 - S). \tag{3.32}$$

These projection operators are hermitian and have unit traces.They can be used to express the measurable combinations of the CKM matrix elements in terms of invariant functions of the mass matrices.

To incorporate the constraint due to the symmetry of CKM,we use Jarlskog's flavour projection [20] operators to express the mod square elements of V in terms of the matrices S and S' as

$$|V_{\alpha j}|^2 = tr[P_\alpha(S)P'_j(S')], \tag{3.33}$$

where the first and second indices denote the up and down quark sectors respectively, and the flavour projection operator in S is given as

$$P_\alpha(S) = \frac{[(S-x_1)(S-x_2)...(S-x_n)]}{[(x_\alpha - x_1)(x_\alpha - x_2)...(x_\alpha - x_n)]'}, \tag{3.34}$$

with $[...]'$ to mean that the factor $(S - x_\alpha)$ in the numerator and the factor $(x_\alpha - x_\alpha)$ in the denominator must be left out. The expression for $P_a(S')$ is obtained by replacing α, S, x_n by j, S' and x'_n respectively. Then, the symmetry condition

$$|V_{\alpha j}|^2 = |V_{j\alpha}|^2 \tag{3.35}$$

is translated into a relation involving the matrices S and S' as

$$tr[P_\alpha(S)P'_j(S')] = tr[P_j(S)P'_\alpha(S')]. \tag{3.36}$$

Since the matrices $M^{(u)}$ and $M^{(d)}$ of our choice are hermitian, we have done all the calculations in terms of invariant functions of the matrices $M^{(u)}$ and $M^{(d)}$ instead of S and S'. Considering, in particular

$$|V_{12}|^2 = |V_{21}|^2, \tag{3.37}$$

we obtain the constraint condition due to symmetry of CKM matrix involving the parameters of the mass matrices as

$$\begin{aligned}&[R_1^2 + R_3^2 + (hm_c + f - m_s)(hm_c + f - m_b)]+ \\ &\tfrac{m_b - m_d}{m_b - m_s}[R_1^2 + R_2^2 + (hm_u - m_d)(hm_u - m_b)] \quad = 0.\end{aligned} \tag{3.38}$$

In general, it was not possible to find out the form of $M^{(d)}$ based on the general consraint involving all the parameters. But, an interesting point was noticed when we calculated the CP violation mesuring plaqutte J in terms of S and S' using [33]

$$\pm J = Im\frac{tr[v_1(S)v'_2(S')v_3(S)v'_1(S')}{vv'}. \tag{3.39}$$

It was found that if any of the R_1, R_2, R_3 is chosen to be zero along with $M^{(u)}$ being diagonal, then J is zero impling such a choice is not allowed for three generations. Thus, we note that in the basis in which $M^{(u)}$ is diagonal, no off-diagonal elements of $M^{(d)}$ can be made zero consistent with the CP violation in the quark sector for three generations.

The numerical calculation was done to find out whether any of the off-diagonal elements of the mass matrix $M^{(d)}$ is consistent with zero. To find out numerically the allowed ranges for the elements of the mass matrix $M^{(d)}$ we note that $M^{(d)}$ can be written as

$$M^{(d)} = D^\dagger \mathcal{M}_d D = V\mathcal{M}_d V^\dagger, \tag{3.40}$$

because a diagonal form for $M^{(u)}$ implies $U = \mathrm{I}$ and $D = V^\dagger$. For a symmetric V it reduces to

$$M^{(d)} = V \mathcal{M}_d V^*. \tag{3.41}$$

Since any unitary matrix that diagonalises a hermitian matrix can be written as the product of an orthogonal matrix and a phase matrix, we write V , in this basis, as

$$V = O_v P_v, \tag{3.42}$$

where the phase matrix P_v carries all the informations regarding the CP violation in quark sector for three generations. Then, the ranges for the elements of the $M^{(d)}$ were calculated using the eigenvalues of $M^{(d)}$ and the mod of the elements of V. The allowed ranges for the elements of $M^{(d)}$ in GeV are found out to be

$$M^{(d)} = \begin{pmatrix} 0.0117-0.0052 & 0.0549-0.0207 & 0.0409-0.0059 \\ 0.0549-0.0207 & 0.2374-0.1186 & 0.3261-0.1591 \\ 0.0409-0.0059 & 0.3261-0.1591 & 5.3962-5.1824 \end{pmatrix}. \tag{3.43}$$

Similarly the allowed ranges for the elements of $M^{(u)}$ are found to be

$$M^{(u)} = \begin{pmatrix} 0.1137-0.0657 & 0.4209-0.2818 & 1.9774-0.1881 \\ 0.4209-0.2818 & 2.2736-1.3885 & 16.312-5.4287 \\ 1.9774-0.1881 & 16.312-5.4287 & 279.78-179.38 \end{pmatrix}, \tag{3.44}$$

in the basis where $M^{(d)}$ is diagonal.

Since the CKM matrix $V = UD^\dagger$, where U and D are unitary matrices that diagonalise the mass matrices $M^{(u)}$ and $M^{(d)}$ respectively, then the symmetry condition for V i.e. $V = V^T$ will be fulfilled by the necessary and sufficient condition involving the matrices U and D

$$U = D^* U^T D. \tag{3.45}$$

Consider the product of the matrix $P(= U^T D)$ with its complex conjugate P^*.

$$P^* P = (U^T D)^* (U^T D) = U^\dagger D^* U^T D. \tag{3.46}$$

Now, the use of symmetry condition and the unitarity of U yields

$$PP^* = U^\dagger U = \mathrm{I}. \tag{3.47}$$

Thus, we have seen that P is a unitary matrix which is also symmetric. Hence, the most general condition for V to be symmetric is

$$D = U^* P, \tag{3.48}$$

which helps us to write the symmetric V as

$$V = UD^\dagger = UP^* U^T. \tag{3.49}$$

Since the unitary matrix U can be written as the product of a phase matrix P_u and an orthogonal matrix O_u i.e.

$$\begin{aligned} U &= O_u P_u, \\ U^\dagger &= P_u^* O_u^T, \end{aligned} \tag{3.50}$$

we reduce the symmetric V to

$$V = U(U^{*}P)^{\dagger} = O_u P_u P^{*} P_u O_u^T. \qquad (3.51)$$

The choice of P to be a phase matrix is a special but interesting case because for such a choice of P we can write either

$$M^{(d)} = f(M^{(u)^*}) \quad \text{or} \quad M^{(u)} = g(M^{(d)^*}). \qquad (3.52)$$

For such a chioce of P, we write the CKM matrix as

$$V = O_u \widetilde{P} O_u^T, \qquad (3.53)$$

where $\widetilde{P}$ is a phase matrix. Then one of the choices for the mass matrix $M^{(d)}$ is a function of $M^{(u)^*}$ as follows:

$$M^{(d)} = p(M^{(u)^*})^2 + qM^{(u)^*} + rI, \qquad (3.54)$$

where the parameters p, q, r are introduced to retain the mass hierarchy for the down quark sector. Upon diagonalisation of both sides of the above equation,we obtain three equations involving six quark masses and three unknown parameters p, q, r which can be determined uniquely. These three parameters are given in terms of the quark masses as

$$\begin{aligned} p &= \frac{m_s}{m_c m_t}, \\ q &= \frac{m_s}{m_c}, \\ r &= m_d - m_u \frac{m_s}{m_c}. \end{aligned} \qquad (3.55)$$

To get the ranges of the mod elements of the mass matrices for the case when P is a phase matrix we proceed with the numerical calculation using a convenient parametrization [34].

Comparing this general form with the form of symmetric V, we see that if Λ is recognised as $\widetilde{P}$ then the general form is reducible to symmetric form only if W is real. Thus we conclude that the reality of W is a necessary and sufficient condition for having a symmetric CKM matrix. Then it is evident that the choice $\alpha = 0$ will make V symmetric within the above parametrization. In this parametrization all the mod elements of V were written in terms of x as well as the angles. Consider the case when $\beta_3 = 0$ and α drops out. Then the mod elements of CKM matrix relevant to our discussion are:

$$\begin{aligned} |V_{11}| &= \sqrt{1 - (1/4)sin^2(2\beta_1)(3 + 2x - x^2)}, \\ |V_{12}| &= (1/2)sin^2(2\beta_1)cos(2\beta_2)\sqrt{(3 + 2x - x^2)}, \\ |V_{13}| &= (1/2)sin^2(2\beta_1)sin(2\beta_2)\sqrt{(3 + 2x - x^2)}. \end{aligned} \qquad (3.56)$$

The experimental constraints i.e.the values of the magnitudes, $\rho = |V_{13}/V_{23}|$ and J imply[22] that x must lie between -0.882 and 0.02 We then solve for β_1 and β_2 by inverting the above equation and using the the magnitudes of the first row of V and found out the allowed ranges to be

$$\begin{aligned} \beta_1 &= 0.1265 \quad to \quad 0.3605, \\ \beta_2 &= 0.0040 \quad to \quad 0.0320. \end{aligned} \tag{3.57}$$

Consider the case of $\alpha = 0$. Then the elements of the matrix W are functions of 3 mixing angles β_1, β_2 and β_3 out of which two are independent and we recognise $O_u = W$. Then, the unitary matrix U is given as

$$\begin{aligned} U &= O_u P_u = W P_u; \\ P_u &= diag(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}). \end{aligned} \tag{3.58}$$

Using the matrix U and assuming the matrix $M^{(u)}$ to be hermitian, we can write the mass matrix $M^{(u)}$ as

$$M^{(u)} = U^\dagger \mathcal{M}_u U = P_u^* W^T \mathcal{M}_u W P_u, \tag{3.59}$$

and the mass matrix $M^{(d)}$ as

$$M^{(d)} = D^\dagger \mathcal{M}_d D = P^* P_u W^T \mathcal{M}_d W P_u^* P. \tag{3.60}$$

In our numerical calculation, we use above mentioned ranges of the angles β_1 and β_2 to calculate the ranges for the mod elements of the mass matrices using the above equations in this two-angle parametrization of CKM matrix. The allowed ranges in GeV for the mod elements of $M^{(u)}$ in GeV are:

$$M^{(u)} = \begin{pmatrix} 0.1799 - 0.0242 & 0.4609 - 0.1617 & 0.0147 - 0.0006 \\ 0.4609 - 0.1617 & 1.6636 - 1.1413 & 8.9172 - 0.7141 \\ 0.0147 - 0.0006 & 8.9172 - 0.7141 & 279.93 - 179.87 \end{pmatrix}. \tag{3.61}$$

and for the matrix $M^{(d)}$ are:

$$M^{(d)} = \begin{pmatrix} 0.0386 - 0.0081 & 0.0738 - 0.0135 & 0.0023 - 0.00005 \\ 0.0738 - 0.0135 & 0.2318 - 0.1059 & 0.1692 - 0.0198 \\ 0.0023 - 0.00005 & 0.1692 - 0.0198 & 5.3995 - 5.1947 \end{pmatrix}. \tag{3.62}$$

Basis independent symmetry constraint

In the previous sections, we have given the ranges of the elements of the mass matrices M_u and M_d allowed by the symmetric CKM in two different bases. In this section we give the symmetry constraint written in a basis independent form. As we have seen in the previous section, the condition $|V_{12}| = |V_{21}|$ implies

$$tr[P_1(M_u)P_2(M_d)] = tr[P_2(M_u)P_1(M_d)] \tag{3.63}$$

which can be rewritten as

$$tr[c_1 V^\dagger \widehat{P}_1(M_u) V \widehat{P}_2(M_d) - c_2 V^\dagger \widehat{P}_2(M_u) V \widehat{P}_1(M_d)] = 0 \tag{3.64}$$

where the constants c_1 and c_2 are functions of the mass eigenvalues and

$$\begin{aligned}\widehat{P}_{1,2}(M_u) &= UP_{1,2}(M_u)U^\dagger; \\ \widehat{P}_{1,2}(M_d) &= DP_{1,2}(M_d)D^\dagger\end{aligned} \tag{3.65}$$

Consider going from a unprimed basis to a primed basis by the following transformations:

$$U' = AU; \quad D' = BD \tag{3.66}$$

where A and B are unitary matrices. Then the CKM matrix in the primed basis is

$$V' = AVB^\dagger \tag{3.67}$$

Requiring $V' = V$ relates A and B through the matrix V as follows:

$$A = VBV^\dagger \tag{3.68}$$

Then use of symmetry of CKM matrix in the primed basis yields

$$A = VBV^*. \tag{3.69}$$

The mass matrices transform under this basis transformation as followes:

$$M'_u = AM_uA^\dagger \quad , M'_d = BM_dB^\dagger \tag{3.70}$$

But the difficulty in using these expressions to find out how the mod elements of the mass matrices transform under this basis transformation is that it is not possible to seperate out the phase from the the mass matrices in the primed basis for any general unitary matrix A and B after the transformation.

Firstly, we write [31] the symmetry constraint as an equation involving the parameters of the mass matrices using flavour projection operators of Jarlskog[20] in a basis where $M^{(u)}$ is diagonal. In general, it was not possible to find out the form of $M^{(d)}$ based on the general consraint involving all the parameters. Also we give the numerical ranges for the mod elements of $M^{(d)}$ in this basis. Then, we wrote the necessary condition for having a symmetric V in terms of the matrices U and D as

$$U = D^* U^T D \tag{3.71}$$

We chose a particularly interesting basis where $U = D^*P$;P being a phase matrix and gave the ranges for the mod elements of $M^{(u)}, M^{(d)}$ in that basis using a convenient parametrization for V. We noticed that non of the off-diagonal elements of $M^{(u)}$ and $M^{(d)}$ is consistent with zero for a symmetric V, which means such forms for mass matrices cannot be obtained from any symmetry. But, in principle there exists infinite number of other bases related to each other by similarity transformations So it is apparent that the numbers we provided for the allowed ranges of the mod elements of mass matrices are not basis independent. Finally the symmetry constraint is written in a basis-independent form

3.2.3 Rank One quark mass matrices and Phenomenological constraints

Recently in an interesting letter[35], the results of the studies on the approximately symmetric KM matrix based on eigenvalues of the KM matrix and the rank-one quark mass matrices were reported In this new scheme, the up and down quark mass matrices are given as

$$M^{(u)} = \kappa_U M_0 + X_U; \quad M^{(d)} = \kappa_D M_0 + X_D \tag{3.72}$$

where κ_U and κ_D are numerical constants; M_0 is a $3X3$ rank-one matrix defined as

$$(M_0)_{ij} = h_i h_j; \quad \mathbf{h} = (g_1, g_2, g_3). \tag{3.73}$$

with $g_i (i = 1,2,3)$ being real and the matrices X_U and X_D are correction terms that have to be added to M_0 to obtain the non-zero masses of the light two generation quarks since the rank-one mass matrix M_0 has only one non-vanishing eigenvalue. These quark mass matrices are diagonalised by unitary matrices as follows

$$\begin{aligned} M^{(u)}(diag) &= U_U U_0 (\kappa_U M_0 + X_U)(U_U U_0)^{-1}, \\ M^{(d)}(diag) &= U_D U_0 (\kappa_D M_0 + X_D)(U_D U_0)^{-1}. \end{aligned} \tag{3.74}$$

where U_0 diagonalises the rank-one matrix M_0 and is given as

$$U_0 = \begin{pmatrix} \frac{g_2}{N_1} & \frac{-g_1}{N_1} & 0 \\ \frac{-g_1 g_3}{N_2} & \frac{-g_2 g_3}{N_2} & \frac{(g_1^2+g_2^2)}{N_2} \\ \frac{g_1}{N_3} & \frac{g_2}{N_3} & \frac{g_3}{N_3} \end{pmatrix}, \tag{3.75}$$

with

$$N_1 = \sqrt{g_1^2 + g_2^2}, \quad N_3 = \sqrt{g_1^2 + g_2^2 + g_3^2}, \quad N_2 = N_1 \times N_3. \tag{3.76}$$

Then the KM matrix V is written in terms of its eigenvalues and unitary matrices U_0, U_U, and U_D as follows

$$V = (U_U U_0) K (U_D U_0)^{-1}; \quad K = diag(e^{i\sigma_1}, e^{i\sigma_2}, e^{i\sigma_3}). \tag{3.77}$$

In this scheme KM matrix is symmetric if U_U and U_D are the unit matrix because then matrix U_0 is real. In this work[3], we mainly comment on the results given in this scheme related to perfectly symmetric KM matrix. We started with the most general parametrization of KM matrix for three generations in terms of three angles and a phase[12]

It is easy to see that the symmetry condition for KM matrix reduces the number of independent parameters from four to three. For example, taking

$$|V_{13}|^2 = |V_{31}|^2 \tag{3.78}$$

puts the constraint

$$s_{13}^2 = s_{23}^2 (s_{12}^2 + R^2 - 2 s_{12} R \cos\delta) \tag{3.79}$$

where $R = |V_{13}/V_{23}|$. Hence the four parameters $\sigma_1 - \sigma_3, \sigma_2 - \sigma_3, g_1/g_3, g_2/g_3$ used in Tanimato's paper[35] to express the matrix elements of the perfectly symmetric KM matrix cannot

[3]This section is based on the work reported in ref [31]

be independent of each other as the above constraint can be translated into an equation relating them. To demonstrate this in a simpler way, consider the generalised two-angle parametrization of KM matrix[34]

To establish the link between this parametrization and the new scheme[35] consider

$$\lambda_i = exp(i\sigma_i); \quad i = 1, 2, 3. \tag{3.80}$$

Then we can write

$$\begin{aligned} e^{i(\sigma_1-\sigma_3)} &= \lambda_1/\lambda_3 \\ e^{i(\sigma_2-\sigma_3)} &= \lambda_2/\lambda_3 \end{aligned} \tag{3.81}$$

Denoting $\sigma_1 - \sigma_3$, $\sigma_2 - \sigma_3$ by δ_1, δ_2 respectively and using the expressions for the eigenvalues we obtain

$$e^{i(\delta_1+\delta_2)} = 1 \tag{3.82}$$

which implies

$$(\delta_1 + \delta_2) = 0 \tag{3.83}$$

Thus we see that the parameters $\sigma_1 - \sigma_3$ and $\sigma_2 - \sigma_3$ are not independent in general and we have to be careful while choosing their values.

Now we relate the angles β_1, β_2 to the parameters $g_1/g_3, g_2/g_3$. Since the eigenvectors of KM matrix are given by eqn (3.12), we compare the elements of the matrix that diagonalises V for the case $\beta_3 = 0$ with that of the matrix U_0 and get

$$\begin{aligned} g_1/g_3 &= -s_1 s_2/c_2 \\ g_2/g_3 &= -c_1 s_2/c_2. \end{aligned} \tag{3.84}$$

Using the expressions for the KM matrix elements given in ref[35], it is easy to see that the CP violation measuring plaquette J can be written in terms of the parameters $G_1(= g_1/g_3)$, $G_2(= g_2/g_3)$, δ_1, δ_2 as follows

$$J = \frac{2[1 - cos(\delta_1 - \delta_2)](G_1^2 sin\delta_1 + G_2^2 sin\delta_2)G_1^2 G_2^2}{(G_1^2 + G_2^2)^2(1 + G_1^2 + G_2^2)^2}. \tag{3.85}$$

The ranges for the parameters $\sigma_1 - \sigma_3, \sigma_2 - \sigma_3, g_1/g_3, g_2/g_3$ can be found out using the allowed ranges of x, β_1, β_2. It has been shown[34] that the experimental constraints i.e. the values of the magnitudes of the KM matrix elements, $R = |V_{13}/V_{23}|$ and CP violation measuring plaquette J imply that x must lie between -0.882 and 0.02. Hence the allowed ranges for β_1 and β_2 was found to be[31]

$$\begin{aligned} \beta_1 &= 0.1265 \quad to \quad 0.3605, \\ \beta_2 &= 0.0040 \quad to \quad 0.0320; \end{aligned} \tag{3.86}$$

which in turn decide the allowed ranges for parameters $g_1/g_3, g_2/g_3$ to be

$$\begin{aligned} g_1/g_3 &= 0.0005 \quad to \quad 0.0112, \\ g_2/g_3 &= 0.003 \quad to \quad 0.031. \end{aligned} \tag{3.87}$$

We also found out that[31] the experimental constraint

$$0.05 \leq q(= \frac{|V_{13}|}{|V_{23}}|) \leq 0.13 \tag{3.88}$$

restricts the allowed range in δ to be

$$8^0 \leq \delta \leq 32^0. \tag{3.89}$$

Cosequently, considering the CP violation measuring rephasing invariant plaquette J to be given as

$$\begin{aligned} J &= s_{12}^2 s_{23} s_{13} c_{12} c_{23} c_{13} sin\delta, \\ &= \frac{4sin(\delta_2)^3 G_1^2 G_2^2 (G_1^2 - G_2^2)}{(G_1^2 + G_2^2)^2 (1 + G_1^2 + G_2^2)^2}, \end{aligned} \tag{3.90}$$

and using the experimental numbers

$$\begin{aligned} s_{12} &= 0.221 \pm 0.002, \\ s_{23} &= 0.044 \pm 0.009, \\ s_{13}/s_{23} &= 0.09 \pm 0.05, \end{aligned} \tag{3.91}$$

the allowed region for the parameter $(\sigma_2 - \sigma_3)$ can be found out. Now consider the R versus δ curve for symmetric KM matrix which is plotted using eqn.(3.79). Then recognising $-ArgV_0^{KM}(ub) = \delta$, it seems from the numbers provided in ref[35] that the solutions A and B correspond to two different points whereas the solution C corresponds to a spread in the allowed ranges of R versus δ curve. In the generalised two angle parametrization, δ is a function of x and consequently the solutions A, B, C seem to correspond to suitable choices of x in the generalised parametrization. For example, the Kielanowski's solution *i.e.* $\delta = 30^0$ corresponds to $x = 0$.

Now let us analyse the solutions provided in ref[35] from the viewpoint of the generalised two angle parametrization. The conclusions regarding the observables should be the same in both the schemes

$$Case \quad A : g_1 << g_2 << g_3, \quad \sigma_1 = \sigma_3$$

This case corresponds to symmetric KM matrix by construction, since two of the eigenvalues are taken to be degenerate[21] The numerical values

$$g_1/g_3 = 0.0024 \quad and \quad g_2/g_3 = 0.021$$

lie well within the allowed ranges for the parameters g_1/g_3 and g_2/g_3. The choices $\sigma_2 - \sigma_3 \approx 180^0$ and $\sigma_1 = \sigma_3$ are also consistent with each other as the constraint $(\delta_1 + \delta_2) = 0$ is not applicable to this case. To see the allowed value of top quark mass (m_t) in this case, we consider the δ versus m_t curve [3] which is consistent with the experimental constraints from $B_0 - \overline{B}_0$ mixing and ϵ parameter in the neutral K meson system. We found out that this case requires $m_t \approx 255\ GeV$ provided the Bag factor $B_K = 1$; otherwise this solution is ruled out experimentally.

$$Case\ B : g_1 << g_2 << g_3, \quad \sigma_1 - \sigma_3 = -(\sigma_2 - \sigma_3) = 120^0$$

This is Kielanowski's solution[23] which has been discussed extensively in the literature. In terms of the parameters of the generalised two angle parametrization this case corresponds to

$$\beta_1 = 0.1285, \quad \beta_2 = 0.0300, \quad x = 0 \tag{3.92}$$

$$Case \quad C : g_1 = g_2 << g_3$$

The condition $g_1 = g_2$ requires $sin\beta_1 = cos\beta_1$ implying $\beta_1 = 45^0$. Then the CP violation measuring rephasing invariant plaquette J vanishes for this case as we have

$$J = \frac{1}{32} cos(2\beta_1) sin^2(2\beta_1) sin^2(2\beta_2)[3 + 2x - x^2]^{3/2} \tag{3.93}$$

Secondly, the given numerical value of the parameter g_1/g_3 does not lie within its allowed region. Hence it is difficult to see the consistency as well as the physical significance of the solution C.

To summarize, we found out that the solutions A and B are special cases of the allowed solutions for symmetric KM matrix corresponding to different values of the parameter x in the generalised two angle parametrization. The solution A predicts $m_t \approx 255 GeV$ only if $B_K = 1$; otherwise it is ruled out experimentally. The solution C was found to be inconsistent with the experimental constraints.