Chapter 4

Studies related to massive neutrinos

The question of whether the neutrino has a non-zero mass is one of the important questions of particle physics today. Neutrino mass has also great significance for astrophysics and cosmology. In this chapter, we present a brief review of neutrino masses and mixing followed by a study related to the 17 keV neutrino mass eigenstate. At the end, we present an analysis of neutrino masses in left-right symmetric extensions of the SM with various choices of the higgs scalars. We minimise the scalar potentials in all these cases.

4.1 Review of neutrino masses and mixing

4.1.1 Neutrino Mass

In the minimal SM the neutrinos are strictly massless due to the absence of right handed neutrinos and lepton-number violating processes This choice is made not by any deeper theoretical motivations (like gauge invariance, which keeps the photon and gluons massless or the spontaneous symmetry breaking of global gauge that renders Goldstone bosons massless) but rather by our limitations regarding the apparatus to conclusively detect any non-zero mass for the neutrinos. Current experiments provide only the upper limits for the neutrino masses which is consistent with any or all of the neutrinos having zero mass and these limits are not very restrictive.

Any discussion of massive neutrinos takes us beyond the minimal SM. Extensions of the SM that allows non-zero neutrino mass can be models involving new $SU(2)$ singlet (such that the anomaly cancellation is not affected) neutral fermions or extension of the higgs sector, or both In this subsection, first, we will be discussing the types of neutrino masses and then some models for neutrino masses.

Types of neutrino mass

Since each LH (RH) particle is necessarily associated with a RH (LH) antiparticle, the RH antiparticle field ψ_R^c is not independent of ψ_L, but is closely related to $\psi_L^\dagger$ as $\psi_R^c \equiv C\overline{\psi_L}^T$. Similarly, for a RH Weyl spinor, $\psi_L^c \equiv \overline{\psi_R}^T$. In the special case that ψ_L is the chiral projection $P_L\psi$ of a Dirac field ψ, ψ_R^c is just the RH projection $P_R\psi^c$ of the antiparticle field $\psi^c = C\overline{\psi}^T$. Since the quarks and charged leptons carry conserved quantum numbers (like colour and electric charge etc.), they must be Dirac fields i.e. ψ_R and ψ_R^c are distinct and have the opposite values for all additive quantum numbers. But, the charge neutrality of the neutrinos (which take part only in weak interactions) leaves only one quantum number, namely lepton number to be associated with the neutrino. This allows the neutrino to have both lepton number conserving as well as lepton number violating mass terms

In general a mass term for a fermion field consists of fields with opposite chirality. Keeping this in mind, we consider all such combinations of the fields ν_L, N_R, ν_R^c, $N_L^c (\equiv C\overline{N_R}^T)$, and their Hermitian conjugates as follows·

$$
\begin{aligned}
&(1) \qquad \overline{\nu_L}N_R + \overline{N_R}\nu_L \\
&(2) \qquad \overline{N_L}^c\nu_R^c + \overline{\nu_R}^c N_L^c \\
&(3) \qquad \overline{\nu_R}^c\nu_R + \overline{N_R}\nu_R^c + \overline{N_L}^c\nu_L + \overline{\nu_L}N_L^c.
\end{aligned}
\tag{4.1}
$$

The first two combinations are invariant under the global gauge transformations, and consequently can be rewritten as a generalised lepton number conserving **Dirac** mass term

$$
-\mathcal{L}_D = m_D\overline{\nu_L}N_R + H.c., \tag{4.2}
$$

which connects N_R and ν_L. The fields ν_L, N_R, ν_R^c and ν_R^c form a 4-component Dirac particle i.e. we can define $\nu \equiv \nu_L + N_R, \nu^c \equiv N_L^c + \nu_R^c = C\overline{\nu}^T$, so that $-\mathcal{L}_D = m_D\overline{\nu}\nu$. Usually, the N_R is an $SU(2)\otimes U(1)$ singlet, with m_D generated by the SM doublet, and $L = L_e + L_\mu + L_\tau$ is conserved in the three family generalisation. For N generations

$$
-\mathcal{L}_D = \overline{\nu_L'}m_D N_R' + h.c., \tag{4.3}
$$

where m_D is an arbitrary $N \times N$ matrix, and ν_L' N_R' are N component vectors; thus $\nu_L' = (\nu_{1L}', \nu_{2L}', ..., \nu_{NL}')^T$, where ν_{iL}' are the weak eigenstate neutrinos

The third combination violates lepton number by $\Delta L = 2$ and is generally known as the **Majorana** mass term

$$
L_{m_D} = L_{m_1} + L_{m_2} = -m_M\overline{\psi}_D\psi_D \tag{4.4}
$$

In fact a Majorana mass term can be written without introducing any new fermion field N_R. This is done by coupling the ν_L to its CP conjugate ν_R^c.

$$
-\mathcal{L}_M = \frac{1}{2}m\overline{\nu_L}\nu_R^c + h.c. = \frac{1}{2}m\overline{\nu_L}C\overline{\nu_L}^T + h.c. \tag{4.5}
$$

For N generations, the Majorana term is

$$
-\mathcal{L}_M = \frac{1}{2}\overline{\nu_L'}m_M\nu_R'^c + h.c., \tag{4.6}
$$

where m_M is a $N \times N$ Majorana mass matrix and ν'_L and ν'^c_R are N-component vectors i.e. $\nu'_L = (\nu'_{1L}, \nu'_{2L}, ..., \nu'_{NL})^T$, $\nu'c_L = (\nu'^c_{1L}, \nu'^c_{2L}, ..., \nu'^c_{NL})^T$ with weak eigenstate neutrinos ν'_{iL} and antineutrinos ν'^c_{NL}, related by

$$\nu'^c_{iR} = C\overline{\nu'^T_{iL}}, \tag{4.7}$$

from which it follows that $\overline{\nu'_{iL}}\nu'^c_{jR} = \overline{\nu'_{jL}}\nu'^c_{iR}$ This identity in turn implies that the Majorana mass matrix M must be symmetric i.e. $m_M = m_M^T$

The most general mass term for any field having no Abelian charge consists of both Dirac and Majorana mass terms. For example, in a model having one doublet neutrino ν'_L (with $\nu'^c_R = C\overline{\nu'^T_L}$) and one singlet neutrino N'_R (with $N'^c_L \equiv C\overline{N_R}'^T$). One could have the general mass term

$$-\mathcal{L}_m = \frac{1}{2}\begin{pmatrix}\overline{\nu_L}' & \overline{N_L}'^c\end{pmatrix}\begin{pmatrix} m_T & m_D \\ m_D^\dagger & m_S \end{pmatrix}\begin{pmatrix}\nu'^c_R \\ N'_R\end{pmatrix} + h.c. \tag{4.8}$$

where $m_D = m_D^T$ is a Dirac mass generated by a higgs doublet, m_T is a Majorana mass for ν'_L generated by a higgs triplet and m_S is a Majorana mass for N'_R, generated by a higgs singlet. The mass eigenstates are the mixed states

$$\begin{aligned} \nu_{1L} &= \cos\theta\nu_L - \sin\theta\nu^c_L \\ \nu_{2L} &= \sin\theta\nu_L + \cos\theta\nu^c_L, \end{aligned} \tag{4.9}$$

with the mixing angle

$$\theta = \frac{1}{2}arctan\frac{2m_D}{m_T - m_S}, \tag{4.10}$$

with eigenvalues

$$m_{1,2} = \frac{1}{2}\{m_T + m_S \pm [(m_T - m_S)^2 + 4m_D{}^2]^{\frac{1}{2}}\} \tag{4.11}$$

Interpreting ν'_L, N'_R, ν'^c_R, N'^c_L as N-component vectors, and m_T, m_D, m_S as $N \times N$ matrices (with $m_T = m_T^\dagger, m_S = m_S^\dagger$ we can generalise the above Lagrangian for N generations as

$$-\mathcal{L}_m = \frac{1}{2}\overline{n'_L}Mn'^c_R + h.c., \tag{4.12}$$

where $n'_L \equiv (\nu'_L \quad N'^c_L)^T$ and $n'^c_R \equiv (\nu'^c_R \quad N'_R)^T$ are $2N$ component vectors and M is the symmetric $2N \times 2N$ Majorana mass matrix.

To see how the **Dirac** case ($m_T = m_S = 0$) emerges as a limiting case of the general mass term we consider only a single family such that $M = m_D\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Since M is Hermitian (for m_D real) it can be diagonalised by a unitary transformation U as

$$U^\dagger MU = m_D\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{4.13}$$

with $U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$, and then the mass eigenstates are

$$\begin{aligned} \nu_{1L} &= \frac{1}{\sqrt{2}}(\nu'_L + N'_L),\ \nu^{0c}_{1R} = \frac{1}{\sqrt{2}}(\nu'^c_R + N'_R); \\ \nu_{2L} &= \frac{1}{\sqrt{2}}(\nu'_L - N'_L),\ \nu^{0c}_{2R} = \frac{1}{\sqrt{2}}(\nu'^c_R - N'_R); \end{aligned} \tag{4.14}$$

The negative mass eigenvalue can be removed by redefining the RH fields $\nu_{1R}^c = \nu_{1R}^{0c}$ and $\nu_{2R}^c = -\nu_{2R}^{0c}$. Since the two Majorana states $\nu_1 = \nu_{1L} + \nu_{1R}^c$ and $\nu_2 = \nu_{2L} + \nu_{2R}^c$ and degenerate, we can rewrite the Lagrangian $\mathcal{L}_m$ in the new basis

$$\begin{aligned} \nu &\equiv \frac{1}{\sqrt{2}}(\nu_1 + \nu_2) = \nu_L' + N_R', \\ \nu^c &\equiv \frac{1}{\sqrt{2}}(\nu_1 - \nu_2) = N_L'^c + \nu_R'^c; \end{aligned} \tag{4 15}$$

yielding

$$\mathcal{L}_m = \frac{1}{2} m_D(\overline{\nu_{1L}}\nu_{1R}^c + \overline{\nu_{2L}}\nu_{2R}^c) + h.c. = m_D\overline{\nu}\nu, \tag{4.16}$$

which is just the standard Dirac mass term, with a conserved lepton number. Therefore, a **Dirac** neutrino is nothing but a pair of degenerate 2-component Majorana neutrinos (ν_1 and ν_2), combined to form a 4-component neutrino with a conserved lepton number.

A **pseudo-Dirac** neutrino is just a Dirac neutrino to which a small lepton number-violating perturbative term has been added. This can be seen by modifying the Dirac mass to

$$M = \begin{pmatrix} \epsilon & m_D \\ m_D & 0 \end{pmatrix}, \tag{4.17}$$

with $\epsilon \ll m_D$. Then we have two Majorana mass eigenstates $\nu_\pm$, with

$$\nu_{+L} = \nu_{1L} + \frac{\epsilon}{4}\nu_{2L}, \nu_{-L} = -\frac{\epsilon}{4}\nu_{1L} + \nu_{2L}, \tag{4.18}$$

with masses

$$m_\pm = m_0 \pm \frac{\epsilon}{2}. \tag{4.19}$$

Models of neutrino mass

There are many models for neutrino mass, all of which have good and bad features. But we will be discussing few models that involve either an enlarged fermion sector or an extended higgs sector.

As we have discussed earlier, an enlargement of the fermion sector through the inclusion of a RH neutrino field N_R that transforms as (1, 1, 0) under $SU(3)_C \otimes SU(2)_L \otimes U(1)$ leads to Dirac mass for neutrinos. This model treats neutrino mass exactly on the same footing as the masses of the other fermions since the mass is generated by the *vev* of the neutral component of a doublet higgs through the Yukawa couplings However, the model has some drawbacks. First, it cannot predict the neutrino mass as it turns out to be proportional to arbitrary Yukawa coupling constants. Secondly, it fails to explain the smallness of neutrino mass. Since $v = 246GeV$, a ν_e mass in the 10 eV range would require an anomalously small Yukawa coupling $h_{\nu_e} \leq 10^{-10}$. Of course, if the coupling constants are fine-tuned such that h_νs are very small compared to the corresponding coupling constants which generates masses for charged leptons or quarks, the neutrinos can have lighter mass in commensurate with experimental bounds. But there is no good reson why h_νs must be small in this model.

If we donot enlarge the fermion sector of the minimal SM, we have, in each generation, only two degrees of freedom corresponding to neutrinos i e ν_L and $\overline{\nu_R}$. Then the mass of the neutrino must be of the Majorana type i.e. violate B-L symmetry irrespective of the mass-generating mechanism. Thus, we are motivated to extend the higgs sector through the inclusion of new higgs which can violate B-L symmetry. Also, the neutrino masses must somehow be induced by the Yukawa coupling. Majorana mass terms for the ordinary $SU(2)$ doublet neutrino involve a transition from $\nu_R^c(T_3 = -\frac{1}{2})$ into $\nu_L(T_3 = \frac{1}{2})$, and therefore must be generated by an operator transforming as a triplet under weak $SU(2)_L$. The simplest possibility is the Gelmini-Roncadelli[38] model, in which one introduces a triplet of higgs fields $\Delta \equiv (\Delta^0, \Delta^-, \Delta^{--})$ into the theory. The Yukawa coupling

$$\mathcal{L}_y = \frac{1}{2}(\overline{\nu_L} \quad \overline{e_L})\begin{pmatrix} \Delta^- & \sqrt{2}\Delta^0 \\ \sqrt{2}\Delta^{--} & -\Delta^- \end{pmatrix}\begin{pmatrix} e_R^c \\ \nu_R^c \end{pmatrix} \tag{4.20}$$

then generates a Majorana mass $m_T = h_\Delta v_\Delta$ for the ν, when the higgs triplet field Δ_L acquires a $v.e.v.$ $v_\Delta = \sqrt{2} < \Delta^0 >$ is the vev of the higgs triplet. Since both h_Δ and v_Δ are unknown, the neutrino mass is unrelated to that of the other fermions and can in principle be arbitrarily small, at least in the tree level There will be massless Goldstone boson called Majoron in this model if the Lagrangian $\mathcal{L}$ conserves lepton number since the vev $< \Delta^0 > \neq 0$ violates lepton number conservation by two units.

There is a popular scheme called the see-saw **mechanism** to explain the smallness of the neutrino mass. The see-saw mechanism[40] for one generation is a special case of the general mass matrix, in which m_D is a typical Dirac mass (comparable to m_u or m_e for the first generation) connecting ν_L' to a new $SU(2)_L$ singlet N_R' and $m_S \gg m_D$ is a Majorana mass for N_R', presumably comparable to some new physics scale. It is usually assumed in this model that $m_T = 0$ i.e. there exists no higgs triplet. Then eqn.4.8 yields two Majorana mass eigenstates ν_1 and ν_2 with

$$\begin{aligned} \nu_L' &= (\nu_{1L}\cos\theta + \nu_{2L}\sin\theta),\ \nu_R'^c = -(\nu_{1R}^c\cos\theta + \nu_{2R}^c\sin\theta), \\ N_L^c &= -(\nu_{1L}\sin\theta + \nu_{2L}\cos\theta),\ N_R' = -(\nu_{1R}^c\sin\theta + \nu_{2R}^c\cos\theta). \end{aligned} \tag{4.21}$$

Then the physical masses (i.e. the eigenvalues) are

$$m_1 \approx \frac{m_D^2}{m_S} \ll m_D, \quad m_2 \approx m_S, \tag{4.22}$$

and the mixing angle is

$$\tan\theta = \frac{m_1}{m_2}^{\frac{1}{2}} \approx \frac{m_D}{m_S} \ll 1.. \tag{4.23}$$

Since $m_S \gg m_D$, it follows that $m_1 \ll M$, which means that there is one very light neutrino compared to the charged fermions, which is mainly the $SU(2)$ doublet $(\nu_L', \nu_R'^c$, and there exists one heavy neutrino, that is mainly the singlet $(N_L'^c, N_R')$.

This mechanism of making one particle light at the expense of making another one heavy is called the see-saw mechanism But, it should be noted that cosmological constraints restricts $m_S \geq 10^8 GeV$ which is much larger that the weak scale. In the case when $m_T \neq 0$ (but

$\ll m_S$) there exist two Majorana neutrinos with masses $|m_T - m_D^2/m_S|$ and m_S respectively, while $\theta \sim m_D/m_S \ll 1$ still holds. However, in such a case one does not have the natural explanation of why m_1 is so small, unless m_T is itself induced by the underlying physics and is of the order as m_D^2/m_S.

4.1.2 Lepton Mixing and Neutrino Oscillations

One immediate consequence of neutrinos being massive is the possibility of lepton mixing and neutrino oscillations Thus if the neutrino oscillations are observed that will be an indication of non-zero mass for neutrinos and of physics beyond the Standard Model.

Lepton Mixing

In the previous section we have seen how neutrinos can be either Dirac or Majorana particles Consequently, unlike the quarks there exists more than one mixing scheme for neutrinos. The mixing scheme for neutrinos are classified according to the types of mass terms, whose diagonalisation leads to the corresponding mixing. Defining the charged lepton mass basis by

$$l_L = L_L l'_L, \quad l_R = L_R l'_R, \tag{4.24}$$

where $L_{L,R}$ diagonalize the lepton mass matrix M^l through the biunitary transformation

$$L_L^\dagger M_l L_R = \widehat{M}_l \tag{4.25}$$

and assuming that all the LH neutrinos are part of $SU(2)_L$ doublets with hypercharge $Y = -\frac{1}{2}$, whereas all RH neutrinos are gauge singlets, we obtain the relevant charged current

$$J_\mu^+ = \sum_{i,j=1}^{n} \sum_{\alpha=1}^{n+m} \overline{l_{Li}} \gamma_\mu \chi_{L\alpha} (L_L^\dagger)_{ij} (U^*)_{j\alpha}. \tag{4.26}$$

This leads to an effective neutrino mixing matrix (analogous to the CKM matrix)

$$(K^\nu)_{i\alpha} = \sum_{j=1}^{n} (L_L^\dagger)_{ij} (U^*)_{j\alpha}. \tag{4.27}$$

which is, unlike in the hadronic case, a non-unitary and rectangular $[n \times (n+m)]$ matrix that satisfies:

$$(K^\nu K^{\nu\dagger})_{ik} = \delta_{ik} \quad \text{but} \quad (K^{\nu\dagger} K^\nu)_{\alpha\beta} = \sum_{k=1}^{n} U_{\alpha k}^T U_{k\beta}^*.$$

The non-orthogonality also manifests itself in the neutral current interactions, the relevant isotriplet part of which is given by

$$J_\mu^3 = \sum_{i=1}^{n} \overline{\nu_{iL}} \gamma_\mu \nu_{iL} = \sum_{\alpha,\beta=1}^{n+m} (K^{\nu\dagger} K^\nu)_{\alpha\beta} \overline{\chi_{\alpha L}} \chi_{\beta L}.$$

Parameter counting in this case is slightly different from that in the hadronic sector. K^{ν} is best recognized as being a rectangular part of a $(n+m)\times(n+m)$ unitary matrix and hence, in the most general case is given by ${}^{n+m}C_2$ angles and ${}^{n+m+1}C_2$ phases. However, we can't proceed as for the quarks and eliminate $2(n+m)-1$ phases by redefinition of wavefunctions, for the Majorana neutrinos obviously cannot absorb phase transformations. At most n phases can be eliminated by redefining only the charged lepton wavefunctions and thus we are left with ${}^{n}C_2+\frac{m(2n+m+1)}{2}$ CP violating phases. Unlike the case of quarks, even for two generations we can have $\mathcal{CP}$ violation. It seems natural then that this difference can be exploited to distinguish a Majorana neutrino from a Dirac one, but Schechter and Valle [41] have shown that these extra CP violating effects are always suppressed by an additional factor of $(m_\nu/E_\nu)^2$, where m_ν and E_ν respectively are the mass and energy of the Majorana neutrino taking part in the process. The suppression is easily understood by appreciating that a process dependent on the Majorana mass must have an amplitude proportional to the latter and hence for dimensional reasons there has to be a suppression factor given by the relevant energy scale in the problem.

As in the case of the $K_0-\overline{K^0}$ system, we have, in the general case, a number of neutrinos with possibly all different masses mixing with each other While the interaction terms in the Lagrangian conserve the individual lepton numbers, the mass terms do not, and in the case of Majorana neutrinos even the total lepton number is not preserved. As a neutrino with definite interaction properties evolves in time, each of its massive modes propagates differently resulting in a periodic variation in their relative proportions in the generic neutrino 'beam'. Analogous to strangeness oscillations for the neutral kaons, we have then the possibility of lepton number oscillations [42].

Neutrino Oscillations

To explore the consequences of the mixing hypothesis for neutrinos consider first the mixing of only two species of neutrino, ν_e and ν_μ. In analogy to the quark sector, we express the weak eigenstates as linear combination of mass eigenstates ν_1 and ν_2 at time $t=0$ as

$$\begin{aligned} |\nu_e(0)> &= |\nu_1(0)>\cos\alpha+|\nu_2(0)>\sin\alpha \\ |\nu_\mu(0)> &= -|\nu_1(0)>\sin\alpha+|\nu_2(0)>\cos\alpha \end{aligned} \tag{4.28}$$

where α is the angle that parametrizes the mixing. A non-zero α implies that some neutrinos masses are non-zero and that the mass eigenstates are non-degenerate. Then in a production process, like $\pi^+\longrightarrow e^+\mu_e$, for example, we start with the weak eigenstate $|\mu_e>$ But it is the mass eigenstates $|\nu_i>$ that have a definite time evolution of the form

$$|\nu_i(t)>=|\nu_i(0)>e^{-iE_it};\ i=1,2, \tag{4.29}$$

where their energies are

$$E_i=(p^2+m_i^2)^{1/2}\approx p+\frac{m_i^2}{2p} \tag{4.30}$$

if $p>>m_i$, since our concern is with spatially coherent states in which the neutrinos have essentially identical momenta p. After a time t has elapsed, the pure ν_e state therefore becomes

$$|\nu(t)>=|\nu_1(0)>\cos\alpha e^{-iE_1t}+|\nu_2(0)>\sin\alpha e^{-iE_2t}. \tag{4.31}$$

Substituting for $\nu_1(0)$ and $\nu_2(0)$ in terms of $\nu_e(0)$ and $\nu_\mu(0)$, we have

$$|\nu(t)>=|\nu_e(0)>[e^{-iE_1t}\cos^2\alpha+e^{-iE_2t}\sin^2\alpha]+|\nu_\mu(0)>\sin\alpha\cos\alpha[e^{-iE_2t}-e^{-iE_1t}]. \quad (4.32)$$

The probability that an initial beam of ν_e later contains some ν_μ is given by

$$\begin{aligned} P(\nu_e \to \nu_\mu, t) &= |<\nu_\mu|\nu(t)>|^2 \\ &= \frac{1}{2}\sin^2 2\alpha[1-\cos(E_2-E_1)t], \end{aligned} \quad (4.33)$$

while that of ν_e is

$$\begin{aligned} P(\nu_e \to \nu_e, t) &= |<\nu_e|\nu(t)>|^2 \\ &= 1-\frac{1}{2}\sin^2 2\alpha[1-\cos(E_2-E_1)t]. \end{aligned} \quad (4.34)$$

Since the difference in energy

$$E_2 - E_1 = \frac{E_1^2-E_2^2}{E_1+E_2} \approx \frac{(m_2^2-m_1^2)}{2p}$$

for $E_1 \approx E_2 \sim E >> m_i$, and since the distance travelled $r \approx ct$ is essentially the same for both ν_1 and ν_2 if the state is to remain coherent spatially, we rewrite the oscillation probalities as

$$\begin{aligned} P(\nu_e \to \nu_\mu, t) &= \sin^2 2\alpha \sin^2\frac{\pi r}{L}, \\ P(\nu_e \to \nu_e, t) &= 1-\sin^2 2\alpha\sin^2\frac{\pi r}{L}, \end{aligned} \quad (4.35)$$

where the so-called "oscillation length" L is defined as

$$L \sim \frac{4\pi E}{\Delta m^2} \quad (4.36)$$

and it is an effective length that determines the distance over which one might expect to detect the neutrino oscillations effect Note that the detectability depends on $(m_2^2-m_1^2)$, not on m_2 or m_1 themselves. Thus the oscillation effect of the species of neutrino that is observed will change as a function of the distance r from the source, provided that (a) the mixing angle $\alpha \neq 0$, and (b) $m_1 \neq m_2$.

For the general case of more number of neutrino flavours $\nu_i (i = e, \mu, \tau, ...)$, we have

$$\begin{aligned} |\nu_i> &= \sum_i U_{li}|\nu_i>, \\ P(\nu_l \longrightarrow \nu_{l'}) &= \delta_{ll'} - \sum_{i>j} 4U_{li}U^*_{l'i}U^*_{lj}U_{l'j}\sin^2(\frac{\pi r}{L_{ij}}), \end{aligned} \quad (4.37)$$

where the U_{ij} are the lepton mixing matrix elements and

$$L_{ij} \sim \frac{4\pi E}{|m_i^2-m_j^2|} = \frac{2.48E/(MeV)}{\Delta m_{ij}^2/(eV)^2} \ meters \quad (4.38)$$

For a direct observation of such oscillations, we need to perform experiments such that

$$\frac{r}{E} \sim \frac{L}{E} \sim \frac{1}{\Delta m^2}, \quad (4.39)$$

though the effect of mixing will be significant for $r \geq L$. The null results obtained so far indicate either that $\Delta m^2 < E/r$ or that the relevant mixing matrix elements U_{ij} are very small.

4.1.3 Experimental Evidences

Accurate measurements of the charged particle momenta in the processes

$$^{3}He \longrightarrow ^{3}He^{+} + e^{-} + \overline{\nu_e}, \quad \pi^{-} \longrightarrow \mu^{-}\overline{\nu_\mu}, \quad \tau^{-} \longrightarrow \pi^{+}\pi^{-}\pi^{+}\pi^{-}\pi^{-}\nu_\tau$$

yields the upper bounds on the masses (in GeV) of the neutrinos

$$\begin{aligned} m_{\nu_e} &\leq 1.8 \times 10^{-8}, \\ m_{\nu_\mu} &\leq 2.5 \times 10^{-4}, \\ m_{\nu_\tau} &\leq 3.5 \times 10^{-2}. \end{aligned} \tag{4.40}$$

There exists no heavy neutrinos, that might be detected in processes such as $\pi^{+} \longrightarrow \mu^{+}\nu_H$ or $\nu_H \longrightarrow e^{+}e^{-}\nu_e$, in the mass range 10 MeV- 10 GeV unless their coupling to e and μ are extraordinarily small ($< 10^{-5}G_F$) Besides these above mentioned kinematics, the process of neutrinoless double β-decays of the nuclei might provide us with a clue for finiteness of neutrino masses. Normally, the double β-decay of a nucleus of mass number A and charge Z is

$$(A, Z) \longrightarrow (A, Z+2) + e^{-} + e^{-} + \overline{\nu_e} + \overline{\nu_e},$$

but with a massive Majorana neutrino the decay

$$(A, Z) \longrightarrow (A, Z+2) + e^{-} + e^{-}$$

is possible through the $\Delta L = 2$ transition $\overline{\nu_R} \rightarrow \nu_L$. Searches for such decays in

$$^{76}Ge \longrightarrow ^{76}Se + e^{-} + e^{-}$$

give the lifetime bounds $\tau \geq 10^{23}$ years, from which a rather model-dependent limit to the Majorana mass $m_{\nu_e} < 1 - 2eV$ can be deduced[39] But such experiments say nothing about the masses of Dirac neutrinos for which these decays are forbidden.

Another possible source of evidence for finite masses for neutrinos would be the observation of neutrino oscillations. The oscillation length L_{ij} and the difference of squares of neutrino masses Δm^2 disscussed earlier decide the way in which such oscillations can be detected. Although various experiments in the context of solar neutrinos, cosmic rays, nuclear reactors and particle accelerators have been performed, there is as yet no convincing evidence that the neutrinos mix under the weak interaction like the quarks.

4.2 17 keV Nondegenerate Majorana Neutrino and neutrino mixing

In 1985, it was observed by Simpson[44] that there exists an anomalous kink in the Curie plot of the β-spectrum in Tritium decay This was interpreted as a mixture (with a 3% mixing) of a $17keV$ neutrino with the ν_e i e. $|U_{17e}|^2 = 0.03$. This was reobserved by others[46] in 1991.

As compared to the original claim, the mixing ($|U_{17e}|^2 = x$) of the 17 kev neutrino with ν_e had changed, the later value being close to 1% i.e.$x = 0.01$ [45, 46]. We have studied[1] the limits on the elements of the neutrino mixing matrix consistent with neutrinoless double beta decay and the neutrino oscillation experiments as a function of the mixing probability (x) of ν_e with the $17keV$ neutrino assuming only three generations of left-handed neutrinos and no sterile neutrinos

We considered all x between 0.003 and 0.03, and found that $x > 0.015$ is not allowed. Stringent limits on m_{ν_μ} (when $m_{\nu_\mu} \gg m_{\nu_\tau}$) and the mass difference ($m_{\nu_\mu} - m_{\nu_\tau}$) (when ν_μ and ν_τ form a pseudo-Dirac particle) are found Allowed values of the various mixing angles are obtained as function of x, when ν_e, ν_μ and ν_τ are nondegenerate Majorana neutrinos.

In our analysis we take x as a parameter and quote limits on other quantities as function of x. Since the present limit on the number of light neutrino species (as obtained from the Z width) is very close to 3, we shall study the constraints on the mixing matrix for three Majorana neutrinos. When the masses are non-degenerate at the tree level i.e.the mass differences are large,we parametrize the mixing matrix by three angles (assuming no CP violation in the leptonic sector).Then the limits on ($\nu_e \to \nu_\mu$) oscillaton, the neutrinoless double beta decay and the value of x can set limits on the three angles and hence on all the elements of the mixing matrix. For consistency we then calculate the ($\nu_\mu \to \nu_\tau$) and ($\nu_e \to \nu_\tau$) oscillation probabilities, and compare them with the experimental limits.We find extremely narrow allowed regions for the three mixing angles and hence the elements of the mixing matrix.

Next we analyse the situation when the $17keV$ neutrino is a pseudo-Dirac particle,that is, at tree level ν_μ and ν_τ combine together to form a Dirac particle,but a small mass difference is generated radiatively.In this case the strongest bound on the mass difference comes from the ν_μ disappearance experiment. If one starts with a ($L_e + L_\tau - L_\mu$) type of symmetry [47], which is broken at low energy,then at the tree level ν_e is massless and 17 kev neutrino is a Dirac neutrino. The symmetry breaking will induce new contributions to the zero elements of the mass matrix.From the limit on the allowed mass difference,we find that the limits on these non-zero elements and found them to be unnaturally small.

We shall first consider that three neutrinos (ν_i) have nondegenerate Majorana masses m_i [48]. The weak eigenstates of the neutrino $\nu_\alpha (\alpha = e, \mu, \tau)$ are related to the mass eigenstates $\nu_i (i = 1, 2, 3)$ through the relation

$$\nu_\alpha = \sum_{i=1}^{3} U_{\alpha i} \nu_i \tag{4.41}$$

where $U_{\alpha i}$ is the mixing matrix.If we assume that there is no CP violation in the leptonic sector, then $U_{\alpha i}$ is real and is an orthogonal matrix. We start with the most general 3x3 orthogonal matrix which has three independent parameters:

$$U = \begin{pmatrix} c_1 c_3 & s_1 c_3 & s_3 \\ -s_1 c_2 - c_1 s_2 s_3 & c_1 c_2 - s_1 s_2 s_3 & s_2 c_3 \\ s_1 s_2 - c_1 c_2 s_3 & -c_1 s_2 - s_1 c_2 s_3 & c_2 c_3 \end{pmatrix} \tag{4.42}$$

[1]This section is based on the work reported in ref [43]

where $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$. In the mass eigen-state basis, the charged-lepton-neutrino-W coupling is of the form

$$L_w = \bar{l}W^- U\nu + h.c. \tag{4.43}$$

The most general neutrino mass Lagrangian L_m has the form

$$L_m = -\frac{1}{2}\overline{(\nu_\alpha)^c}M\nu_\alpha + h.c. \tag{4.44}$$

Using $\nu_\alpha = \sum_{i=1}^{3} U_{\alpha i}\nu_i$, in the expression for L_m, we obtain the mass matrix M in terms of the mixing matrix U and M^{diag} where $M^{diag} = diag(m_1, m_2, m_3)$, and m_1, m_2 and m_3 being the mass eigenvalues, as

$$M = UM^{diag}U^T \tag{4.45}$$

Recognising s_1, s_2, s_3 as the mixing angles we shall try to find out the allowed regions for each of them satisfying the following constraints:

a)The mixing of the $17keV$ neutrino with the ν_e should be commensurate with the latest experimental result. But for completeness we take it as a parameter in our analysis i.e.

$$|U_{17\,e}|^2 = x \tag{4.46}$$

and vary x between 0.003 to 0.030, which includes the present experimental value.

b)The 17 keV neutrino cannot be ν_μ, since $(\nu_e - \nu_\mu)$ oscillation will be too fast in that case So, the third eigenvalue m_3 is dominantly the ν_μ mass We shall thus use the present experimental limit [49] on ν_μ for m_3 i.e.,

$$m_3 < 250keV \tag{4.47}$$

and $m_2 = 17keV$ is mostly ν_τ mass. The limits for m_1 comes from the limit [17] on ν_e mass i.e.

$$m_1 < 17eV \tag{4.48}$$

The non-observation of neutrinoless double beta decay implies limit [17] on,

$$\left|\sum_{i=1}^{3} U_{ei}^2 m_i\right| < 1.8eV \tag{4.49}$$

c)In addition to these, we have, further constraints coming from neutrino oscillation. When

$$|m_i^2 - m_j^2| > 100ev^2 \tag{4.50}$$

the best limits [50] for various neutrino oscillations are

$$P_{\nu_e\to\nu_\mu} < 2\times10^{-3},\quad P_{\nu_\mu\to\nu_\tau} < 3\times10^{-3},\quad P_{\nu_\tau\to\nu_e} < 0.21 \tag{4.51}$$

The probability for a neutrino of flavour a to oscillate into a neutrino of flavour b is given by

$$P_{a\to b} = \left|\sum_{j=1}^{3} U_{aj}U_{bj}^* e^{(im_j^2 L/2E)}\right|^2 \tag{4.52}$$

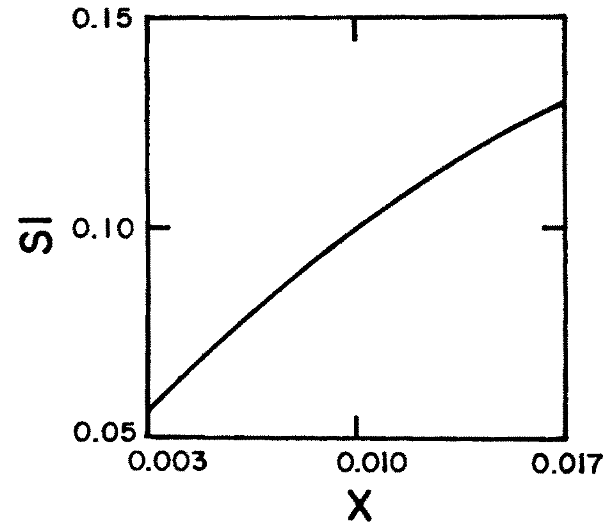

Figure 4.1: The allowed region for s_1 versus x. Note that the curves corresponding to s_{1max} and s_{1min} coincide.

Using the above mentioned parametrization for U [eqn(4.42)], we can express $P_{a\to b}$ as a function of the mixing angles s_1, s_2 and s_3. The $\nu_e \to \nu_\mu$ oscillation probabillity is expressed as

$$P_{\nu_e\to\nu_\mu} = 2[s_2^2\{s_3^2c_3^2(1-c_1^2s_1^2)-(s_1c_1c_3)^2\} + s_2c_2s_1c_1s_3c_3^2(c_1^2-s_1^2)+(s_1c_1c_3)^2], \quad (4.53)$$

since m_i are nondegenerate [they satisfy eqn(4.50)] and the interference terms average out to zero

In our numerical analysis, we vary s_2 and s_3 between 0 and 1 and the parameter x in the range of 0.003 to 0.03 and calculate s_1 using

$$s_1 = (x/c_3^2)^{\frac{1}{2}}. \quad (4.54)$$

Taking $m_2 = 17keV$, m_3 was calculated using the constraint from the neutrinoless double beta decay as

$$m_3 = m_2\frac{|U_{e\,17}|^2}{|U_{e\,3}|^2} = m_2\frac{x}{s_3^2}. \quad (4.55)$$

Corresponding values of the $\nu_e - \nu_\mu$, $\nu_\mu - \nu_\tau$, and $\nu_\tau - \nu_e$ oscillation probabilities were calculated It was found out that the upper limit of $P_{\nu_e-\nu_\mu}$ rules out most of the allowed regions of the various angles. The allowed region for each of the angles s_1, s_2, s_3 and m_3, are plotted versus the mixing parameter x (figs (4.1,4.2,4 3,4.4).

The allowed region of the parameter space is extremely narrow. The fig.(4.1) shows the allowed region of s_1 as a function of x, in which both the curves coresponding to upper and

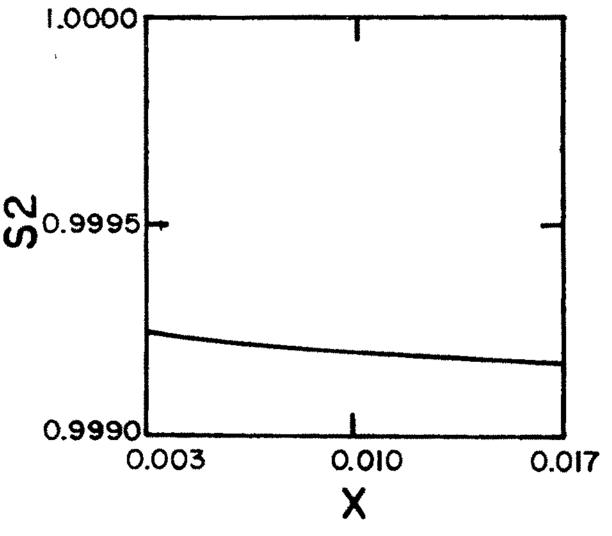

Figure 4 2: The lower limit for s_2 as a function of x. The upper limit is 1.

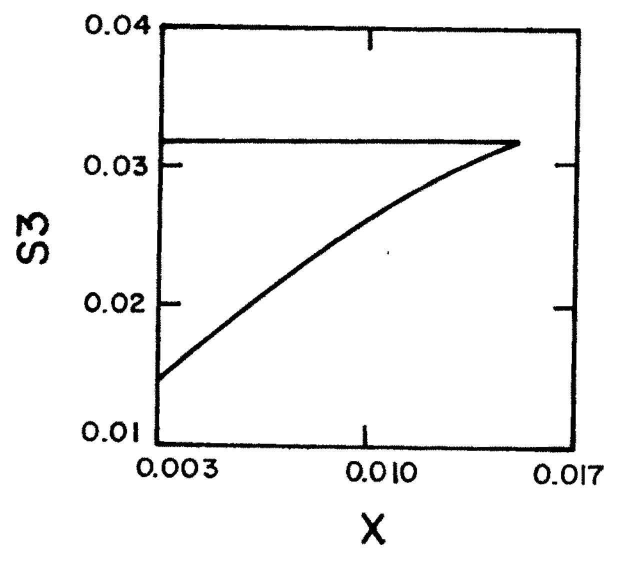

Figure 4.3: Allowed region of s_3 as a function of x.

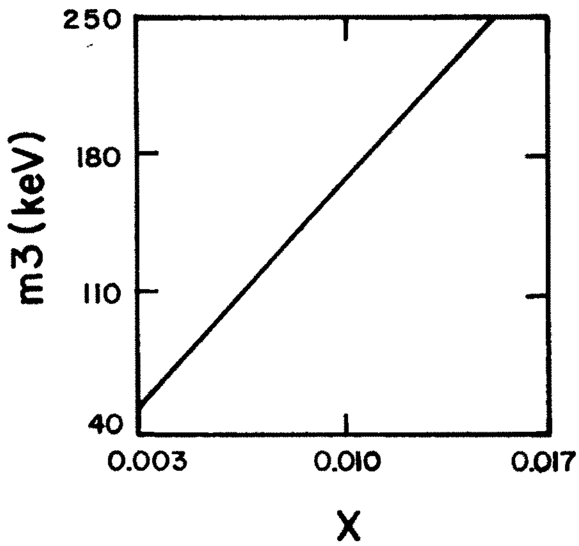

Figure 4.4: Allowed region of m_3 as a function of x. The lower limit is given by the curve, and the upper limit is the experimental upper limit of 250 keV.

lower limits merge As it is shown in fig.(4.2), where we plot the lower limit for s_2 as a function of x, we find the allowed region to be extremely narrow. The upper limit of s_2 is always 1 and as a result, the upper limit of s_3 can be obtained from the expression

$$P_{\nu_e - \nu_\mu} = 2 s_3^2 c_3^2 (1 - s_1^2 c_1^2), \tag{4.56}$$

where s_1 and c_1 are also given in terms of s_3 [eq. 4.54]. The allowed region of s_3 and m_3 as a function of x are shown in figs.(4.3,4.4) and respectively. For $x \geq 0.015$ there is no allowed region in the parameter space. To get an idea of how much restriction is imposed on the various elements of the mixing matrix, we give the allowed ranges of the elements of the mixing matrix U for $x = 0.1$,

$$U = \begin{pmatrix} 0.9945 - 0.9946 & 0.0999 - 0.0999 & 0.0261 - 0.0318 \\ (-0.0259) - (-0.0356) & (-0.0026) - (+0.0365) & 0.9988 - 0.9995 \\ 0.0988 - 0.0999 & (-0.9942) - (-0.9951) & 0.0000 - 0.0398 \end{pmatrix} \tag{4.57}$$

We shall now consider the case, when ν_μ and ν_τ combine to form a pseudo-Dirac neutrino with the tree level mass of 17 keV and about 1 % mixing with ν_e. This scenario can be incorporated in a model with just three conventional neutrinoes in the low energy theory, which has $(L_e + L_\tau - L_\mu)$ as a good approximate symmetry. If the lepton mass matrices have a good approximate global symmetry $(L_e + L_\tau - L_\mu)$ then in the basis in which the first (second,third) row and column refers to the $e(\mu, \tau)$ weak eigenstates and in which the charged lepton mass matrix is diagonal, the most general mass matrix [47] is given by

$$M\begin{pmatrix} 0 & sin\theta & 0 \\ sin\theta & 0 & cos\theta \\ 0 & cos\theta & 0 \end{pmatrix} \tag{4.58}$$

where $M = 17keV$ to reproduce the massive neutrino states and $sin^2(\theta) = 0.03$ to reproduce 3% mixing. But the symmetry breaking will induce new contributions to the the zero elements of the mass matrix Thus the limit on the mass difference ϵ will fix the bounds of the contributions to the zero elements of the mass matrix M after the symmetry breaking.

To consider the constraints on the lepton weak mixing matrix U for the case of

$$|m_3 - m_2| < 3 \times 10^{-03} eV \tag{4.59}$$

we write, in the flavour basis, the neutrino mass matrix as

$$M' = U^* M'^{diag} U^\dagger \tag{4.60}$$

where

$$M'^{diag} = diag(\delta, 17keV + \epsilon, 17keV - \epsilon) \tag{4.61}$$

Using the parametrization [48], which is ideal for the limit $\epsilon \to 0$

$$U_M = \begin{pmatrix} c_\gamma e^{i\sigma} & s_\gamma B^* & s_\gamma A^* \\ s_\alpha s_\gamma e^{i\sigma} & e^{i\rho} c_\alpha A - s_\alpha c_\gamma B^* & -e^{i\rho} c_\alpha B - s_\alpha c_\gamma A^* \\ -s_\gamma c_\alpha e^{i\sigma} & e^{i\rho} s_\alpha A + c_\alpha c_\gamma B^* & -e^{i\rho} s_\alpha B + c_\alpha c_\gamma A^* \end{pmatrix} \tag{4.62}$$

where

$$\begin{pmatrix} A & -B \\ B^* & A^* \end{pmatrix} = \begin{pmatrix} c_\beta & is_\beta \\ is_\beta & c_\beta \end{pmatrix} \begin{pmatrix} c_\lambda & -s_\lambda \\ s_\lambda & c_\lambda \end{pmatrix} \tag{4.63}$$

and $m_1 \leq 13eV$, we have, for the mixing parameter x,

$$C_{2\beta} < \frac{13eV}{17keV}\frac{1}{x} \tag{4.64}$$

The $\nu_\mu \to \nu_e$ oscillation gives

$$s_\alpha^2 < \frac{2 \times 10^{-3}}{x(1-x)} \tag{4.65}$$

The $\nu_\mu \to \nu_\tau$ oscillations and ν_μ disappearance experiments fix the limit on the ν_τ and ν_μ mass difference as $4 \times 10^{-5} eV$ [51] and $9 \times 10^{-8} eV$ [52] respectively

We found out that to satisfy such a strong limit the bounds on $ee, e\tau, \tau e$, $\mu\mu$ and $\tau\tau$ elements after the symmetry breaking is unnaturally small compared to the bounds suggested in Dugan et.al's paper. It was noted in their paper that for the lightest neutrino to be less than $40eV$, the bound on $e\tau$ and τe entries is about $250eV$ and limits from neutrinoless double beta decay require the ee element to be less than $1eV$. It was also noted that rapid oscillations $\nu_\mu \to \nu_\tau$ caused by small non-degeneracy restrict the $\mu\mu$ and $\tau\tau$ elements to be less than $1eV$. But, we notice that $\epsilon < 9 \times 10^{-08} eV$ fixes the bounds for all the zero elements at $10^{-10} keV$. This conclusion is based on a numerical calculation in which $ee, e\tau, \tau e, \mu\mu, \tau\tau$ elements were varied slowly so far as $|m_3 - m_2| < 10^{-08} eV$.

To summarize, we studied constraints on the neutrino mixing matrix from the various oscillation data, neutrinoless double beta decay and the limit on the ν_e, ν_μ and ν_τ masses assuming only three generations of left-handed neutrinos and no sterile neutrinos. In the limit when all the three eigenvalues are nondegenerate and the mass differences are larger than $100 eV^2$, we identify ν_τ with the 17 keV neutrino and vary the mixing probability between 3% and 0.3%. We find a very narrow allowed region for the various mixing angles. The allowed values of m_{ν_μ} lie between 145 keV and 205 keV for 1% mixing and between 135 keV and 240 keV for 3% mixing (our result differs from the earlier similar works with 3% mixing [48], where some approximations were made). The $\nu_e \rightarrow \nu_\mu$ oscillation probability is found to lie between .001 and .002 for consistency. We then considered the allowed amount of the symmetry breaking when ν_μ and ν_τ form a pseudo-Dirac particle. We found the mass difference to be less than $9 \times 10^{-08} eV$, which puts stringent limits on the symmetry breaking effect.

4.3 Potential Minimisation in Left-Right symmetric models and neutrino masses

4.3.1 Introduction

Although neutrinos are massless in the SM, they can be given masses in extensions of the SM based on either an extended fermion sector or an extended higgs sector. In the previous section we have seen how to accomodate the 17 keV massive neutrino without extending the fermion sector of the SM The models incorporating an extension of the higgs sector have the advantage of explaining the smallness of neutrino mass through the see-saw mechanism. The see-saw mechanism can be very easily incorporated into the left-right symmetric extension[54] of the SM. In this section we present an analysis[2] of the minimization of the scalar potentials in left-right symmetric extensions of the SM with various choices of the higgs scalars and subsequently study their phenomenological implications regarding the neutrino masses.

4.3.2 Rudiments of Left-Right symmetric model

Left-Right symmetric models[55] are considered to be the most natural extensions of the standard model. Popularly one chooses the gauge group $G_{3221} = SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ or $G_{224} = SU(2)_L \times SU(2)_R \times SU(4)_c$ to describe the invariance properties of the model. When G_{3221} or G_{224} admits spontaneous symmetry breaking one recovers the standard model. Spontaneous symmetry breakdown takes place when the higgs fields transforming nontrivially under the higher symmetry group but not transforming under the lower symmetry group acquires a vacuum expectation value (vev). If one embeds the group G_{3221} or G_{224} in a grand unified theory or a partially unified theory then LEP constraints on $sin^2\theta_w$[56] can put strong bounds

[2] This section is based on the work reported in ref.[53]

on the breaking scale of the right handed $SU(2)_R$ group. On the other hand if one considers the left-right symmetric model with $g_L \neq g_R$ the right handed breaking scale can be lowered[57]. In this case the model becomes interesting as a rich set of phenomenological consequences can be directly tested in the next generation colliders. To achieve the inequality of the couplings a D-odd singlet higgs field η is introduced which on acquiring vev breaks the left-right parity (D-parity).

We are interested in the following symmetry breaking pattern:

$$\begin{aligned} SU(2)_L \times SU(2)_R \times SU(4)_c \ \underset{M_X}{\longrightarrow} & \ SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \\ \underset{M_R}{\longrightarrow} & \ SU(2)_L \times SU(3)_c \times U(1)_Y \\ \underset{M_W}{\longrightarrow} & \ SU(3)_c \times U(1)_Q \end{aligned} \tag{4.66}$$

If G_{224} is embedded in any higher symmetry group, then also most of the analysis will not change. In this sense our analysis is quite general. The advantage of starting with the group G_{224} instead of the group G_{3221} is that, we can discriminate between the fields which do and donot distinguish between quarks and leptons. This is important to understand the mass ratios of quarks and leptons.

We will also assume that $M_X = M_R$ which will imply that the scale of breaking of SU(4) color is the same as that of the breaking of the left right symmetry. This will not cause any loss of generality of our analysis. To specify the model further let us state the transformation properties of the fermions.

$$\begin{aligned} \psi_L &= \begin{pmatrix} \nu_L \\ e^-{}_L \end{pmatrix} : (2,1,4) \ ; \ \psi_R = \begin{pmatrix} \nu_R \\ e^-{}_R \end{pmatrix} : (1,2,4) \\ Q_L &= \begin{pmatrix} u_L \\ d_L \end{pmatrix} : (2,1,4) \ ; \ Q_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix} : (1,2,4) \end{aligned} \tag{4.67}$$

The scalar fields which may acquire vev are stated below.

$$\begin{aligned} \phi_1 &\equiv (2,2,1) \ ; \ \phi_2 \equiv \tau_2 \phi_1^* \tau_2 \ ; \ \xi_1 \equiv (2,2,15) \ ; \ \xi_2 \equiv \tau_2 \xi_1^* \tau_2 \\ \Delta_L &\equiv (3,1,10) \ ; \ \Delta_R \equiv (1,3,10) \ , \ \eta \equiv (1,1,0) \end{aligned}$$

It has been shown in recent past that the LEP constraints on $sin^2\theta_w$[56] can put strong lower bound on the scale M_R. From renormalization group equations one can show that the right handed breaking scale has to be greater than 10^9 GeV. However one can show that when the D-Parity is broken the right handed breaking scale can be lowered. In that case a rich set of phenomenological predictions can be experimentally tested in high energy colliders. Here we consider the singlet field η which is odd under D-Parity It breaks D-Parity when it acquires vev[57].

If we consider an underlying GUT, and start with the masses of the quarks and leptons to be the same at the unification scale, then, in the absence of ξ the low energy mass relations of fermions are not correct. This is because the field (2,2,1) contributes equally to the masses of the quarks and leptons. The situation can be corrected by the introduction of the field (2,2,15)[58]. This is the initial motivation to introduce the field ξ. Once it is there it allows new interesting baryon number violating decay modes which we discuss below.

Recently a lot of interest has been generated in the three lepton decay of the proton in SU(4) color gauge theory[59]. It can be shown that if the SU(3) triplet component of ξ remains sufficiently light it can mediate the three lepton decay mode of proton with a lifetime of 4×10^{31} years. In that case sufficient number of extra electron type neutrinos can be produced in the detector which can explain atmospheric neutrino anomaly. To keep the SU(3) triplet component of ξ sufficiently light, the following mechanism was proposed by Pati, Salam and Sarkar. If an extra (2,2,15) or (2,2,6) higgs field (henceforth called ξ' and χ) is introduced, its SU(3) triplet component will mix with the triplet component of ξ and hence there will be a light triplet in the model. These extra fields do not acquire *vev*. However the terms in the scalar potential which are linear in these extra fields can strongly constrain the other parameters of the model. In this paper we introduce such extra fields which do not acquire *vev* and study the terms in the scalar potential which are linear in these extra fields. The extra fields we consider here are,

$$\xi' = (2,2,15) \; ; \; \chi = (2,2,6) \; ; \; \delta = (3,3,0). \tag{4.68}$$

We shall see below that the linear term in the extra field δ will constrain the ratio of the D-parity breaking scale and the right handed symmetry breaking scale. We emphasise that in different models with extra scalars such study is necessary as it points out the extra scalar which is not favourable by the existing phenomenology.

4.3.3 Minimization of potential

Minimal choice of higgs scalars

The general procedure we adopt here is the following. First we write down the most general scalar potential which is allowed by renormalizability and gauge invariance. Next we substitute the vacuum expectation values in the potential and find out the minimization conditions. Here let us first write down the scalar potential with the scalar fields ϕ, Δ and η[60],

$$V(\phi_1, \phi_2, \Delta_L, \Delta_R, \eta) = V_\phi + V_\Delta + V_\eta + V_{\eta\Delta} + V_{\eta\Delta} + V_{\eta\phi}, \tag{4.69}$$

where the different terms in this expression are given by,

$$\begin{aligned} V_\phi &= -\sum_{i,j} \mu_{ij}^2 \, tr(\phi_i^\dagger \phi_j) + \sum_{i,j,k,l} \lambda_{ijkl} \, tr(\phi_i^\dagger \phi_j) \, tr(\phi_k^\dagger \phi_l) \\ &\quad + \sum_{i,j,k,l} \lambda_{ijkl} \, tr(\phi_i^\dagger \phi_j \phi_k^\dagger \phi_l) \\ V_\Delta &= -\mu^2 \, (\Delta_L^\dagger \Delta_L + \Delta_R^\dagger \Delta_R) + \rho_1 \, [tr(\Delta_L^\dagger \Delta_L)^2 + tr(\Delta_R^\dagger \Delta_R)^2] \\ &\quad + \rho_2 \, [tr(\Delta_L^\dagger \Delta_L \Delta_L^\dagger \Delta_L) + tr(\Delta_R^\dagger \Delta_R \Delta_R^\dagger \Delta_R)] + \rho_3 \, tr(\Delta_L^\dagger \Delta_L \Delta_R^\dagger \Delta_R) \end{aligned}$$

$$
\begin{aligned}
V_\eta &= -\mu_\eta^2\,\eta^2 + \beta_1\,\eta^4 \\
V_{\Delta\phi} &= +\sum_{i,j}\alpha_{ij}\,(\Delta_L^\dagger\Delta_L + \Delta_R^\dagger\Delta_R)\,tr(\phi_i^\dagger\phi_j) + \sum_{i,j}\beta_{ij}\,[\,tr(\Delta_L^\dagger\Delta_L\phi_i\phi_j^\dagger) \\
&\quad + tr(\Delta_R^\dagger\Delta_R\phi_i^\dagger\phi_j)] \\
&\quad + \sum_{i,j}\gamma_{ij}\,tr(\Delta_L^\dagger\phi_i\Delta_R\phi_j^\dagger) \\
V_{\eta\Delta} &= M\,\eta\,(\Delta_L^\dagger\Delta_L - \Delta_R^\dagger\Delta_R) + \beta_2\,\eta^2\,(\Delta_L^\dagger\Delta_L + \Delta_R^\dagger\Delta_R) \\
V_{\eta\phi} &= \sum_{i,j}\delta_{ij}\,\eta^2\,tr(\phi_i^\dagger\phi_j) \quad .
\end{aligned}
$$

The vacuum expectation values of the fields have the following form:

$$
\begin{aligned}
<\phi> &= \begin{pmatrix} k & 0 \\ 0 & k' \end{pmatrix} \;;\; <\Delta_L>= \begin{pmatrix} 0 & 0 \\ v_L & 0 \end{pmatrix} \;;\; <\eta>=\eta_0; \\
<\tilde{\phi}> &= \begin{pmatrix} k' & 0 \\ 0 & k \end{pmatrix} \;;\; <\Delta_R>= \begin{pmatrix} 0 & 0 \\ v_R & 0 \end{pmatrix}
\end{aligned}
$$

The phenomenological consistency requires the hierarchy $<\Delta_R> \;>>\; <\phi> \;>>\; <\Delta_L>$ and also that $k' << k$. Now the minimization conditions of the potential V are found out by differentiating it with respect to the parameters k, k', v_L, v_R and η_0 and separately equating them to zero. This will give us five equations for five parameters present. Solving the equations involving the derivatives with respect to v_L and v_R we get the relation between v_L and v_R :

$$
v_L v_R = \frac{\beta k^2}{[(\rho-\rho') + \frac{4M\eta_0}{(v_L^2 - v_R^2)}]},
$$

where we have defined $\beta = 2\gamma_{12}$. The details of the derivation are presented in a subsequent section. We get in the M=0 limit,

$$
v_L v_R \simeq \frac{\beta k^2}{[\rho-\rho']} \simeq \gamma k^2 \tag{4.70}
$$

Here γ is a function of the couplings. However when the field η is present, v_L becomes differently related to v_R in the limit of large η_0.

$$
v_L \simeq -(\frac{\beta k^2}{4M\eta_0})v_R \simeq (\frac{\beta k^2}{\eta_0^2})v_R \tag{4.71}
$$

Here we see the important difference between the D-conserving and D-breaking scenarios.

This result was discussed in details in ref. [61]. In the D-parity conserving case, when the η field is absent one has to fine tune parameters to make γ arbitrarily snall so that the see-saw neutrino mass can be comparable to the Majorana mass of the left-handed neutrinos given by v_L. This fine tuning becomes redundant when the field η acquires *vev*.

In the presence ξ=(2,2,15)

When ξ is present, the most general scalar potential takes the following form.

$$V(\phi_1, \phi_2, \Delta_L, \Delta_R, \xi_1, \xi_2, \eta) = V_\phi + V_\Delta + V_\eta + V_\xi + V_{\phi\eta} + V_{\eta\Delta} + V_{\eta\phi} + V_{\phi\xi} + V_{\Delta\xi} + V_{\eta\xi} \quad (4.72)$$

The explicit forms of the terms involving ξ are listed below:

$$\begin{aligned}
V_\xi &= -\sum_{i,j} m_{ij}^2 \; tr(\xi_i^\dagger \xi_j) + \sum_{i,j,k,l} n_{ijkl} \; tr(\xi_i^\dagger \xi_j \xi_k^\dagger \xi_l) + \sum_{i,j,k,l} p_{ijkl} \; tr(\xi_i^\dagger \xi_j) \; tr(\xi_k^\dagger \xi_l) \\
V_{\phi\xi} &= \sum_{i,j,k,l} u_{ijkl} \; tr(\phi_i^\dagger \phi_j \xi_k^\dagger \xi_l) + \sum_{i,j,k,l} v_{ijkl} \; tr(\phi_i^\dagger \phi_j) \; tr(\xi_k^\dagger \xi_l) \\
V_{\Delta\xi} &= +\sum_{i,j} a_{ij} \; [\; tr(\Delta_L^\dagger \Delta_L) + tr(\Delta_R^\dagger \Delta_R)] \; tr(\xi_i^\dagger \xi_j) \\
&\quad + \sum_{i,j} b_{ij} \; [\; tr(\Delta_L^\dagger \Delta_L \xi_i \xi_j^\dagger) + tr(\Delta_R^\dagger \Delta_R \xi_i^\dagger \xi_j)] \\
&\quad + \sum_{i,j} c_{ij} \; tr(\Delta_L^\dagger \xi_i \Delta_R \xi_j^\dagger) \\
V_{\eta\xi} &= \sum_{i,j} d_{ij} \; \eta^2 \; tr(\xi_i^\dagger \xi_j)
\end{aligned}$$

The vacuum expectation value of ξ has the following form,

$$< \xi >= \begin{pmatrix} \tilde{k} & 0 \\ 0 & \tilde{k}' \end{pmatrix} \times (1,1,1,-3). \quad (4.73)$$

Here we may briefly mention the need to introduce the field ξ. The vacuum expectation value of the field ϕ is given by,

$$< \phi >= \begin{pmatrix} k & 0 \\ 0 & k' \end{pmatrix} \times (1,1,1,1). \quad (4.74)$$

Note that in the SU(4) color space the fourth entry is 1 for the *vev* of ϕ whereas it is -3 for the *vev* of ξ. Hence the *vev* of ϕ treats the quarks and the leptons on the same footing, whereas the *vev* of ξ differentiates between the quarks and the leptons. For example in the absence of ξ one gets $m_e^0 = m_d^0$, $m_\mu^0 = m_s^0$ and $m_\tau^0 = m_b^0$. Now including the QCD and electroweak renormalization effects in the symmetric limit it leads to the relation $\frac{m_e}{m_\mu} = \frac{m_d}{m_s}$. However when the field ξ is included in the masses in the symmetric limit they take the form $m_e^0 = m_e^\phi - 3m_e^\xi$ and $m_d^0 = m_d^\phi - m_d^\xi$.

The minimization conditions are again found by taking the derivatives of V with respect to the parameters $k, k', v_L, v_R, \tilde{k}^2$ and $\tilde{k'}^2$ and separately equating them to zero. Solving the equations involving the derivatives of v_L and v_R yields in the limit $\tilde{k}' << \tilde{k}$:

$$v_L v_R = \frac{[(w\tilde{k}^2 + \beta k^2)\,(v_L^2 - v_R^2)}{[(\rho - \rho')(v_L^2 - v_R^2) + 4M\eta_0]}. \quad (4.75)$$

Here we have defined $w = 2c_{12}$. Let us again check the special cases. Firstly the case without ξ can be recovered in the limit w=0, on the other hand the case with unbroken D-parity can be restored in the limit M=0; which is,

$$v_L v_R \simeq \frac{w\tilde{k}^2 + \beta k^2}{[(\rho - \rho')]}. \quad (4.76)$$

When D-parity is broken the v_L can be suppressed by η_0,

$$v_L = \frac{w\tilde{k}^2 + \beta k^2}{\eta_0^2} v_R. \quad (4.77)$$

We infer that the field ξ is allowed by the potential minimization and its introduction does not alter the general features of the see-saw condition between v_L and v_R.

4.3.4 Introduction of extra fields

Introduction of ξ'=(2,2,15)

We have already mentioned that there exist interesting models in the literature where the field ξ' is introduced to induce a sufficiantly large amplitude of the three lepton decay width of the proton In these models the field ξ does not acquire *vev*. Hence after the minimization all terms other than the ones which are linear in ξ' drops out whereas the ones which are linear in ξ' puts constraints on the parameters of the model. Usually when any new fields are introduced in any model, which do not acquire *vevs*, it is assumed that it will not change the minimization conditions. As a result potential minimization with such fields were not done so far.

In this section we will first write down the linear couplings of the field ξ' *i.e.*

$$\begin{aligned}
V'_\xi &= -\sum_{i,j} \tilde{m}_{ij}^2 \, tr(\xi_i^\dagger \xi'_j) + \sum_{i,j,k,l} n_{ijkl} \, tr(\xi_i^\dagger \xi_j \xi_k^\dagger \xi'_l) \\
&\quad + \sum_{i,j,k,l} p_{ijkl} \, tr(\xi_i^\dagger \xi_j) \, tr(\xi_k^\dagger \xi'_l) \\
&+ \sum_{i,j,k,l} u_{ijkl} \, tr(\phi_i^\dagger \phi_j \xi_k^\dagger \xi'_l) + \sum_{i,j,k,l} v_{ijkl} \, tr(\phi_i^\dagger \phi_j) \, tr(\xi_k^\dagger \xi'_l) \\
&+ \sum_{i,j} \tilde{a}_{ij} \, (\Delta_L^\dagger \Delta_L + \Delta_R^\dagger \Delta_R) \, tr(\xi_i^\dagger \xi'_j) \\
&\quad + \sum_{i,j} \tilde{b}_{ij} \, [\, tr(\Delta_L^\dagger \Delta_L \xi'_i \xi_j^\dagger) + tr(\Delta_R^\dagger \Delta_R \xi'_i \xi_j^\dagger)] \\
&+ \sum_{i,j} \tilde{c}_{ij} \, tr(\Delta_L^\dagger \xi'_i \Delta_R \xi_j^\dagger) \\
&+ \sum_{i,j} \tilde{d}_{ij} \, \eta^2 \, tr(\xi_i^\dagger \xi'_j)
\end{aligned}$$

When this potential is minimised with respect to ξ' we get a relation between the couplings and the *vev* s. Obviously in this case due to large number of couplings of the field ξ' (which are independent parameters) this condition can be easily satisfied A more stringent and interesting situation is the case where an extra field χ is introduced instead of ξ'.

Introduction of χ=(2,2,6)

It has been pointed out by Pati[62] that the field χ is a very economical choice for the mechanism that leads to appreciable three lepton decay of proton. The field χ is contained in the field 54-plet of SO(10) which has to be present for the breaking of SO(10). The terms linear in χ can be written as:

$$V_\chi = P\ \eta\xi\chi(\Delta_R - \Delta_L) + M\ \chi\xi(\Delta_R + \Delta_L). \tag{4.78}$$

These terms upon minimization give the condition

$$v_L = \frac{P\eta_0 - M}{P\eta_0 + M} v_R. \tag{4.79}$$

This means that to get $v_R >> v_L$ one has to fine tune $P\eta_0 - M << P\eta_0 + M$. This is interesting in the context of the three lepton decay of proton which will be discussed elsewhere [63].

Introduction of δ=(3,3,0)

In this case we first write down the linear couplings of the field δ:

$$V_\delta = M_1\delta(\Delta_L\Delta_R^\dagger + \Delta_R\Delta_L^\dagger) + M_2\delta\phi\phi^\dagger + C_1\eta\delta(\Delta_L\Delta_R^\dagger + \Delta_R\Delta_L^\dagger) + C_2\eta\delta\phi^\dagger\phi \tag{4.80}$$

These terms upon minimization gives the following conditions,

$$v_L v_R = -\frac{M_2 + C_2\eta_0}{2M_1 + C_1\eta_0} k^2 \tag{4.81}$$

In the limit of very large η_0 we can write,

$$v_L v_R \simeq k^2 \tag{4.82}$$

If we compare this relation with the see-saw relation of eqn(4.77) we get,

$$\frac{v_R^2}{\eta_0^2} = \frac{k^2}{w\tilde{k}^2 + \beta k^2} \simeq O(1). \tag{4.83}$$

Thus due to the introduction of δ the left-right parity and the left right symmetry gets broken almost at the same scale.

4.3.5 Details of potential minimization

When the spontaneous symmetry breakdown (SSB) occurs the scalar fiields acquire *vev*. Let us first consider the case when we include only the fields ϕ, Δ_L and Δ_R. After the SSB, the potential looks like :

$$\begin{aligned} V_1 &= -\mu^2\,(v_L^2+v_R^2)+\frac{\rho}{4}\,(v_L^4+v_R^4)+\frac{\rho'}{4}\,(v_L^2v_R^2)+2v_Lv_R[(\gamma_{11} \\ &\quad +\gamma_{22})kk'+\gamma_{12}(k^2+k'^2)]+(v_L^2+v_R^2)\,[(\alpha_{11}+\alpha_{22}+\beta_{11})\,k^2 \\ &\quad +(\alpha_{11}+\alpha_{22}+\beta_{22})\,k'^2+(4\alpha_{12}+2\beta_{12})\,kk'] \\ &\quad +\text{terms containing k and k}'\text{ only} \end{aligned} \tag{4.84}$$

We have defined the new parameters as $\rho = 4(\rho_1+\rho_2)$ and $\rho' = 2\rho_3$. Minimisation with respect to v_L and v_R yields,

$$v_Lv_R = \frac{2\,[(\gamma_{11}+\gamma_{22})kk'+\gamma_{12}(k^2+k'^2)]}{\rho-\rho'} \tag{4.85}$$

This expression simplifies in the limit $k' << k$ to

$$v_Lv_R = \frac{2\,\gamma_{12}}{\rho-\rho'}k^2 \tag{4.86}$$

Here let us introduce the new scalar η which has a *vev* η_0. After the SSB, the scalar potential will be,

$$V_2 = V_1-\mu_\eta^2\,\eta_0^2+\beta_1\,\eta_1\,\eta_0^4+M\,\eta_0\,(v_L^2-v_R^2)+\beta_2\,\eta_0^2\,(v_L^2+v_R^2)+\gamma\,\eta_0^2\,(k^2+k'^2) \tag{4.87}$$

Now the minimization with respect to v_L and v_R gives the following relation in the limit $k' << k$,

$$v_Lv_R = \frac{2\,\gamma_{12}k^2}{[\rho-\rho'+\frac{4M\eta_0}{(v_L^2-v_R^2)}]} = \frac{\beta k^2}{[\rho-\rho'+\frac{4M\eta_0}{(v_L^2-v_R^2)}]} \tag{4.88}$$

We have defined the new parameter $\beta = 2\gamma_{12}$. At this stage let us introduce the scalar field ξ. This will again introduce new terms in the scalar potential. The scalar potential after SSB now becomes,

$$\begin{aligned} V_3 &= V_2+(v_L^2+v_R^2)\,[(a_{11}+a_{22}+b_{11})\,\tilde{k}^2+(a_{11}+a_{22}+b_{22})\,\tilde{k'}^2 \\ &\quad +(4a_{12}+b_{12})\,\tilde{k}\tilde{k'}]+2v_Lv_R[(c_{11}+c_{22})\tilde{k}\tilde{k'}+c_{12}(\tilde{k}^2+\tilde{k'}^2)] \\ &\quad +\text{terms containing }\tilde{\text{k}}\text{ and }\tilde{\text{k}'}\text{ only} \end{aligned} \tag{4.89}$$

Now we minimise V_3 with respect to v_L and v_R. The see-saw relation becomes,

$$v_Lv_R = \frac{\beta k^2+2c_{12}\,(\tilde{k}^2+\tilde{k'}^2)}{[\rho-\rho'+\frac{4M\eta_0}{(v_L^2-v_R^2)}]} \tag{4.90}$$

This relation in the limit $\tilde{k'} << \tilde{k}$ becomes,

$$v_Lv_R = \frac{\beta k^2+w\,\tilde{k}^2}{[\rho-\rho'+\frac{4M\eta_0}{(v_L^2-v_R^2)}]} \tag{4.91}$$

Here we have defined $w = 2c_{12}$. This is the see-saw condition in the presence of ξ .

4.3.6 Neutrino mass matrix

The fermions acquire masses through the Yukawa terms in the lagrangian when the higgs fields acquire *vev*. The Yukawa part in the Lagrangian written in terms of fermionic and higgs fields is given by,

$$\begin{aligned} L_{Yukawa} &= y_1(\bar{f}_L f_R \phi_1) + y_2(\bar{f}_L f_R \phi_2) + y_3(\bar{f}_L^c f_L \Delta_L + \bar{f}_R^c f_R \Delta_R) \\ &+ y_4(\bar{f}_L f_R \xi_1) + y_5(\bar{f}_L f_R \xi_2) \end{aligned} \tag{4.92}$$

where y_i (i=1,5) are Yukawa couplings. With this notation neutrino mass matrix written in the basis (ν_L, ν_L^c) is

$$M = \begin{pmatrix} m_{M_L} & m_D \\ m_D & m_{M_R} \end{pmatrix} \tag{4.93}$$

where m_{M_L} (m_{M_R}) is the left (right) handed Majorana mass term whereas m_D is the Dirac mass term. These terms can be related to the Yukawa couplings and *vevs* through the following relation,

$$\begin{aligned} m_{M_L} &= y_3 v_L \\ m_D &= (y_1 + y_2)(k + k') + (y_4 + y_5)(\tilde{k} + \tilde{k}') \\ m_{M_R} &= y_3 v_R \end{aligned} \tag{4.94}$$

Upon diagonalization of the mass matrix we obtain the mass eigenvalues. Now let us consider the simplifying assumption that all the Yukawa couplings are of order h and the *vev* s k' and $\tilde{k}'$ are much smaller than the *vev* s k and $\tilde{k}$ respectively. Under this assumption the eigenvalues become,

$$\begin{aligned} m_1 &= y_3 v_R \\ m_2 &= m_{M_L} - \frac{M_D^2}{m_{M_R}} = y_3 v_L - \frac{h^2(k^2 + \tilde{k}^2)}{y_3 v_R} \end{aligned}$$

We substitute for v_L from the see-saw condition to get in the D-parity conserving $g_L = g_R$ case,

$$m_2 = y_3 \frac{(\beta k^2 + w\tilde{k}^2)}{v_R} - \frac{h^2(k^2 + \tilde{k}^2)}{y_3 v_R} \tag{4.95}$$

We notice that the second term in the right hand side is suppressed by the square of the Yukawa coupling. Due to this the first term dominates. If we want to make the first term small compared to the second we need to fine tune the parameters. Hence one has to fine tune such that $\beta k^2 + w\tilde{k}^2 \simeq 0$ to get acceptable value of the the light neutrino mass. However in the presence of the *vev* of η we get,

$$m_2 = y_3 \frac{w\tilde{k}^2 + \beta k^2}{\eta_0^2} v_R - \frac{h^2(k^2 + \tilde{k}^2)}{y_3 v_R}. \tag{4.96}$$

In the limit of very large η_0 the first term drops out of the expression and one gets rid of the fine tuning problem. However, if the field δ (which does not acquire any *vev*) is present, we cannot get away with the fine tuning problem, since it is difficult to maintain $v_R << \eta_0$.

4.3.7 Conclusion

We have incorporated the scalar field ξ=(2,2,15) in the scalar potential of the $SU(4)_{color}$ left-right symmetric extension of the standard model. This field is necessary to predict correct mass relationships of the quarks and the leptons. After including the field ξ in the scalar potential we have carried out the minimization of potential, and worked out the relationship between the *vevs* of the left-handed and the right-handed triplets (see-saw relationship). We have shown that the field ξ is allowed by potential minimization and its inclusion does not change the qualitative nature of the see-saw relationship existing in literature. Once the see-saw relationship between the v_L and v_R is known we have gone ahead to construct the neutrino mass matrix. We have shown that even after the inclusion of the field ξ one needs to fine tune the parameters in the $g_L = g_R$ case to pŕedict small mass for the left handed neutrino, while in the $g_L \neq g_R$ case one naturally gets a large suppression for the left handed neitrino mass. This happens because even after the inclusion of the field ξ the light neutrino mass gets suppressed by the *vev* of the D-odd singlet η rather than the *vev* of Δ_R.

If there are new scalar fields which donot acquire any *vev*, then to check the consistency one has to write down their linear couplings with other fields and after minimizing the potential use the appropriate *vevs* of the various fields. In some cases the presence of such fields can give new interesting phenomenology. We studied some such cases for demonstration.

In recent past it has been shown that the three lepton decay of the proton can successfully explain the atmospheric neutrino anomaly by producing excess of electron type neutrino in the detector. To produce phenomenologically acceptable decay rate in the three lepton decay mode a mechanism was suggested by Pati, Salam and Sarkar, and later by Pati. In this mechanism one has to include extra scalars ξ'=(2,2,15) or χ=(2,2,6) which do not acquire *vevs*. We have calculated the linear couplings of such terms in the scalar potential and shown that these terms give relations that constrain the values of parameters and *vevs* of the model. In this work, we have given these constraints. We have also included, as a special case, the extra scalar δ=(3,3,0) and shown that its inclusion forces the right handed breaking scale and the D-parity breaking scale to become almost equal.