Chapter 7

Higher Order Traces and their Applications

7.1 Introduction

Embedded ensembles operating in many-particle spaces generate forms for distributions of various physical quantities with respect to energy and other quantum numbers; several examples for these are already discussed in Chapters 2-6. The separation of the energy evolution of various observables into a smoothed and a fluctuating part provides a basis for statistical spectroscopy. In statistical spectroscopy, methods are developed to determine various moments defining the distributions (predicted by EGEs) for the smoothed parts (valid in the chaotic region) without recourse to many-particle Hamiltonian construction. Parameters defining many of the important spectral distributions, generated by EGEs, involve traces of product of four (or even more) two-body (or one-body or a mixture of one and two-body) operators [Da-80, Ko-10]. For example, they are required for calculating nuclear structure matrix elements for β and $0\nu-\beta\beta$ decay and also for establishing Gaussian density of states generated by various extended two-body ensembles.

Propagation formulas for the moments $M_r = \langle H^r \rangle^m$, $r = 3, 4$ and also for traces over multi-orbit configurations for a given one plus two-body Hamiltonian $H = h(1) + V(2)$ follow from the results, derived using diagrammatic methods, given in [Wo-86, No-72, Ay-74, Po-75, Ch-78, Ka-95] many years back. These results extend to traces of product of four operators each of maximum body-rank 2. From now on, we refer to these traces as fourth order traces or averages. The propagation formulas derived using diagrammatic methods contain very large number of complicated terms (in particular for fourth order averages) and carrying out analytically ensemble averaging of all these terms is proved to be impractical (we are not aware if anyone was successful in the past). Some idea of the difficulty in carrying out simplifications can be seen from the attempt in [Pl-97]. Ensemble averages from trace propagation formulas is feasible for the second order moments and we have already presented examples for these in Chapters 2, 5 and 6. An alternative is to program the exact formulas and evaluate the moments numerically for each member of EGE's by considering say 500 members in two-particle spaces. However, as pointed out by Terán and Johnson [Te-06] in their most recent attempt in this direction, these calculations for the fourth order averages are time consuming if not impractical. All the problems with the exact formulas have been emphasized in [Ko-10]. Because of these (in future with much faster computers it may be possible to use the exact formulas), we have adopted the binary correlation approximation, first used by Mon and French [Mo-73, Mo-75] and later by French et al [Fr-88, To-86] for deriving formulas for ensemble averaged traces and they are good in the dilute limit. All the "basic" binary correlation results for averages over one orbit and two orbit configurations are available in literature and for easy reference, we discuss these in Appendix H. Extending the binary correlation approximation method for two different operators and for traces over two orbit configurations, we have addressed two applications: (i) derived formulas for the skewness γ_1 and excess γ_2 parameters for EGOE(1+2)-π ensemble in the dilute limit; and (ii) we have derived formula for the fourth order trace defining correlation coefficient and sixth order traces defining the fourth order cumulants of the bivariate transition strength density generated by the transition operator relevant for 0ν-$\beta\beta$ decay (also β decay). The results for (i) and (ii) are presented in Secs. 7.2 and 7.3. In addition, we have derived formulas for cumulants (they also involve fourth order traces) over m-particle spaces that enter into the expansions for the energy centroids and spectral variances, up to order $[J(J+1)]^2$, for EGOE(2)-J i.e., embedded Gaussian orthogonal ensemble generated by random two-body interactions with angular momentum J symmetry for fermions in a single-j shell. The expansions for fixed-J centroids and variances involve traces of powers of operators H and J^2. As H preserves J symmetry, we use exact methods to evaluate these traces. More specifically, we have derived trace propagation formulas for the bivariate moments $\langle H^P (J^2)^Q \rangle^m$, $P+Q \leq 4$ and the results are presented in Sec. 7.4. All the results in Secs. 7.2 and 7.4 are published in [Ma-11a] and [Ko-08], respectively.

7.2 Application to EGOE(1+2)-π: Formulas for Skewness and Excess Parameters

For the EGOE(1+2)-π Hamiltonian, we have $H = h(1) + V(2) = h(1) + X(2) + D(2)$ with $X(2) = A \oplus B \oplus C$ is the direct sum of the spreading matrices A, B and C and $D(2) = D + \widetilde{D}$ is the off-diagonal mixing matrix as defined in Chapter 5. Here, $\widetilde{D}$ is the transpose of the matrix D. The operator form for D is

$$D(2) = \sum_{\gamma,\delta} v_D^{\gamma\delta} \gamma_1^\dagger(2) \delta_2(2) \,, \tag{7.2.1}$$

with $\overline{[v_D^{\gamma\delta}]^2} = v_D^2$. Note that the operator form of $X(2)$ is given by Eq. (H33) and then $v_X^2(i,j) = \tau^2$ with $i+j=2$ and similarly, $v_D^2 = \alpha^2$; see Chapter 5 for further discussion on the (α,τ) parameters. Using this and the property that $h(1)$ conserves (m_1,m_2) symmetry and X preserves (m_1,m_2) symmetry, we apply the results in Appendix H and derive formulas for $M_r(m_1,m_2)$ with $r \leq 4$. These results are good in the dilute limit: $m_1, N_1, m_2, N_2 \to \infty$, $m/N_1 \to 0$ and $m/N_2 \to 0$ with $m = m_1$ or m_2. With the sp energies defining the mean field $h(1)$ as in Chapter 5, the first moment M_1 of the partial densities $\rho^{m_1,m_2}(E)$ is trivially,

$$M_1(m_1,m_2) = \overline{\langle (h+V) \rangle^{m_1,m_2}} = m_2 \,, \tag{7.2.2}$$

as $\langle h^r \rangle^{m_1,m_2} = (m_2)^r$ and $\overline{\langle V \rangle^{m_1,m_2}} = 0$. Applying the results in Appendix H in different ways, we derive formulas for the second, third and fourth order traces giving $M_r(m_1,m_2)$, $r = 2-4$. However, the presence of the mixing matrix D makes the application involved. The second moment M_2 is,

$$\begin{aligned} M_2(m_1,m_2) &= \overline{\left\langle (h+V)^2 \right\rangle^{m_1,m_2}} \\ &= \left\langle h^2 \right\rangle^{m_1,m_2} + \overline{\left\langle V^2 \right\rangle^{m_1,m_2}} = (m_2)^2 + \overline{\left\langle V^2 \right\rangle^{m_1,m_2}} \,; \end{aligned}$$

Table 7.1: Exact results for skewness and excess parameters for fixed-π eigenvalue densities $I_{\pm}(E)$ compared with the binary correlation results (in the table, called 'Approx'). For exact results, we have used the eigenvalues obtained from EGOE(1+2)-π ensembles with 100 members. The binary correlation results are obtained using Eqs. (7.2.2)-(7.2.17) and extension of Eq. (5.3.7). See text for details.

		$\gamma_1(m,\pi)$				$\gamma_2(m,\pi)$			
(N_+,N_-,m)	$(\tau,\alpha/\tau)$	Exact		Approx		Exact		Approx	
		$\pi=+$	$\pi=-$	$\pi=+$	$\pi=-$	$\pi=+$	$\pi=-$	$\pi=+$	$\pi=-$
(8,8,4)	(0.05,0.5)	0.01	0	0	0	−0.05	−0.99	−0.05	−1.00
	(0.05,1.0)	0.01	0	0	0	0.12	−1.08	0.13	−1.08
	(0.05,1.5)	0.01	0	0	0	0.33	−1.16	0.34	−1.17
	(0.1,0.5)	0	0	0	0	−0.84	−0.66	−0.84	−0.67
	(0.1,1.0)	0	0	0	0	−0.70	−0.79	−0.71	−0.79
	(0.1,1.5)	0	0	0	0	−0.51	−0.90	−0.51	−0.91
	(0.2,0.5)	0	0	0	0	−0.83	−0.74	−0.84	−0.75
	(0.2,1.0)	0	0	0	0	−0.84	−0.81	−0.84	−0.81
	(0.2,1.5)	0	0	0	0	−0.74	−0.87	−0.74	−0.87
	(0.3,1.0)	0	0	0	0	−0.85	−0.83	−0.85	−0.84

Table 7.1 – (continued)

(N_+, N_-, m)	$(\tau, \alpha/\tau)$	$\gamma_1(m,\pi)$				$\gamma_2(m,\pi)$			
		Exact		Approx		Exact		Approx	
		$\pi=+$	$\pi=-$	$\pi=+$	$\pi=-$	$\pi=+$	$\pi=-$	$\pi=+$	$\pi=-$
(8,8,5)	(0.05,0.5)	0.15	−0.15	0.15	−0.15	−0.52	−0.52	−0.52	−0.52
	(0.05,1.0)	0.16	−0.16	0.16	−0.16	−0.50	−0.50	−0.50	−0.50
	(0.05,1.5)	0.18	−0.17	0.18	−0.18	−0.46	−0.46	−0.46	−0.46
	(0.2,0.5)	−0.03	0.03	−0.03	0.03	−0.71	−0.71	−0.71	−0.71
	(0.2,1.0)	−0.01	0.01	−0.01	0.01	−0.73	−0.73	−0.74	−0.74
	(0.2,1.5)	0.02	−0.02	0.02	−0.02	−0.72	−0.72	−0.73	−0.73
(10,6,5)	(0.05,0.5)	−0.06	0.09	−0.07	0.09	−0.26	−0.76	−0.26	−0.75
	(0.05,1.5)	−0.04	0.15	−0.05	0.15	−0.01	−0.86	−0.01	−0.86
	(0.2,0.5)	0.01	−0.04	0.01	−0.04	−0.73	−0.69	−0.73	−0.69
	(0.2,1.5)	0.01	0.02	0.01	0.02	−0.69	−0.75	−0.70	−0.75
(6,10,5)	(0.05,0.5)	−0.09	0.07	−0.09	0.07	−0.76	−0.26	−0.75	−0.26
	(0.05,1.5)	−0.15	0.05	−0.15	0.05	−0.86	−0.01	−0.86	−0.01
	(0.2,0.5)	0.04	−0.01	0.04	−0.01	−0.68	−0.73	−0.69	−0.73
	(0.2,1.5)	−0.02	−0.01	−0.02	−0.01	−0.75	−0.69	−0.75	−0.70

$$\overline{\langle V^2\rangle^{m_1,m_2}} = \overline{\langle X^2\rangle^{m_1,m_2}} + \overline{\langle D\widetilde{D}\rangle^{m_1,m_2}} + \overline{\langle \widetilde{D}D\rangle^{m_1,m_2}}\,, \tag{7.2.3}$$

$$\overline{\langle X^2\rangle^{m_1,m_2}} = \tau^2 \sum_{i+j=2} T(m_1,N_1,i)\, T(m_2,N_2,j)\,,$$

$$\overline{\langle D\widetilde{D}\rangle^{m_1,m_2}} = \alpha^2 \binom{m_1}{2}\binom{\widetilde{m}_2}{2},\quad \overline{\langle \widetilde{D}D\rangle^{m_1,m_2}} = \alpha^2 \binom{\widetilde{m}_1}{2}\binom{m_2}{2}.$$

The second line in Eq. (7.2.3) follows by using the fact that $X(2)$ and $D(2)$ are independent and $D(2)$ can correlate only with $\widetilde{D}(2)$. In Eq. (7.2.3), the expression for $\overline{\langle X^2\rangle^{m_1,m_2}}$ follows directly from Eq. (H34). The last two equations in Eq. (7.2.3) can be derived using Eq. (7.2.1) giving the definition of the operator $D(2)$ and using Eqs. (H2) and (H3) appropriately to contract the operators $\gamma^\dagger$ with γ and δ with $\delta^\dagger$. For the $T(\cdots)$'s in Eq. (7.2.3), we use Eq. (H8). Note that, Eq. (7.2.3) gives the binary correlation formula for $\overline{\sigma^2(m_1,m_2)}$. Similarly, the third moment M_3 is

$$\begin{aligned}
M_3(m_1,m_2) &= \overline{\langle (h+V)^3\rangle^{m_1,m_2}} \\
&= \langle h^3\rangle^{m_1,m_2} + 2\langle h\rangle^{m_1,m_2}\,\overline{\langle V^2\rangle^{m_1,m_2}} + \overline{\langle XhX\rangle^{m_1,m_2}} \\
&+ \overline{\langle Dh\widetilde{D}\rangle^{m_1,m_2}} + \overline{\langle \widetilde{D}hD\rangle^{m_1,m_2}} \\
&= (m_2)^3 + 2\,m_2\,\overline{\langle V^2\rangle^{m_1,m_2}} + m_2\,\overline{\langle X^2\rangle^{m_1,m_2}} \\
&+ (m_2+2)\,\overline{\langle D\widetilde{D}\rangle^{m_1,m_2}} + (m_2-2)\,\overline{\langle \widetilde{D}D\rangle^{m_1,m_2}}\,.
\end{aligned} \tag{7.2.4}$$

In Eq. (7.2.4), the last three terms on the RHS are evaluated by using the following properties of the operators X, D and $\widetilde{D}$,

$$\begin{aligned}
& X(2)\,|m_1,m_2\rangle \rightarrow |m_1,m_2\rangle\,,\quad D(2)\,|m_1,m_2\rangle \rightarrow |m_1+2,m_2-2\rangle\,, \\
& \widetilde{D}(2)\,|m_1,m_2\rangle \rightarrow |m_1-2,m_2+2\rangle\,.
\end{aligned} \tag{7.2.5}$$

Also, the fixed-(m_1, m_2) averages involving X^2, V^2, $D\widetilde{D}$ and $\widetilde{D}D$ in Eq. (7.2.4) follow from Eq. (7.2.3). Now, the formula for the fourth moment M_4 is,

$$
\begin{aligned}
M_4(m_1,m_2) &= \overline{\left\langle (h+V)^4 \right\rangle^{m_1,m_2}} \\
&= \left\langle h^4 \right\rangle^{m_1,m_2} + 3\left\langle h^2 \right\rangle^{m_1,m_2} \overline{\left\langle V^2 \right\rangle^{m_1,m_2}} + \left\langle h^2 \right\rangle^{m_1,m_2} \overline{\left\langle X^2 \right\rangle^{m_1,m_2}} \\
&+ \overline{\left\langle D h^2 \widetilde{D} \right\rangle^{m_1,m_2}} + \overline{\left\langle \widetilde{D} h^2 D \right\rangle^{m_1,m_2}} + 2\,\overline{\left\langle hXhX \right\rangle^{m_1,m_2}} \\
&+ 2\,\overline{\left\langle hDh\widetilde{D} \right\rangle^{m_1,m_2}} + 2\,\overline{\left\langle h\widetilde{D}hD \right\rangle^{m_1,m_2}} + \overline{\left\langle V^4 \right\rangle^{m_1,m_2}} \\
&= (m_2)^4 + 3\,(m_2)^2\, \overline{\left\langle V^2 \right\rangle^{m_1,m_2}} + (m_2)^2\, \overline{\left\langle X^2 \right\rangle^{m_1,m_2}} \\
&+ (m_2+2)^2\, \overline{\left\langle D\widetilde{D} \right\rangle^{m_1,m_2}} + (m_2-2)^2\, \overline{\left\langle \widetilde{D}D \right\rangle^{m_1,m_2}} \\
&+ 2\,(m_2)^2\, \overline{\left\langle X^2 \right\rangle^{m_1,m_2}} + 2\, m_2(m_2+2)\, \overline{\left\langle D\widetilde{D} \right\rangle^{m_1,m_2}} \\
&+ 2\, m_2(m_2-2)\, \overline{\left\langle \widetilde{D}D \right\rangle^{m_1,m_2}} + \overline{\left\langle V^4 \right\rangle^{m_1,m_2}}\,.
\end{aligned}
\tag{7.2.6}
$$

The first term in Eq. (7.2.6) is trivial. The next two terms follow from Eq. (7.2.3). The terms $4-8$ in Eq. (7.2.6) are also simple and follow from Eq. (7.2.5). The expression for $\overline{\left\langle V^4 \right\rangle^{m_1,m_2}}$, which is non-trivial, is,

$$
\begin{aligned}
\overline{\left\langle V^4 \right\rangle^{m_1,m_2}} &= \overline{\left\langle X^4 \right\rangle^{m_1,m_2}} + 3\,\overline{\left\langle X^2 \right\rangle^{m_1,m_2}} \left\{ \overline{\left\langle D\widetilde{D} \right\rangle^{m_1,m_2}} + \overline{\left\langle \widetilde{D}D \right\rangle^{m_1,m_2}} \right\} \\
&+ \overline{\left\langle DX^2\widetilde{D} \right\rangle^{m_1,m_2}} + \overline{\left\langle \widetilde{D}X^2D \right\rangle^{m_1,m_2}} \\
&+ 2\,\overline{\left\langle XDX\widetilde{D} \right\rangle^{m_1,m_2}} + 2\,\overline{\left\langle X\widetilde{D}XD \right\rangle^{m_1,m_2}} + \overline{\left\langle (D+\widetilde{D})^4 \right\rangle^{m_1,m_2}}\,.
\end{aligned}
\tag{7.2.7}
$$

The formula for the first term in Eq. (7.2.7) follows from Eq. (H39),

$$
\begin{aligned}
\overline{\langle X^4\rangle^{m_1,m_2}} &= 2\left\{\overline{\langle X^2\rangle^{m_1,m_2}}\right\}^2 + T_1\,; \\
T_1 &= \tau^4 \sum_{i+j=2,\; t+u=2} F(m_1,N_1,i,t)\, F(m_2,N_2,j,u)\,.
\end{aligned}
\tag{7.2.8}
$$

Combining Eqs. (7.2.7) and (7.2.8), we have,

$$
\begin{aligned}
&\overline{\langle V^4\rangle^{m_1,m_2}} \\
&= 2\left\{\overline{\langle X^2\rangle^{m_1,m_2}}\right\}^2 + T_1 + 3\,\overline{\langle X^2\rangle^{m_1,m_2}}\left\{\overline{\langle D\widetilde{D}\rangle^{m_1,m_2}} + \overline{\langle \widetilde{D}D\rangle^{m_1,m_2}}\right\} \\
&+\left\{\overline{\langle DX^2\widetilde{D}\rangle^{m_1,m_2}} + \overline{\langle \widetilde{D}X^2D\rangle^{m_1,m_2}}\right\} \\
&+2\,\left\{\overline{\langle XDX\widetilde{D}\rangle^{m_1,m_2}} + \overline{\langle X\widetilde{D}XD\rangle^{m_1,m_2}}\right\} + \overline{\langle (D+\widetilde{D})^4\rangle^{m_1,m_2}} \\
&= 2\left\{\overline{\langle X^2\rangle^{m_1,m_2}}\right\}^2 + 3\,\overline{\langle X^2\rangle^{m_1,m_2}}\left\{\overline{\langle D\widetilde{D}\rangle^{m_1,m_2}} + \overline{\langle \widetilde{D}D\rangle^{m_1,m_2}}\right\} \\
&+T_1 + T_2 + 2\,T_3 + T_4\,.
\end{aligned}
\tag{7.2.9}
$$

To simplify the notations, we have introduced T_1, T_2, T_3 and T_4 in Eq. (7.2.9). The first and second terms in the RHS of the last step in Eq. (7.2.9) are completely determined by Eq. (7.2.3). Also, expression for T_1 is given in Eq. (7.2.8). Now, we will evaluate the terms T_2, T_3 and T_4. Firstly, using Eq. (7.2.5), we have

$$
\begin{aligned}
T_2 &= \overline{\langle DX^2\widetilde{D}\rangle^{m_1,m_2}} + \overline{\langle \widetilde{D}X^2D\rangle^{m_1,m_2}} \\
&= \left\{\overline{\langle D\widetilde{D}\rangle^{m_1,m_2}}\right\}\left\{\overline{\langle X^2\rangle^{m_1-2,m_2+2}}\right\} \\
&+ \left\{\overline{\langle \widetilde{D}D\rangle^{m_1,m_2}}\right\}\left\{\overline{\langle X^2\rangle^{m_1+2,m_2-2}}\right\}.
\end{aligned}
\tag{7.2.10}
$$

Formulas for the averages involving X^2, $D\widetilde{D}$ and $\widetilde{D}D$ in Eq. (7.2.10) are given by Eq. (7.2.3). Using Eqs. (H4) and (H5) appropriately to contract the operators D with $\widetilde{D}$

across operator X along with the expression for $\overline{\langle X^2\rangle^{m_1,m_2}}$ in Eq. (7.2.3), we have

$$
\begin{aligned}
T_3 &= \overline{\langle XDX\widetilde{D}\rangle^{m_1,m_2}} + \overline{\langle X\widetilde{D}XD\rangle^{m_1,m_2}} \\
&= \tau^2\,\alpha^2 \sum_{i+j=2} \left[\binom{m_1-i}{2}\binom{\widetilde{m}_2-j}{2} + \binom{\widetilde{m}_1-i}{2}\binom{m_2-j}{2} \right] \\
&\times\;\; T(m_1,N_1,i)\,T(m_2,N_2,j)\,.
\end{aligned}
\tag{7.2.11}
$$

Similarly, the expression for T_4 is as follows,

$$
\begin{aligned}
T_4 &= \overline{\langle (D+\widetilde{D})^4\rangle^{m_1,m_2}} \\
&= \overline{\langle D^2\widetilde{D}^2\rangle^{m_1,m_2}} + \overline{\langle \widetilde{D}^2D^2\rangle^{m_1,m_2}} + \overline{\langle D\widetilde{D}D\widetilde{D}\rangle^{m_1,m_2}} \\
&+ \overline{\langle \widetilde{D}D\widetilde{D}D\rangle^{m_1,m_2}} + \overline{\langle D\widetilde{D}^2D\rangle^{m_1,m_2}} + \overline{\langle \widetilde{D}D^2\widetilde{D}\rangle^{m_1,m_2}}\,.
\end{aligned}
\tag{7.2.12}
$$

As, in leading order, D can correlate only with $\widetilde{D}$, we have

$$
\begin{aligned}
&\overline{\langle D^2\widetilde{D}^2\rangle^{m_1,m_2}} = \overline{\langle DD\widetilde{D}\widetilde{D}\rangle^{m_1,m_2}} + \overline{\langle DD\widetilde{D}\widetilde{D}\rangle^{m_1,m_2}} \\
&= \alpha^4 \sum_{\gamma,\delta,\kappa,\eta} \left\langle \gamma_1^\dagger(2)\delta_2(2)\kappa_1^\dagger(2)\eta_2(2)\delta_2^\dagger(2)\gamma_1(2)\eta_2^\dagger(2)\kappa_1(2)\right\rangle^{m_1,m_2} \\
&+\alpha^4 \sum_{\gamma,\delta,\kappa,\eta} \left\langle \gamma_1^\dagger(2)\delta_2(2)\kappa_1^\dagger(2)\eta_2(2)\eta_2^\dagger(2)\kappa_1(2)\delta_2^\dagger(2)\gamma_1(2)\right\rangle^{m_1,m_2} \\
&= \alpha^4 \sum_{\gamma,\delta,\kappa,\eta} \left\langle \gamma_1^\dagger(2)\kappa_1^\dagger(2)\gamma_1(2)\kappa_1(2)\right\rangle^{m_1} \left\langle \delta_2(2)\eta_2(2)\delta_2^\dagger(2)\eta_2^\dagger(2)\right\rangle^{m_2} \\
&+\alpha^4 \sum_{\gamma,\delta,\kappa,\eta} \left\langle \gamma_1^\dagger(2)\kappa_1^\dagger(2)\kappa_1(2)\gamma_1(2)\right\rangle^{m_1} \left\langle \delta_2(2)\eta_2(2)\eta_2^\dagger(2)\delta_2^\dagger(2)\right\rangle^{m_2} \\
&= 2\,\alpha^4 \sum_{\gamma,\kappa} \left\langle \gamma_1^\dagger(2)\kappa_1^\dagger(2)\kappa_1(2)\gamma_1(2)\right\rangle^{m_1} \sum_{\delta,\eta} \left\langle \delta_2(2)\eta_2(2)\eta_2^\dagger(2)\delta_2^\dagger(2)\right\rangle^{m_2} \\
&= 2\,\overline{\langle D\widetilde{D}\rangle^{m_1,m_2}}\;\overline{\langle D\widetilde{D}\rangle^{m_1-2,m_2+2}}\,.
\end{aligned}
\tag{7.2.13}
$$

In order to obtain the last step in Eq. (7.2.13), the operators $\kappa^\dagger\kappa$ and $\gamma^\dagger\gamma$ are contracted using Eq. (H2) that gives $\binom{m_1-2}{2}$ and $\binom{m_1}{2}$ respectively. Similarly, contracting operators $\eta\eta^\dagger$ and $\delta\delta^\dagger$ using Eq. (H3) gives $\binom{\tilde{m}_2-2}{2}$ and $\binom{\tilde{m}_2}{2}$ respectively. Combining these gives the last step in Eq. (7.2.13). Note that the correlated pairs of operators are represented using same color in Eq. (7.2.13). Also, the third binary pattern $\overline{\langle DD\tilde{D}\tilde{D}\rangle^{m_1,m_2}}$ is not considered as it will be $1/N_1$ or $1/N_2$ order smaller compared to the other two binary patterns shown in Eq. (7.2.13). Similarly, we obtain

$$\begin{aligned}
\overline{\langle \tilde{D}^2 D^2\rangle^{m_1,m_2}} &= \overline{\langle \tilde{D}\tilde{D}DD\rangle^{m_1,m_2}} + \overline{\langle \tilde{D}\tilde{D}DD\rangle^{m_1,m_2}} \\
&= 2\,\overline{\langle \tilde{D}D\rangle^{m_1,m_2}}\;\overline{\langle \tilde{D}D\rangle^{m_1+2,m_2-2}}\,, \\
\overline{\langle D\tilde{D}D\tilde{D}\rangle^{m_1,m_2}} &= \overline{\langle D\tilde{D}D\tilde{D}\rangle^{m_1,m_2}} + \overline{\langle D\tilde{D}D\tilde{D}\rangle^{m_1,m_2}} \\
&= \left\{\overline{\langle D\tilde{D}\rangle^{m_1,m_2}}\right\}^2 + \overline{\langle D\tilde{D}\rangle^{m_1,m_2}}\;\overline{\langle \tilde{D}D\rangle^{m_1-2,m_2+2}}\,, \\
\overline{\langle D\tilde{D}\tilde{D}D\rangle^{m_1,m_2}} &= \overline{\langle D\tilde{D}\tilde{D}D\rangle^{m_1,m_2}} + \overline{\langle D\tilde{D}\tilde{D}D\rangle^{m_1,m_2}} \\
&= 2\,\overline{\langle D\tilde{D}\rangle^{m_1,m_2}}\;\overline{\langle \tilde{D}D\rangle^{m_1,m_2}}\,, \\
\overline{\langle \tilde{D}DD\tilde{D}\rangle^{m_1,m_2}} &= \overline{\langle \tilde{D}DD\tilde{D}\rangle^{m_1,m_2}} + \overline{\langle \tilde{D}DD\tilde{D}\rangle^{m_1,m_2}} \\
&= 2\,\overline{\langle D\tilde{D}\rangle^{m_1,m_2}}\;\overline{\langle \tilde{D}D\rangle^{m_1,m_2}}\,, \\
\overline{\langle \tilde{D}D\tilde{D}D\rangle^{m_1,m_2}} &= \overline{\langle \tilde{D}D\tilde{D}D\rangle^{m_1,m_2}} + \overline{\langle \tilde{D}D\tilde{D}D\rangle^{m_1,m_2}} \\
&= \left\{\overline{\langle \tilde{D}D\rangle^{m_1,m_2}}\right\}^2 + \overline{\langle \tilde{D}D\rangle^{m_1,m_2}}\;\overline{\langle D\tilde{D}\rangle^{m_1+2,m_2-2}}\,.
\end{aligned} \tag{7.2.14}$$

Combining Eqs. (7.2.12)-(7.2.14), we have

$$\begin{aligned} T_4 &= \left\{\overline{\langle D\tilde{D}\rangle^{m_1,m_2}}\right\}^2 + \left\{\overline{\langle \tilde{D}D\rangle^{m_1,m_2}}\right\}^2 \\ &+ \overline{\langle D\tilde{D}\rangle^{m_1,m_2}}\left[2\,\overline{\langle D\tilde{D}\rangle^{m_1-2,m_2+2}} + \overline{\langle \tilde{D}D\rangle^{m_1-2,m_2+2}}\right] \\ &+ \overline{\langle \tilde{D}D\rangle^{m_1,m_2}}\left[2\,\overline{\langle \tilde{D}D\rangle^{m_1+2,m_2-2}} + \overline{\langle D\tilde{D}\rangle^{m_1+2,m_2-2}}\right] \\ &+ 4\left\{\overline{\langle D\tilde{D}\rangle^{m_1,m_2}}\right\}\left\{\overline{\langle \tilde{D}D\rangle^{m_1,m_2}}\right\}. \end{aligned} \tag{7.2.15}$$

Therefore, combining Eqs. (7.2.6), (7.2.8), (7.2.9), (7.2.10), (7.2.11) and (7.2.15), the expression for the fourth moment is,

$$\begin{aligned} &M_4(m_1,m_2) = (m_2)^4 + 3\,(m_2)^2\,\overline{\langle V^2\rangle^{m_1,m_2}} + 3\,(m_2)^2\,\overline{\langle X^2\rangle^{m_1,m_2}} \\ &+(m_2+2)^2\,\overline{\langle D\tilde{D}\rangle^{m_1,m_2}} + (m_2-2)^2\,\overline{\langle \tilde{D}D\rangle^{m_1,m_2}} \\ &+2\,m_2(m_2+2)\,\overline{\langle D\tilde{D}\rangle^{m_1,m_2}} + 2\,m_2(m_2-2)\,\overline{\langle \tilde{D}D\rangle^{m_1,m_2}} + 2\left\{\overline{\langle X^2\rangle^{m_1,m_2}}\right\}^2 \\ &+3\,\overline{\langle X^2\rangle^{m_1,m_2}}\left\{\overline{\langle D\tilde{D}\rangle^{m_1,m_2}} + \overline{\langle \tilde{D}D\rangle^{m_1,m_2}}\right\} \\ &\quad +\tau^4 \sum_{i+j=2,\ t+u=2} F(m_1,N_1,i,t)\,F(m_2,N_2,j,u) \\ &\quad +\left\{\overline{\langle D\tilde{D}\rangle^{m_1,m_2}}\right\}\left\{\overline{\langle X^2\rangle^{m_1-2,m_2+2}}\right\} \\ &\quad +\left\{\overline{\langle \tilde{D}D\rangle^{m_1,m_2}}\right\}\left\{\overline{\langle X^2\rangle^{m_1+2,m_2-2}}\right\} \\ &\quad +2\,\tau^2\,\alpha^2 \sum_{i+j=2}\left[\binom{m_1-i}{2}\binom{\tilde{m}_2-j}{2} + \binom{\tilde{m}_1-i}{2}\binom{m_2-j}{2}\right] \\ &\quad \times T(m_1,N_1,i)\,T(m_2,N_2,j) \end{aligned} \tag{7.2.16}$$

$$+\left\{\overline{\langle D\widetilde{D}\rangle^{m_1,m_2}}\right\}^2+\left\{\overline{\langle\widetilde{D}D\rangle^{m_1,m_2}}\right\}^2$$

$$+\overline{\langle D\widetilde{D}\rangle^{m_1,m_2}}\left[2\,\overline{\langle D\widetilde{D}\rangle^{m_1-2,m_2+2}}+\overline{\langle\widetilde{D}D\rangle^{m_1-2,m_2+2}}\right]$$

$$+\overline{\langle\widetilde{D}D\rangle^{m_1,m_2}}\left[2\,\overline{\langle\widetilde{D}D\rangle^{m_1+2,m_2-2}}+\overline{\langle D\widetilde{D}\rangle^{m_1+2,m_2-2}}\right]$$

$$+4\left\{\overline{\langle D\widetilde{D}\rangle^{m_1,m_2}}\right\}\left\{\overline{\langle\widetilde{D}D\rangle^{m_1,m_2}}\right\}.$$

Equations (7.2.2), (7.2.3), (7.2.4), and (7.2.16), respectively give the first four non-central moments [$M_1(m_1,m_2)$, $M_2(m_1,m_2)$, $M_3(m_1,m_2)$ and $M_4(m_1,m_2)$]. In Eq. (7.2.16), we use Eq. (H8) for $T(\cdots)$'s and for $F(\cdots)$'s, we use Eq. (H14) and also Eq. (H23) in applications. The first four cumulants [$k_1(m_1,m_2)$, $k_2(m_1,m_2)$, $k_3(m_1,m_2)$, $k_4(m_1,m_2)$] can be calculated from these non-central moments using the formulas [St-87],

$$\begin{aligned}
k_1(m_1,m_2) &= M_1(m_1,m_2)\,,\quad k_2(m_1,m_2)=M_2(m_1,m_2)-M_1^2(m_1,m_2)\,,\\
k_3(m_1,m_2) &= M_3(m_1,m_2)-3\,M_2(m_1,m_2)\,M_1(m_1,m_2)+2\,M_1^3(m_1,m_2)\,,\\
k_4(m_1,m_2) &= M_4(m_1,m_2)-4\,M_3(m_1,m_2)\,M_1(m_1,m_2)-3\,M_2^2(m_1,m_2)\\
&+\ 12\,M_2(m_1,m_2)\,M_1^2(m_1,m_2)-6\,M_1^4(m_1,m_2)\,.
\end{aligned} \tag{7.2.17}$$

Then, the skewness and excess parameters are,

$$\gamma_1(m_1,m_2)=\frac{k_3(m_1,m_2)}{[k_2(m_1,m_2)]^{3/2}}\,,\quad \gamma_2(m_1,m_2)=\frac{k_4(m_1,m_2)}{[k_2(m_1,m_2)]^2}\,. \tag{7.2.18}$$

After carrying out the simplifications using Eqs. (7.2.2), (7.2.3), (7.2.4), (7.2.16) and (7.2.17), it is easily seen that,

$$\gamma_1(m_1,m_2)=\frac{2\left[\overline{\langle D\widetilde{D}\rangle^{m_1,m_2}}-\overline{\langle\widetilde{D}D\rangle^{m_1,m_2}}\right]}{\left\{\overline{\langle D\widetilde{D}\rangle^{m_1,m_2}}+\overline{\langle\widetilde{D}D\rangle^{m_1,m_2}}+\overline{\langle X^2\rangle^{m_1,m_2}}\right\}^{3/2}}\,. \tag{7.2.19}$$

Thus, γ_1 will be non-zero only when $\alpha\neq 0$ and the τ dependence appears only in the denominator. Also, it is seen that for $N_+=N_-$, $\gamma_1(m_1,m_2)=-\gamma_1(m_2,m_1)$. The expression for γ_2 is more cumbersome. Denoting $\mathscr{D}=\overline{\langle D\widetilde{D}\rangle^{m_1,m_2}}$, $\widetilde{\mathscr{D}}=\overline{\langle\widetilde{D}D\rangle^{m_1,m_2}}$

and $\mathscr{X} = \overline{\langle X^2 \rangle^{m_1,m_2}}$ for brevity, we have

$$\gamma_2(m_1,m_2)+1 = \frac{T_1+T_2+2\,T_3+T_4+\left(\widetilde{\mathscr{D}}+\mathscr{D}\right)(4-\mathscr{X})-2\,\left(\widetilde{\mathscr{D}}+\mathscr{D}\right)^2}{\left\{\widetilde{\mathscr{D}}+\mathscr{D}+\mathscr{X}\right\}^2}\,. \tag{7.2.20}$$

The formulas for T's, $\mathscr{D}$, $\widetilde{\mathscr{D}}$ and $\mathscr{X}$ given before together with Eq. (7.2.20) show that, for $N_+ = N_-$, $\gamma_2(m_1,m_2) = \gamma_2(m_2,m_1)$. With, $T_1 \sim \mathscr{X}^2 + C_1$, $T_2 = T_3 \sim \mathscr{X}(\widetilde{\mathscr{D}}+\mathscr{D})$ and $T_4 \sim 3(\widetilde{\mathscr{D}}+\mathscr{D})^2 + C_2$ which are good in the dilute limit ($|C1/T_1|$ and $|C_2/T_4|$ will be close to zero), we have

$$\gamma_2(m_1,m_2) = \frac{C_1+C_2+4\,(\widetilde{\mathscr{D}}+\mathscr{D})}{\left\{\widetilde{\mathscr{D}}+\mathscr{D}+\mathscr{X}\right\}^2}\,. \tag{7.2.21}$$

Note that C_1 and $\mathscr{X}$ depend only on τ. Similarly, C_2 and $(\widetilde{\mathscr{D}},\mathscr{D})$ depend only on α. The $(\widetilde{\mathscr{D}}+\mathscr{D})$ term in the numerator will contribute to $\gamma_2(m_1,m_2)$ when $\tau = 0$ and α is very small. The approximation $T_2 = T_3 \sim \mathscr{X}(\widetilde{\mathscr{D}}+\mathscr{D})$ is crucial in obtaining the numerator in Eq. (7.2.21) with no cross-terms involving the α and τ parameters. With this, we have k_4 to be the sum of k_4's coming from $X(2)$ and $D(2)$ matrices [note that, as mentioned before, $X(2) = A \oplus B \oplus C$ and $D(2) = D + \widetilde{D}$].

To test the accuracy of the formulas for M_r given by Eqs. (7.2.2), (7.2.3), (7.2.4) and (7.2.16), the binary correlation results for $\gamma_1(m,\pm)$ and $\gamma_2(m,\pm)$ are compared with exact results obtained using the eigenvalues from EGOE(1+2)-π ensembles with 100 members for several values of (N_+, N_-, m) and (τ, α) parameters in Table 7.1. Extension of Eq. (5.3.7) along with the results derived for $M_r(m_1,m_2)$ will give the binary correlation results for $\gamma_1(m,\pm)$ and $\gamma_2(m,\pm)$. It is clearly seen from the results in the Table that in all the examples considered, the binary correlation results are quite close to the exact results. Similar agreements are also seen in many other examples which are not shown in the table.

7.3 Application to $\beta\beta$ Decay: Formulas for the Bivariate Correlation Coefficient and Fourth Order Cumulants for the Transition Strength Density

7.3.1 Transition matrix elements and bivariate strength densities

Given a transition operator $\mathcal{O}$, the transition matrix elements are given by $|\langle f \mid \mathcal{O} \mid i\rangle|^2$, with i and f being the initial and final states. These are also generally called transition strengths. Operation of EGEs in many-particle spaces will lead to a theory for the smoothed part of transition strengths and the fluctuations in the locally renormalized strengths follow Porter-Thomas form for systems in the chaotic region. The transition matrix elements are needed in many applications. Examples are one-particle transfer [Po-91], E2 and M1 transition strengths in nuclei [Ha-82], dipole strengths in atoms [Fl-98], beta-decay [Ma-07], giant dipole resonances [Ma-98] and problems involving time-reversal non-invariance and parity [Fr-88, To-00]. Here, our focus is on $0\nu-\beta\beta$ decay. Half-life for 0ν double-beta decay (NDBD), for the 0_i^+ gs of a initial even-even nucleus decay to the 0_f^+ gs of the final even-even nucleus, with a few approximations, is given by [El-02]

$$
\begin{aligned}
\left[T_{1/2}^{0\nu}(0_i^+ \to 0_f^+)\right]^{-1} &= G^{0\nu}\left|M^{0\nu}\right|^2 \frac{\langle m_\nu\rangle^2}{m_e^2}, \\
M^{0\nu} &= M_{GT}^{0\nu} - \frac{g_V^2}{g_A^2} M_F^{0\nu} = \left\langle 0_f^+ \,||\, \mathcal{O}(2:0\nu) \,||\, 0_i^+ \right\rangle,
\end{aligned}
\tag{7.3.1}
$$

where $\langle m_\nu\rangle$ is effective neutrino mass and the $G^{0\nu}$ is an accurately calculable phase space integral [Bo-92, Do-93]. Similarly g_A and g_V are the weak axial-vector and vector coupling constants ($g_A/g_V = 1$ to 1.254). The nuclear matrix elements M_{GT} and M_F are matrix elements of Gamow-Teller and Fermi like two-body operators respectively. Forms for them will follow from the closure approximation which is well justified for NDBD [El-02]. As seen from Eq. (7.3.1), the NDBD half-lives are generated by the two-body transition operator $\mathcal{O}(2:0\nu)$. An experimental value of (bound on) $T_{1/2}^{0\nu}$ will give a value for (bound on) neutrino mass via Eq. (7.3.1) provided we know the value of $|M^{0\nu}|^2$ generated by the NDBD two-body transition operator $\mathcal{O}(2:0\nu)$, connecting the ground states of the initial and final even-even nuclei involved.

Transition strengths multiplied by the eigenvalue densities at the two energies involved define the transition strength densities. With EGOE(1+2) operating in the Gaussian domain, it was established in the past that transition strength densities follow close to bivariate Gaussian form for spinless fermion systems and for operators that preserve particle number with the additional assumption that the transition operator and the Hamiltonian operator can be represented by independent EGOEs. With extensions of these results (without a EGOE basis), the bivariate Gaussian form is used in practical applications. Our purpose is to establish that for the $0\nu-\beta\beta$ decay (also for β decay), transition strength densities are close to bivariate Gaussian form and also to derive a formula for the bivariate correlation coefficient. We will address these two important questions so that the EGOE results can be applied to formulate a theory for calculating 0ν-$\beta\beta$ transition matrix elements [Ko-08a]. With space #1 denoting protons and similarly space #2 neutrons, the general form of the transition operator $\mathcal{O}$ is,

$$\mathcal{O}(k_{\mathcal{O}}) = \sum_{\gamma,\delta} v_{\mathcal{O}}^{\gamma\delta}(k_{\mathcal{O}})\, \gamma_1^\dagger(k_{\mathcal{O}})\delta_2(k_{\mathcal{O}})\,; \quad k_{\mathcal{O}} = 2 \text{ for NDBD}\,. \tag{7.3.2}$$

Therefore, in order to derive the form for the transition strength densities generated by $\mathcal{O}$, it is necessary to deal with two-orbit configurations denoted by (m_1, m_2), where m_1 is the number of particles in the first orbit (protons) and m_2 in the second orbit (neutrons). Now, the transition strength density $I_{\mathcal{O}}(E_i, E_f)$ is

$$\begin{array}{l} I_{\mathcal{O}}^{(m_1^f,m_2^f),(m_1^i,m_2^i)}(E_i,E_f) \\ = I^{(m_1^f,m_2^f)}(E_f)\left|\left\langle (m_1^f,m_2^f)E_f \mid \mathcal{O} \mid (m_1^i,m_2^i)E_i\right\rangle\right|^2 I^{(m_1^i,m_2^i)}(E_i)\,, \end{array} \tag{7.3.3}$$

and the corresponding bivariate moments are

$$\widetilde{M}_{PQ}(m_1^i, m_2^i) = \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) H^Q(k_H) \mathcal{O}(k_{\mathcal{O}}) H^P(k_H)\rangle}^{m_1^i,m_2^i}\,. \tag{7.3.4}$$

Note that $\widetilde{M}$ are in general non-central and non-normalized moments. The general form of the operator $H(k_H)$ is given by Eq. (H33) and it preserves (m_1^i, m_2^i)'s. However, $\mathcal{O}$ and its hermitian conjugate $\mathcal{O}^\dagger$ do not preserve (m_1, m_2) i.e., $\mathcal{O}(k_\mathcal{O})\,|m_1, m_2\rangle = |m_1 + k_\mathcal{O}, m_2 - k_\mathcal{O}\rangle$ and $\mathcal{O}^\dagger(k_\mathcal{O})\,|m_1, m_2\rangle = |m_1 - k_\mathcal{O}, m_2 + k_\mathcal{O}\rangle$. Thus, given a (m_1^i, m_2^i) for an initial state, the (m_1^f, m_2^f) for the final state generated by the action of $\mathcal{O}$ is uniquely defined and therefore, in the bivariate moments defined in Eq. (7.3.4), only the initial (m_1^i, m_2^i) is specified. For completeness, let us mention that given the marginal centroids (ϵ_i, ϵ_f), widths (σ_i, σ_f) and the bivariate correlation coefficient ζ_{biv}, the normalized bivariate Gaussian is defined by,

$$
\begin{aligned}
&\rho_{\text{biv-}\mathcal{G};\mathcal{O}}(E_i, E_f) = \rho_{\text{biv-}\mathcal{G};\mathcal{O}}(E_i, E_f; \epsilon_i, \epsilon_f, \sigma_i, \sigma_f, \zeta_{biv}) \\
&= \frac{1}{2\pi\sigma_i\sigma_f\sqrt{(1-\zeta_{biv}^2)}} \\
&\times \exp -\frac{1}{2(1-\zeta_{biv}^2)}\left\{\left(\frac{E_i-\epsilon_i}{\sigma_i}\right)^2 - 2\zeta_{biv}\left(\frac{E_i-\epsilon_i}{\sigma_i}\right)\left(\frac{E_f-\epsilon_f}{\sigma_f}\right) + \left(\frac{E_f-\epsilon_f}{\sigma_f}\right)^2\right\}.
\end{aligned}
\tag{7.3.5}
$$

7.3.2 Formulas for the bivariate moments

Using binary correlation approximation, we derive formulas for the first four moments $\widetilde{M}_{PQ}(m_1^i, m_2^i)$, $P+Q \le 4$ of $I_\mathcal{O}^{(m_1^f, m_2^f),(m_1^i, m_2^i)}(E_i, E_f)$ for any $k_\mathcal{O}$ by representing $H(k_H)$ and $\mathcal{O}(k_\mathcal{O})$ operators by independent EGOEs and assuming $H(k_H)$ is a k_H-body operator preserving (m_1, m_2)'s. Note that the ensemble averaged k_H-particle matrix elements of $H(k_H)$ are $v_H^2(i,j)$ with $i + j = k_H$ [see Eq. (H33)] and similarly the ensemble average of $(v_\mathcal{O}^{\gamma\delta})^2$ is $v_\mathcal{O}^2$. From now on, we use $(m_1^i, m_2^i) = (m_1, m_2)$. Using Eq. (7.3.2) and applying the basic rules given by Eqs. (H2) and (H3), we have

$$
\begin{aligned}
\widetilde{M}_{00}(m_1, m_2) &= \overline{\langle \mathcal{O}^\dagger(k_\mathcal{O})\mathcal{O}(k_\mathcal{O})\rangle^{m_1, m_2}} \\
&= \sum_{\gamma,\delta} \overline{\left\{v_\mathcal{O}^{\gamma\delta}\right\}^2} \left\langle \delta_2^\dagger(k_\mathcal{O})\gamma_1(k_\mathcal{O})\gamma_1^\dagger(k_\mathcal{O})\delta_2(k_\mathcal{O})\right\rangle^{m_1, m_2} \\
&= v_\mathcal{O}^2 \binom{\widetilde{m}_1}{k_\mathcal{O}}\binom{m_2}{k_\mathcal{O}}.
\end{aligned}
\tag{7.3.6}
$$

Trivially, $\widetilde{M}_{10}(m_1,m_2)$ and $\widetilde{M}_{01}(m_1,m_2)$ will be zero as $H(k_H)$ is represented by EGOE(k_H). Thus, $\widetilde{M}_{PQ}(m_1,m_2)$ are central moments. Moreover, by definition, all the odd-order moments, i.e., $\widetilde{M}_{PQ}(m_1,m_2)$ with $\mod(P+Q,2) \neq 0$, will be zero. Now, the $\widetilde{M}_{11}$ is given by,

$$
\begin{aligned}
\widetilde{M}_{11}(m_1,m_2) &= \overline{\left\langle \mathcal{O}^\dagger(k_\mathcal{O})H(k_H)\mathcal{O}(k_\mathcal{O})H(k_H)\right\rangle^{m_1,m_2}} \\
&= v_\mathcal{O}^2 \sum_{\alpha_1,\beta_1,\alpha_2,\beta_2,\gamma_1,\delta_2;\, i+j=k_H} v_H^2(i,j) \\
&\times \left\langle \gamma_1^\dagger(k_\mathcal{O})\alpha_1(i)\beta_1^\dagger(i)\gamma_1(k_\mathcal{O})\beta_1(i)\alpha_1^\dagger(i)\right\rangle^{m_1} \\
&\times \left\langle \delta_2(k_\mathcal{O})\alpha_2(j)\beta_2^\dagger(j)\delta_2^\dagger(k_\mathcal{O})\beta_2(j)\alpha_2^\dagger(j)\right\rangle^{m_2} .
\end{aligned}
\tag{7.3.7}
$$

Then, contracting over the $\gamma^\dagger\gamma$ and $\delta\delta^\dagger$ operators, respectively in the first and second traces in Eq. (7.3.7) using Eqs. (H4) and (H5) appropriately, we have

$$
\begin{aligned}
\widetilde{M}_{11}(m_1,m_2) &= v_\mathcal{O}^2 \sum_{i+j=k_H} v_H^2(i,j) \binom{\widetilde{m}_1-i}{k_\mathcal{O}}\binom{m_2-j}{k_\mathcal{O}} \\
&\times\; T(m_1,N_1,i)\; T(m_2,N_2,j) .
\end{aligned}
\tag{7.3.8}
$$

Note that the formulas for the functions $T(\cdots)$'s appearing in Eq. (7.3.8) are given by Eqs. (H8), (H9) and (H10). Similarly, the functions $F(\cdots)$'s appearing ahead are given by Eqs. (H14) and (H23). For the marginal variances, we have

$$
\begin{aligned}
\widetilde{M}_{20}(m_1,m_2) &= \overline{\left\langle \mathcal{O}^\dagger(k_\mathcal{O})\mathcal{O}(k_\mathcal{O})H^2(k_H)\right\rangle^{m_1,m_2}} \\
&= \widetilde{M}_{00}(m_1,m_2)\,\overline{\left\langle H^2(k_H)\right\rangle^{m_1,m_2}} , \\
\widetilde{M}_{02}(m_1,m_2) &= \overline{\left\langle \mathcal{O}^\dagger(k_\mathcal{O})H^2(k_H)\mathcal{O}(k_\mathcal{O})\right\rangle^{m_1,m_2}} \\
&= \widetilde{M}_{00}(m_1,m_2)\,\overline{\left\langle H^2(k_H)\right\rangle^{m_1+k_\mathcal{O},m_2-k_\mathcal{O}}} .
\end{aligned}
\tag{7.3.9}
$$

In Eq. (7.3.9), the ensemble averages of $H^2(k_H)$ are given by Eq. (H34). Now, the correlation coefficient ζ_{biv} is

$$\zeta_{biv}(m_1, m_2) = \frac{\widetilde{M}_{11}(m_1, m_2)}{\sqrt{\widetilde{M}_{20}(m_1, m_2)\,\widetilde{M}_{02}(m_1, m_2)}} . \tag{7.3.10}$$

Clearly, ζ_{biv} will be independent of $v_{\mathscr{O}}^2$.

Proceeding further, we derive formulas for the fourth order moments $\widetilde{M}_{PQ}$, $P + Q = 4$. The results are as follows. Firstly, for $(PQ) = (40)$ and (04), we have

$$\begin{aligned}
\widetilde{M}_{40}(m_1, m_2) &= \overline{\langle \mathscr{O}^\dagger(k_{\mathscr{O}})\mathscr{O}(k_{\mathscr{O}})H^4(k_H)\rangle}^{m_1,m_2} \\
&= \widetilde{M}_{00}(m_1, m_2)\,\overline{\langle H^4(k_H)\rangle}^{m_1,m_2} , \\
\widetilde{M}_{04}(m_1, m_2) &= \overline{\langle \mathscr{O}^\dagger(k_{\mathscr{O}})H^4(k_H)\mathscr{O}(k_{\mathscr{O}})\rangle}^{m_1,m_2} \\
&= \widetilde{M}_{00}(m_1, m_2)\,\overline{\langle H^4(k_H)\rangle}^{m_1+k_{\mathscr{O}},m_2-k_{\mathscr{O}}} .
\end{aligned} \tag{7.3.11}$$

In Eq. (7.3.11), the ensemble averages of $H^4(k_H)$ are given by Eq. (H39). For $(PQ) = (31)$, we have

$$\begin{aligned}
\widetilde{M}_{31}(m_1, m_2) &= \overline{\langle \mathscr{O}^\dagger(k_{\mathscr{O}})H(k_H)\mathscr{O}(k_{\mathscr{O}})H^3(k_H)\rangle}^{m_1,m_2} \\
&= \overline{\langle \mathscr{O}^\dagger(k_{\mathscr{O}})H(k_H)\mathscr{O}(k_{\mathscr{O}})H(k_H)H(k_H)H(k_H)\rangle}^{m_1,m_2} \\
&+ \overline{\langle \mathscr{O}^\dagger(k_{\mathscr{O}})H(k_H)\mathscr{O}(k_{\mathscr{O}})H(k_H)H(k_H)H(k_H)\rangle}^{m_1,m_2} \\
&+ \overline{\langle \mathscr{O}^\dagger(k_{\mathscr{O}})H(k_H)\mathscr{O}(k_{\mathscr{O}})H(k_H)H(k_H)H(k_H)\rangle}^{m_1,m_2} .
\end{aligned} \tag{7.3.12}$$

Note that in Eq. (7.3.12), we use the same color for the binary correlated pairs of operators. First and last terms on RHS of Eq. (7.3.12) are simple and this gives,

$$\widetilde{M}_{31}(m_1,m_2) = 2\,\overline{\langle H^2(k_H)\rangle}^{m_1,m_2}\,\widetilde{M}_{11}(m_1,m_2)$$

$$+\overline{\langle \mathscr{O}^\dagger(k_\mathscr{O}) H(k_H) \mathscr{O}(k_\mathscr{O}) H(k_H) H(k_H) H(k_H)\rangle}^{m_1,m_2}$$

$$= 2\,\overline{\langle H^2(k_H)\rangle}^{m_1,m_2}\,\widetilde{M}_{11}(m_1,m_2) + v_\mathscr{O}^2 \sum_{i+j=k_H, t+u=k_H} v_H^2(i,j)\; v_H^2(t,u) \tag{7.3.13}$$

$$\times \binom{m_2-j}{k_\mathscr{O}} \binom{\widetilde{m}_1-i}{k_\mathscr{O}} F(m_1,N_1,i,t)\; F(m_2,N_2,j,u)\,.$$

Similarly, we have

$$\widetilde{M}_{13}(m_1,m_2) = \overline{\langle \mathscr{O}^\dagger(k_\mathscr{O}) H^3(k_H) \mathscr{O}(k_\mathscr{O}) H(k_H)\rangle}^{m_1,m_2}$$

$$= \overline{\langle \mathscr{O}^\dagger(k_\mathscr{O}) H(k_H) H(k_H) H(k_H) \mathscr{O}(k_\mathscr{O}) H(k_H)\rangle}^{m_1,m_2}$$

$$+\overline{\langle \mathscr{O}^\dagger(k_\mathscr{O}) H(k_H) H(k_H) H(k_H) \mathscr{O}(k_\mathscr{O}) H(k_H)\rangle}^{m_1,m_2}$$

$$+\overline{\langle \mathscr{O}^\dagger(k_\mathscr{O}) H(k_H) H(k_H) H(k_H) \mathscr{O}(k_\mathscr{O}) H(k_H)\rangle}^{m_1,m_2}$$

$$= 2\,\overline{\langle H^2(k_H)\rangle}^{m_1+k_\mathscr{O},m_2-k_\mathscr{O}}\,\widetilde{M}_{11}(m_1,m_2) \tag{7.3.14}$$

$$+v_\mathscr{O}^2 \sum_{i+j=k_H, t+u=k_H} v_H^2(i,j)\; v_H^2(t,u)\; G(t,u)$$

$$\times \binom{\widetilde{m}_1-k_\mathscr{O}-t+i}{i} \binom{m_1+k_\mathscr{O}-t}{i} \binom{\widetilde{m}_2-u+k_\mathscr{O}+j}{j} \binom{m_2-k_\mathscr{O}-u}{j};$$

$$G(t,u) = \binom{\widetilde{m}_1-t}{k_\mathscr{O}} \binom{m_2-u}{k_\mathscr{O}} T(m_1,N_1,t)\; T(m_2,N_2,u)\,.$$

Finally, $\widetilde{M}_{22}(m_1, m_2)$ is given by,

$$
\begin{aligned}
\widetilde{M}_{22}(m_1,m_2) &= \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) H^2(k_H) \mathcal{O}(k_{\mathcal{O}}) H^2(k_H)\rangle}^{m_1,m_2} \\
&= \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) H(k_H) H(k_H) \mathcal{O}(k_{\mathcal{O}}) H(k_H) H(k_H)\rangle}^{m_1,m_2} \\
&+ \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) H(k_H) H(k_H) \mathcal{O}(k_{\mathcal{O}}) H(k_H) H(k_H)\rangle}^{m_1,m_2} \\
&+ \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) H(k_H) H(k_H) \mathcal{O}(k_{\mathcal{O}}) H(k_H) H(k_H)\rangle}^{m_1,m_2} \\
&= \widetilde{M}_{00}(m_1,m_2)\, \overline{\langle H^2(k_H)\rangle}^{m_1+k_{\mathcal{O}}, m_2-k_{\mathcal{O}}}\; \overline{\langle H^2(k_H)\rangle}^{m_1,m_2} \\
&+ v_{\mathcal{O}}^2 \sum_{i+j=k_H, t+u=k_H} v_H^2(i,j)\, v_H^2(t,u) \binom{\widetilde{m}_1 - i - t}{k_{\mathcal{O}}} \binom{m_2 - u - j}{k_{\mathcal{O}}} \\
&\times \left[F(m_1,N_1,i,t)\, F(m_2,N_2,j,u) \right. \\
&\left. + T(m_1,N_1,i)\, T(m_1,N_1,t)\, T(m_2,N_2,j)\, T(m_2,N_2,u) \right] .
\end{aligned}
\tag{7.3.15}
$$

Given the $\widetilde{M}_{PQ}(m_1, m_2)$, the normalized central moments M_{PQ} are $M_{PQ} = \widetilde{M}_{PQ}/\widetilde{M}_{00}$.

7.3.3 Numerical results for bivariate correlation coefficient and fourth order cumulants

Firstly, the scaled moments $\widehat{M}_{PQ}$ are

$$
\widehat{M}_{PQ} = \frac{M_{PQ}(m_1,m_2)}{[M_{20}(m_1,m_2)]^{P/2}\,[M_{02}(m_1,m_2)]^{Q/2}}\,; \quad P+Q \geq 2\,. \tag{7.3.16}
$$

Now the fourth order cumulants are [St-87],

$$
\begin{aligned}
& k_{40}(m_1,m_2) = \widehat{M}_{40}(m_1,m_2) - 3\,, k_{04}(m_1,m_2) = \widehat{M}_{04}(m_1,m_2) - 3\,, \\
& k_{31}(m_1,m_2) = \widehat{M}_{31}(m_1,m_2) - 3\,\widehat{M}_{11}(m_1,m_2)\,, \\
& k_{13}(m_1,m_2) = \widehat{M}_{13}(m_1,m_2) - 3\,\widehat{M}_{11}(m_1,m_2)\,, \\
& k_{22}(m_1,m_2) = \widehat{M}_{22}(m_1,m_2) - 2\,\widehat{M}_{11}^2(m_1,m_2) - 1\,.
\end{aligned}
\tag{7.3.17}
$$

Table 7.2: Correlation coefficients $\zeta_{biv}(m_1, m_2)$ for some nuclei with $k_{\mathcal{O}} = 2$ as appropriate for $0\nu - \beta\beta$ decay operator. Note that space #1 is for protons and space #2 for neutrons. The configuration spaces corresponding to N_1 or $N_2 = 20, 22, 30, 32, 44$ and 58 are $r_3 f$, $r_3 g$, $r_4 g$, $r_4 h$, $r_5 i$, and $r_6 j$, respectively with $f = {}^1f_{7/2}$, $g = {}^1g_{9/2}$, $h = {}^1h_{11/2}$, $i = {}^1i_{13/2}$, $j = {}^1j_{15/2}$, $r_3 = {}^1f_{5/2}\ {}^2p_{3/2}\ {}^2p_{1/2}$, $r_4 = {}^1g_{7/2}\ {}^2d_{5/2}\ {}^2d_{3/2}\ {}^3s_{1/2}$, $r_5 = {}^1h_{9/2}\ {}^2f_{7/2}\ {}^2f_{5/2}\ {}^3p_{3/2}\ {}^3p_{1/2}$ and $r_6 = {}^1i_{11/2}\ {}^2g_{9/2}\ {}^2g_{7/2}\ {}^3d_{5/2}\ {}^3d_{3/2}\ {}^4s_{1/2}$. See text for details.

Nuclei	N_1	m_1	N_2	m_2	$\zeta_{biv}(m_1, m_2)$
${}^{76}_{32}Ge_{44}$	22	4	22	16	0.64
${}^{82}_{34}Se_{48}$	22	6	22	20	0.6
${}^{100}_{42}Mo_{58}$	30	2	32	8	0.57
${}^{128}_{52}Te_{76}$	32	2	32	26	0.62
${}^{130}_{52}Te_{78}$	32	2	32	28	0.58
${}^{150}_{60}Nd_{90}$	32	10	44	8	0.72
${}^{154}_{62}Sm_{92}$	32	12	44	10	0.76
${}^{180}_{74}W_{106}$	32	24	44	24	0.77
${}^{238}_{92}U_{146}$	44	10	58	20	0.83

Assuming $v_H^2(i,j)$ defining $H(2)$ are independent of (i,j) so that ζ_{biv} is independent of v_H^2, we have calculated the value of ζ_{biv} with $k_{\mathcal{O}} = 2$ for several $0\nu - \beta\beta$ decay candidate nuclei using Eq. (7.3.10) along with Eqs. (7.3.6), (7.3.8), (7.3.9) and (H34). For the function $T(\cdots)$, we use Eq. (H8). Note that $v_H^2(i,j)$ correspond to the variance of two-particle matrix elements from the p-p ($i = 2, j = 0$), n-n ($i = 0, j = 2$) and p-n ($i = 1, j = 1$) interactions. Results are given in Table 7.2. It is seen that $\zeta_{biv} \sim 0.6$–0.8. It is important to mention that $\zeta_{biv} = 0$ for GOE. Therefore, the transition strength density will be narrow in (E_i, E_f) plane. In order to establish the bivariate Gaussian form for the $0\nu - \beta\beta$ decay transition strength density, we have examined k_{PQ}, $P + Q = 4$. For a good bivariate Gaussian, $|k_{PQ}| \lesssim 0.3$. Using Eqs. (7.3.6), (7.3.8), (7.3.9), (7.3.11), (7.3.13)-(7.3.17) along with Eqs. (H34) and (H39), we have calculated the cumulants $k_{PQ}(m_1, m_2)$, $P + Q = 4$. These involve $T(\cdots)$ and $F(\cdots)$ functions. For set #1 calculations in Table 7.3, we use Eq. (H8) for $T(\cdots)$ and Eq. (H23) for $F(\cdots)$. For the set #2

Table 7.3: Cumulants k_{PQ}, $P+Q=4$ for some nuclei listed in Table 7.2. The numbers in the brackets are for the strict dilute limit as explained in the text. Just as in the construction of Table 7.2, we use $v_H^2(i,j)$ independent of (i,j). See Table 7.2 and text for details.

Nuclei	N_1	m_1	N_2	m_2	k_{40}	k_{04}	k_{13}	k_{31}	k_{22}
$^{100}_{42}Mo_{58}$	30	2	32	8	−0.45(−0.39)	−0.42(−0.38)	−0.24(−0.23)	−0.26(−0.25)	−0.20(−0.22)
$^{150}_{60}Nd_{90}$	32	10	44	8	−0.27(−0.22)	−0.29(−0.23)	−0.22(−0.18)	−0.20(−0.17)	−0.19(−0.18)
$^{154}_{62}Sm_{92}$	32	12	44	10	−0.24(−0.18)	−0.25(−0.18)	−0.19(−0.15)	−0.18(−0.15)	−0.17(−0.15)
$^{180}_{74}W_{106}$	32	24	44	24	−0.19(−0.08)	−0.20(−0.08)	−0.17(−0.08)	−0.15(−0.08)	−0.15(−0.08)
$^{238}_{92}U_{146}$	44	10	58	20	−0.18(−0.13)	−0.18(−0.13)	−0.15(−0.11)	−0.15(−0.11)	−0.13(−0.11)

calculations, shown in 'brackets' in Table 7.3, we use Eq. (H9) for $T(\cdots)$, Eq. (H14) for $F(\cdots)$ and replace everywhere $\binom{\tilde{m}_i+r}{s} \to \binom{N_i}{s}$ for any (r,s) with $i = 1,2$. Then we have the strict dilute limit. We show in Table 7.3, bivariate cumulants for five heavy nuclei for both sets of calculations and they clearly establish that bivariate Gaussian is a good approximation. We have also examined this analytically in the dilute limit with $N_1, N_2 \to \infty$ and assuming $v_H^2(i,j)$ independent of (i,j). With these, we have expanded k_{PQ} in powers of $1/m_1$ and $1/m_2$ using Mathematica. It is seen that all the k_{PQ}, $P+Q=4$ behave as,

$$k_{PQ} = -\frac{4}{m_1} + O\left(\frac{1}{m_1^2}\right) + O\left(\frac{m_2^2}{m_1^3}\right) + \ldots . \tag{7.3.18}$$

Therefore, for $m_1 >> 1$ and $m_2 << m_1^{3/2}$, the strength density approaches bivariate Gaussian form in general. It is important to recall that the strong dependence on m_1 in Eq. (7.3.18) is due to the nature of the operator $\mathcal{O}$ i.e., $\mathcal{O}(k_{\mathcal{O}})|m_1,m_2\rangle = |m_1+k_{\mathcal{O}}, m_2-k_{\mathcal{O}}\rangle$. Thus, we conclude that bivariate Gaussian form is a good approximation for $0\nu-\beta\beta$ decay transition strength densities. With this, one can apply the formulation given in [Ko-08a] with the bivariate correlation coefficient ζ_{biv} given by Eqs. (7.3.10), (7.3.9) and (7.3.8). The values given by the two-orbit binary correlation theory for ζ_{biv} can be used as starting values in practical calculations.

For completeness, we have also calculated the correlation coefficient and fourth order moments for the transition operator relevant for β decay and the results presented in Table 7.4 confirm that bivariate Gaussian form is a good approximation for β decay transition strength densities. These results justify the assumptions made in [Ko-95].

7.4 EGOE(2)-J Ensemble: Structure of Centroids and Variances for Fermions in a Single-j Shell

7.4.1 Definition and construction of EGOE(2)-J

Shell-model corresponds to m fermions occupying sp j-orbits $j_1, j_2, \ldots$ and interacting via a two body interaction $H = V(2)$ that preserves total m-particle angular momenta J. For simplicity we restrict to identical nucleons and degenerate sp ener-

Table 7.4: Correlation coefficients $\zeta_{biv}(m_1, m_2)$ and cumulants k_{PQ}, $P+Q=4$ for some nuclei relevant for β decay [$k_{\mathcal{O}}=1$ in Eq. (7.3.2)]. The first four nuclei in the table are relevant for β^- transitions, next four nuclei are relevant for electron capture and the last two nuclei are relevant for β^+ transitions. The numbers in the brackets for k_{PQ} are for the strict dilute limit as in Table 7.3. We assume $v_H^2(i,j)$ are independent of (i,j) just as in the calculations for generating Tables 7.2 and 7.3. See caption to Table 7.2 for other details.

Nuclei	N_1	m_1	N_2	m_2	$\zeta_{biv}(m_1, m_2)$	k_{40}	k_{04}	k_{13}	k_{31}	k_{22}
$^{62}_{27}Co_{35}$	20	7	30	15	0.72	−0.26(−0.18)	−0.27(−0.18)	−0.24(−0.16)	−0.23(−0.16)	−0.22(−0.16)
$^{64}_{27}Co_{37}$	20	7	30	17	0.73	−0.27(−0.16)	−0.27(−0.16)	−0.24(−0.15)	−0.23(−0.15)	−0.21(−0.15)
$^{62}_{26}Fe_{36}$	20	6	30	16	0.72	−0.28(−0.18)	−0.28(−0.18)	−0.24(−0.16)	−0.24(−0.16)	−0.22(−0.16)
$^{68}_{28}Ni_{40}$	20	8	30	20	0.72	−0.27(−0.14)	−0.27(−0.14)	−0.24(−0.13)	−0.23(−0.13)	−0.21(−0.13)
$^{65}_{32}Ge_{33}$	36	5	36	4	0.55	−0.45(−0.41)	−0.46(−0.42)	−0.35(−0.33)	−0.34(−0.32)	−0.34(−0.34)
$^{69}_{34}Se_{35}$	36	7	36	6	0.66	−0.36(−0.29)	−0.34(−0.30)	−0.28(−0.25)	−0.28(−0.25)	−0.27(−0.25)
$^{73}_{36}Kr_{37}$	36	9	36	8	0.72	−0.28(−0.23)	−0.28(−0.23)	−0.24(−0.20)	−0.24(−0.20)	−0.23(−0.20)
$^{77}_{38}Sr_{39}$	36	11	36	10	0.76	−0.24(−0.19)	−0.24(−0.19)	−0.21(−0.17)	−0.21(−0.17)	−0.20(−0.17)
$^{85}_{42}Mo_{43}$	36	15	36	14	0.79	−0.20(−0.14)	−0.21(−0.14)	−0.19(−0.13)	−0.18(−0.13)	−0.17(−0.13)
$^{93}_{46}Pd_{47}$	36	19	36	18	0.80	−0.19(−0.11)	−0.19(−0.11)	−0.18(−0.10)	−0.17(−0.10)	−0.16(−0.10)

gies. Firstly, the $V(2)$ matrix $[V(2)]$ in two-particle spaces is a direct sum of matrices, $[V(2)] = [V^{J_{12}}(2)] \oplus [V^{J'_{12}}(2)] \oplus [V^{J''_{12}}(2)] \oplus \ldots$ where J_{12} are two-particle angular momenta. Now the $[V^{J_{12}}(2)]$ matrices are represented by GOE, i.e., $V(2)$ in two-particle spaces is a direct sum of GOE's. Let us consider the example of $j = (7/2, 5/2, 3/2, 1/2)$, i.e., the nuclear $2p1f$ shell. Here $J_{12} = 0-6$ and the corresponding matrix dimensions are 4, 3, 8, 5, 6, 2, and 2, respectively. This gives 94 independent matrix elements for the $\{V(2)\}$ ensemble and they are chosen to be Gaussian variables with zero center and variance unity (variance of the diagonal elements being 2); see Eq. (1.2.4). The EGOE(2)-J ensemble in m-particle spaces is then generated by propagating this $\{V(2)\}$ ensemble to a given (m, J) space by using the shell-model geometry, i.e., by the algebra $U(N) \supset SO_J(3)$ with a suitable sub-algebra in between, where $N = \sum_i (2j_i + 1)$. Then, the m-particle H matrix elements are linear combinations of two-particle matrix elements with the expansion coefficients being essentially fractional parentage coefficients. For the $(2p1f)^{m=8}$ system, the dimensions $D(m, J)$ are 347, 880, 1390, 1627, 1755, 1617, 1426, 1095, 808, 514, 311, 151, 73, 22, 6 for $J = 0$ to 14, respectively. As the shell-model geometry is complex, EGOE(2)-J is mathematically a difficult ensemble. In the case of a single-j shell, $J_{12} = 0, 2, 4, \ldots, (2j-1)$ and $\{V^{J_{12}}(2)\}$ are one dimensional. In general, nuclear shell-model codes can be used to construct EGOE(2)-J [Br-81, Ze-04, Zh-04, Pa-07].

For a $(j)^m$ system with H's preserving angular momentum J symmetry, the operator form for a two-body H is,

$$H = \sum_{J_2 = \text{even}, M_2} V_{J_2}\, A\left(j^2; J_2 M_2\right) \left[A\left(j^2; J_2 M_2\right)\right]^\dagger \,, \tag{7.4.1}$$

where $V_{J_2} = \left\langle (j^2) J_2 M_2 \mid H \mid (j^2) J_2 M_2 \right\rangle$ are independent of M_2 and $J_2 = 0, 2, 4, \ldots, (2j-1)$. The operator $A(j^2; J_2 M_2)$ creates a two-particle state. The EGOE(2)-J ensemble for the $(j)^m$ system is generated by assuming V_{J_2}'s to be independent Gaussian random variables with zero center and variance unity,

$$\rho_{V_{J_{2a}}, V_{J_{2b}}, \ldots}(x_a, x_b, \ldots) dx_a dx_b \ldots = \rho_{V_{J_{2a}};\mathscr{G}}(x_a)\, \rho_{V_{J_{2b}};\mathscr{G}}(x_b) \ldots\, dx_a dx_b \ldots \tag{7.4.2}$$

One simple way to construct the EGOE(2)-J ensemble in m-particle spaces with a fixed-J value is as follows. Consider the $N = (2j+1)$ sp states $|jm\rangle$, $m = -j, -j+1, \ldots, j$. Now distributing m fermions in the $|jm\rangle$ orbits in all possible ways will give the configurations $[m_\nu] = |n_{\nu_1}, n_{\nu_2}, \ldots, n_{\nu_m}\rangle$ where $(\nu_1, \nu_2, \ldots, \nu_m)$ are the filled orbits so that $n_{\nu_i} = 1$. We can select configurations such that $M = \sum_{i=1}^{m} n_{\nu_i} m_{\nu_i} = 0$ for even m and $M = 1/2$ for odd m. The number of $[m_\nu]$'s for $M = 0$, with m even, is $D(m, M=0) = \sum_{J=0}^{J_{max}} d(m,J)$ and similarly for odd m, $D(m, M=1/2) = \sum_{J=1/2}^{J_{max}} d(m,J)$. Converting V_{J_2} into the $|jm\rangle\,|jm'\rangle$ basis will give,

$$
\begin{aligned}
V_{m_1,m_2,m_3,m_4} &= \langle jm_3\, jm_4 | V | jm_1\, jm_2 \rangle \\
&= 2 \sum_{J_2=\text{even}, M_2} \langle jm_1\, jm_2 | J_2 M_2 \rangle \langle jm_3\, jm_4 | J_2 M_2 \rangle V_{J_2}\,,
\end{aligned}
\tag{7.4.3}
$$

where $M_2 = m_1 + m_2 = m_3 + m_4$. The V matrix in the $[m_\nu]$ basis follows easily from the formalism used for EGOE(2) for spinless fermion systems when we use V_{m_1,m_2,m_3,m_4} matrix elements; see Chapter 1 for details. Starting with the J^2 operator and writing its one and two-body matrix elements in the $|jm\rangle\,|jm'\rangle$ basis, it is possible to construct the J^2 matrix in the $[m_\nu]$ basis. Diagonalizing this matrix will give (with $M_0 = 0$ for even m and $1/2$ for odd m) the C-coefficients in

$$
|(j)^m \alpha J M_0\rangle = \sum_{[m_\nu]} C^{\alpha J}_{[m_\nu]}\, \phi_{[m_\nu]}
\tag{7.4.4}
$$

and we can identify the J-value of the eigenfunctions by using the eigenvalues $J(J+1)$. With this, the H matrix in the $|(j)^m \alpha J M_0\rangle$ basis is

$$
\langle (j)^m \beta J M_0 | H | (j)^m \alpha J M_0 \rangle = \sum_{[m_\nu]_i, [m_\nu]_f} C^{\alpha J}_{[m_\nu]_i}\, C^{\beta J}_{[m_\nu]_f}\, \langle \phi_{[m_\nu]_f} \,|\, V \,|\, \phi_{[m_\nu]_i} \rangle\,.
\tag{7.4.5}
$$

The above procedure can be implemented on a computer easily. In our study we analyze EGOE(2)-J without explicitly constructing the H matrices in the m-particle spaces. In particular, we analyze the structure of fixed-J energy centroids $E_c(m,J)$ and spectral variances $\sigma^2(m,J)$ for $(j)^m$ systems.

Exact formulas for $E_c(m,J)$ and $\sigma^2(m,J)$ can be obtained from the results in [Ja-79, Ja-79a, Wo-86, Ve-81, Ve-82, Ve-84, No-72]. However, they are too complicated and computationally extensive. An alternative is to use the bivariate Edgeworth form for $\rho(E,M)$ and seek expansions for the centroids and variances. The expansion coefficients then will involve fourth order traces over fixed-m spaces. We will derive the expansions in Sec. 7.4.2. Trace propagation formulas for the expansion coefficients are given in Sec. 7.4.3. Finally, in Sec. 7.4.4, we will discuss the structure of $E_c(m,J)$ and $\sigma^2(m,J)$ for $(j)^m$ systems.

7.4.2 Expansions for centroids $E_c(m,J)$ and variances $\sigma^2(m,J)$

Firstly, fixed-J averages of a J invariant operator $\mathcal{O}$ follow from fixed-M averages using,

$$
\begin{aligned}
\langle \mathcal{O} \rangle^{m,J} &= \frac{\langle\langle \mathcal{O} \rangle\rangle^{m,M=J} - \langle\langle \mathcal{O} \rangle\rangle^{m,M=J+1}}{\mathcal{D}(m,M=J) - \mathcal{D}(m,M=J+1)} \\
&\simeq \left[-\left.\frac{\partial \mathcal{D}(m,M)}{\partial M}\right|_{M=J+1/2} \right]^{-1} \left[-\left.\frac{\partial \langle\langle \mathcal{O} \rangle\rangle^{M}}{\partial M}\right|_{M=J+1/2} \right].
\end{aligned}
\tag{7.4.6}
$$

Here, $\mathcal{D}(m,M)$ is fixed-M dimension. We use an expansion for the bivariate distribution $\rho^{H,m}(E,M)$ and obtain the expansion for various quantities in Eq. (7.4.6). Applying this to H and H^2 operators, we have derived expansions to order $[J(J+1)]^2$ for $E_c(m,J)$ and $\sigma^2(m,J)$. Now we present these results.

The operators H and J_z whose eigenvalues are E and M, respectively, commute and therefore the bivariate moments of $\rho^{H,m}(E,M)$ are just $M_{rs}(m) = \langle H^r J_z^s \rangle^m$; note that nuclear effective Hamiltonians are all J invariant. Now some important results are: (i) $M_{rs}(m) = 0$ for s odd and therefore all the cumulants $k_{rs}(m) = 0$ for s odd; (ii) the marginal densities $\rho(E)$ and $\rho(M)$ are close to Gaussian, the first one is a result of the fact that nuclear H's can be represented by two-body random matrix ensembles giving $k_{40}(m) \sim -4/m$ and the second as J_z is a one-body operator giving $k_{04}(m) \sim -1/m$; (iii) the correlation coefficient $\zeta_{biv}(m) = k_{11}(m) = 0$ and hence the bivariate Gaussian in (E,M) is just $\rho_{\mathcal{G}}(E)\rho_{\mathcal{G}}(M)$; (iv) random matrix representation of H shows that $k_{22}(m) \sim -2/3m$ in the dilute limit and this follows from the results in Eqs. (7.4.15), (7.4.16) and (7.4.22); (v) as $k_{rs}(m) \sim 1/m$ for $r+s=4$, one can assume further that $k_{rs}(m) \sim 1/[m^{(r+s-2)/2}]$. With (i)-(v), it is possible to use bivariate

ED expansion for $\rho(E,M)$ and the system parameter that decides the convergence of the expansion is the particle number m; see [Ko-01, Ko-84, St-87]. The expansion for $\eta(E,M)$ up to order $1/m^2$ follow from Eq. (12) and Table 2 of [Ko-84]. Using these and noting that $\widehat{E} = He_1(\widehat{E})$ and $\widehat{E}^2 - 1 = He_2(\widehat{E})$, the traces $\langle\langle(\widehat{H})^p\rangle\rangle^{m,M}$, $p = 0, 1, 2$ are given by

$$
\begin{aligned}
\langle\langle\widehat{H}\rangle\rangle^{m,M} &= \mathscr{D}_{\mathscr{G}}(m,M)\left\{\frac{k_{12}(m)}{2}He_2(\widehat{M}) + \frac{k_{14}(m)}{24}He_4(\widehat{M})\right. \\
&+ \left.\frac{k_{04}(m)k_{12}(m)}{48}He_6(\widehat{M}) + O\left(\frac{1}{m^{5/2}}\right)\right\}, \\
\langle\langle\widehat{H}^2 - 1\rangle\rangle^{m,M} &= \mathscr{D}_{\mathscr{G}}(m,M)\left\{\frac{k_{22}(m)}{2}He_2(\widehat{M}) + \frac{[k_{12}(m)]^2}{4}He_4(\widehat{M})\right. \\
&+ \frac{k_{24}(m)}{24}He_4(\widehat{M}) + \frac{k_{14}(m)k_{12}(m)}{24}He_6(\widehat{M}) \\
&+ \frac{k_{22}(m)k_{04}(m)}{48}He_6(\widehat{M}) + \frac{k_{04}(m)[k_{12}(m)]^2}{96}He_8(\widehat{M}) \\
&+ \left.O\left(\frac{1}{m^3}\right)\right\}, \\
\mathscr{D}(m,M) &= \mathscr{D}_{\mathscr{G}}(m,M)\left\{He_0(\widehat{M}) + \frac{k_{04}(m)}{24}He_4(\widehat{M})\right. \\
&+ \left.\frac{k_{06}(m)}{720}He_6(\widehat{M}) + \frac{[k_{04}(m)]^2}{1152}He_8(\widehat{M}) + O\left(\frac{1}{m^3}\right)\right\}.
\end{aligned}
\tag{7.4.7}
$$

Here we have used the results that $\int He_r(\widehat{E})He_s(\widehat{E})\eta_{\mathscr{G}}(\widehat{E})d\widehat{E} = r!\;\delta_{rs}$ and $\widehat{M} = M/\sigma_{J_z}(m)$ with $\sigma^2_{J_z}(m) = \langle J_z^2\rangle^m$.

Using Eqs. (7.4.6) and (7.4.7) and carrying out some tedious algebra (and also verified using Mathematica) will give the following expansions to order $[J(J+1)]^2$,

$$
\begin{aligned}
D(m,J) &\simeq \binom{N}{m}\frac{(2J+1)}{\sqrt{8\pi\sigma^3_{J_z}(m)}}\exp-\frac{(J+1/2)^2}{2\sigma^2_{J_z}(m)} \\
&\times \left[1 + \frac{k_{04}(m)}{24}\left\{\left[\frac{J(J+1)}{\sigma^2_{J_z}(m)}\right]^2 - 10\frac{J(J+1)}{\sigma^2_{J_z}(m)} + 15\right\}\right],
\end{aligned}
\tag{7.4.8}
$$

$$\langle \widehat{H} \rangle^{m,J} = \frac{E_c(m,J) - E_c(m)}{\sigma(m)}$$

$$= \left[\frac{k_{12}(m)}{2}\left(-3 + \frac{1}{4\sigma_{J_z}^2(m)}\right) + \frac{k_{14}(m)}{8}\left(5 - \frac{5}{6\sigma_{J_z}^2(m)} + \frac{1}{48\sigma_{J_z}^4(m)}\right)\right.$$

$$\left. + \frac{k_{04}(m)k_{12}(m)}{4}\left(-5 + \frac{5}{4\sigma_{J_z}^2(m)} - \frac{1}{24\sigma_{J_z}^4(m)}\right)\right]$$

$$+\frac{J(J+1)}{\sigma_{J_z}^2(m)}\left\{\frac{k_{12}(m)}{2} + \frac{k_{14}(m)}{12}\left(-5 + \frac{1}{4\sigma_{J_z}^2(m)}\right)\right. \tag{7.4.9}$$

$$\left. + \frac{k_{04}(m)k_{12}(m)}{4}\left(5 - \frac{1}{3\sigma_{J_z}^2(m)}\right)\right\}$$

$$+\frac{[J(J+1)]^2}{\sigma_{J_z}^4(m)}\left\{\frac{k_{14}(m)}{24} - \frac{k_{04}(m)k_{12}(m)}{6}\right\}$$

$$\simeq \left[-\frac{3k_{12}(m)}{2}\right] + \frac{k_{12}(m)}{2}\frac{J(J+1)}{\sigma_{J_z}^2(m)} + \left\{\frac{k_{14}(m)}{24} - \frac{k_{04}(m)k_{12}(m)}{6}\right\}\frac{[J(J+1)]^2}{\sigma_{J_z}^4(m)},$$

$$\frac{\sigma^2(m,J)}{\sigma^2(m)} = \langle \widehat{H}^2 \rangle^{m,J} - \left(\langle \widehat{H} \rangle^{m,J}\right)^2$$

$$= \left[1 - \frac{3k_{22}(m)}{2} + \frac{3[k_{12}(m)]^2}{2} + \frac{5k_{24}(m)}{8} - \frac{5k_{14}(m)k_{12}(m)}{2}\right.$$

$$\left. - \frac{5k_{22}(m)k_{04}(m)}{4} + \frac{15k_{04}(m)[k_{12}(m)]^2}{4}\right] + \left[\frac{J(J+1)}{\sigma_{J_z}^2(m)} + \frac{1}{4\sigma_{J_z}^2(m)}\right]$$

$$\left\{\frac{k_{22}(m)}{2} - [k_{12}(m)]^2 - \frac{5k_{24}(m)}{12} + \frac{5k_{14}(m)k_{12}(m)}{2} - 5k_{04}(m)[k_{12}(m)]^2\right. \tag{7.4.10}$$

$$\left. + \frac{5k_{22}(m)k_{04}(m)}{4}\right\} + \left[\frac{J(J+1)}{\sigma_{J_z}^2(m)} + \frac{1}{4\sigma_{J_z}^2(m)}\right]^2 \left\{\frac{k_{24}(m)}{24} - \frac{k_{14}(m)k_{12}(m)}{3}\right.$$

$$\left. - \frac{k_{22}(m)k_{04}(m)}{6} + \frac{5k_{04}(m)[k_{12}(m)]^2}{6}\right\}$$

$$\simeq \left[1-\frac{3k_{22}(m)}{2}\right]+\left[\frac{k_{22}(m)}{2}\right]\frac{J(J+1)}{\sigma^2_{J_z}(m)}+\left\{\frac{k_{24}(m)}{24}-\frac{k_{22}(m)k_{04}(m)}{6}\right\}\left[\frac{J(J+1)}{\sigma^2_{J_z}(m)}\right]^2 .$$

The last step in Eq. (7.4.9) follows from the assumption that $\sigma^2_{J_z}(m) >> 1$. Similarly, in the last step in Eq. (7.4.10), assuming that $\sigma^2_{J_z}(m) >> 1$, we have neglected $1/4\sigma^2_{J_z}(m)$ terms and so also the terms with squares and products of cumulants that are expected to be small. Note that the expansions to order $J(J+1)$ were given before [Ko-02a] and the terms with $[J(J+1)]^2$ are new. From now on, we use the last forms in Eqs. (7.4.9) and (7.4.10) and apply them to $(j)^m$ systems in the present section. To proceed further, we need to define and evaluate the bivariate cumulants $k_{rs}(m)$.

Bivariate cumulants $k_{rs}(m)$ are defined in terms of the bivariate moments $\langle \widetilde{H}^r J_z^s\rangle^m$ with $\widetilde{H} = H - \langle H\rangle^m$,

$$\begin{aligned}
k_{04}(m) &= \frac{\langle J_z^4\rangle^m}{\sigma^4_{J_z}(m)} - 3\,, \quad k_{12}(m) = \frac{\langle \widetilde{H} J_z^2\rangle^m}{\sigma(m)\sigma^2_{J_z}(m)}\,, \\
k_{14}(m) &= \frac{\langle \widetilde{H} J_z^4\rangle^m}{\sigma(m)\sigma^4_{J_z}(m)} - \frac{6\,\langle \widetilde{H} J_z^2\rangle^m}{\sigma(m)\sigma^2_{J_z}(m)}\,, \qquad (7.4.11) \\
k_{22}(m) &= \frac{\langle \widetilde{H}^2 J_z^2\rangle^m}{\sigma^2_{J_z}(m)\sigma^2(m)} - 1\,, \\
k_{24}(m) &= \frac{\langle \widetilde{H}^2 J_z^4\rangle^m}{\sigma^4_{J_z}(m)\,\sigma^2(m)} - \frac{\langle J_z^4\rangle^m}{\sigma^4_{J_z}(m)} - 6\,\frac{\langle \widetilde{H}^2 J_z^2\rangle^m}{\sigma^2_{J_z}(m)\,\sigma^2(m)} \\
&- 6\,\frac{\left[\langle \widetilde{H} J_z^2\rangle^m\right]^2}{\sigma^4_{J_z}(m)\,\sigma^2(m)} + 6\,.
\end{aligned}$$

Note that, $\sigma^2(m) = \langle \widetilde{H}^2\rangle^m$.

7.4.3 Propagation equations for bivariate cumulants $k_{rs}(m)$ for $(j)^m$ systems

To begin with, let us mention that the tensorial decomposition of the H and J^2 operators with respect to the $U(N)$, $N = 2j+1$, algebra will be useful for deriving propagation equations for $k_{rs}(m)$. For the single-j shell situation, the H operator is defined by the two-body matrix elements $V_{J_2} = \langle (j)^2 J_2 \mid H \mid (j)^2 J_2\rangle$ with J_2 being even taking values $0, 2, \ldots, 2j-1$. Using the results in Appendix A, unitary decomposition for the operators H and J^2 are,

$$H = H^{\nu=0} + H^{\nu=2}\,;$$

$$H^{\nu=0} = \frac{\hat{n}(\hat{n}-1)}{2}\,\overline{V}\,, \quad \overline{V} = \binom{N}{2}^{-1} \sum_{J_2} (2J_2+1)\, V_{J_2}\,, \tag{7.4.12}$$

$$H^{\nu=2} \Longleftrightarrow V_{J_2}^{\nu=2} = V_{J_2} - \overline{V}\,.$$

$$\begin{aligned} J^2 &= (J^2)^{\nu=0} + (J^2)^{\nu=2}\,; \\ (J^2)^{\nu=0} &= \frac{\hat{n}(N-\hat{n})}{N(N-1)}\, j(j+1)(2j+1)\,, \\ (J^2)^{\nu=2} &\Longleftrightarrow (J^2)_{J_2}^{\nu=2} = J_2(J_2+1) - (2j-1)(j+1)\,. \end{aligned} \tag{7.4.13}$$

To proceed further, we write the cumulants defined in Eq. (7.4.11) in terms of $H^{\nu=2}$ and $(J^2)^{\nu=2}$. For this purpose, we use the equalities $\langle H^p J_z^2\rangle^m = \langle H^p J^2\rangle^m/3$ and $\langle H^p J_z^4\rangle^m = \langle H^p (J^2)^2\rangle^m/5 - \langle H^p (J^2)\rangle^m/15$. Then the formulas are,

$$\begin{aligned} k_{12}(m) &= \frac{\langle H^{\nu=2} (J^2)^{\nu=2}\rangle^m}{3\,\sigma(m)\sigma_{J_z}^2(m)}\,, \\ k_{14}(m) &= \frac{1}{\sigma(m)\,\sigma_{J_z}^4(m)} \left\{ \frac{1}{5} \langle (J^2)^{\nu=2} (J^2)^{\nu=2} H^{\nu=2}\rangle^m \right. \\ &- \left. \left[\frac{4}{5}\sigma_{J_z}^2(m) + \frac{1}{15}\right] \langle (J^2)^{\nu=2} H^{\nu=2}\rangle^m \right\} \\ &\simeq \frac{1}{24\,\sigma_{J_z}^8(m)} \left\{ \frac{1}{5} \langle (J^2)^{\nu=2} (J^2)^{\nu=2} H^{\nu=2}\rangle^m \right. \\ &- \left. \frac{4}{5}\sigma_{J_z}^2(m)\ \langle (J^2)^{\nu=2} H^{\nu=2}\rangle^m \right\}\,, \end{aligned} \tag{7.4.14}$$

$$k_{22}(m) = \frac{\left\langle (H^{\nu=2})^2 (J^2)^{\nu=2} \right\rangle^m}{3\sigma^2_{J_z}(m)\,\sigma^2(m)},$$

$$\begin{aligned} k_{24}(m) &= \frac{9}{5} - \frac{1}{5\,\sigma^2_{J_z}(m)} - \frac{\left\langle J_z^4 \right\rangle^m}{\sigma^4_{J_z}(m)} + \frac{\left\langle ((J^2)^{\nu=2})^2 (H^{\nu=2})^2 \right\rangle^m}{5\,\sigma^4_{J_z}(m)\,\sigma^2(m)} \\ &- \frac{2\left[\left\langle (J^2)^{\nu=2} H^{\nu=2} \right\rangle^m\right]^2}{3\,\sigma^4_{J_z}(m)\,\sigma^2(m)} - \frac{\left\langle (J^2)^{\nu=2} (H^{\nu=2})^2 \right\rangle^m}{15\,\sigma^4_{J_z}(m)\,\sigma^2(m)} \\ &- \frac{4\left\langle (J^2)^{\nu=2} (H^{\nu=2})^2 \right\rangle^m}{5\,\sigma^2_{J_z}(m)\,\sigma^2(m)}. \end{aligned}$$

Simple trace propagation formulas that follows from the results in Appendix A are as follows,

$$\begin{aligned} &\sigma^2_{J_z}(m) = \left\langle J_z^2 \right\rangle^m = \frac{1}{3}\left\langle (J^2)^{\nu=0} \right\rangle^m = \frac{m(N-m)}{N(N-1)}\,\frac{1}{3}\,j(j+1)(2j+1)\,, \\ &\left\langle J_z^4 \right\rangle^m = \frac{9}{5}\sigma^4_{J_z}(m) - \frac{1}{5}\sigma^2_{J_z}(m) + \frac{1}{5}\left\langle (J^2)^{\nu=2} (J^2)^{\nu=2} \right\rangle^m ; \\ &\left\langle X^{\nu=2} Y^{\nu=2} \right\rangle^m = \frac{m(m-1)(N-m)(N-m-1)}{N(N-1)(N-2)(N-3)} \sum_{J_2} (2J_2+1) X^{\nu=2}_{J_2}\, Y^{\nu=2}_{J_2}\,. \end{aligned} \tag{7.4.15}$$

Note that for $\sigma^2(m) = \left\langle H^{\nu=2} H^{\nu=2} \right\rangle^m$ is given by $X = Y = H$ in last equality in Eq. (7.4.15). Similarly, $\left\langle H^{\nu=2} (J^2)^{\nu=2} \right\rangle^m$ and $\left\langle (J^2)^{\nu=2} (J^2)^{\nu=2} \right\rangle^m$ are given by $X = H$, $Y = J^2$ and $X = Y = J^2$, respectively. From now on, we use the symbols $m^\times = (N-m)$ and $[X]_r = X(X-1)\ldots(X-r+1)$, $X = m, N, m^\times$. Then, the propagation equation for $\left\langle (J^2)^{\nu=2} (J^2)^{\nu=2} H^{\nu=2} \right\rangle^m$ is [Ko-01],

$$\left\langle (J^2)^{\nu=2} (J^2)^{\nu=2} H^{\nu=2} \right\rangle^m = \frac{[m]_3 [m^\times]_3}{[N]_6} A + \left[\frac{[m]_2 [m^\times]_4 + [m]_4 [m^\times]_2}{[N]_6}\right] B\,, \tag{7.4.16}$$

where

$$\begin{aligned} A &= \sum_{\Delta} (-1)^{\Delta} (2\Delta+1)^{-1/2} [\beta^{\Delta}((J^2)^{\nu=2})]^2 \beta^{\Delta}(H^{\nu=2})\,, \\ B &= \left\langle \left\langle (J^2)^{\nu=2} (J^2)^{\nu=2} H^{\nu=2} \right\rangle \right\rangle^2 = \sum_{J_2} (2J_2+1) \left[(J^2)^{\nu=2}_{J_2}\right]^2 V^{\nu=2}_{J_2}\,. \end{aligned} \tag{7.4.17}$$

Note that Δ symbol in Eq. (7.4.17) should not be confused with 'Δ' used in Chapters 2-6. In Eq. (7.4.17), the term A is more complicated involving particle-hole matrix elements β^{Δ} of the $(J^2)^{v=2}$ and $H^{v=2}$ operators. The β^{Δ} for a $v = 2$ operator $\boldsymbol{V}$, in the example of a single j shell is

$$\beta^{\Delta}(V) = -2 \sum_{J_2 = even} (-1)^{\Delta}\sqrt{2\Delta+1}\ (2J_2+1) \left\{ \begin{array}{ccc} j & j & J_2 \\ j & j & \Delta \end{array} \right\} V_{J_2}\,. \tag{7.4.18}$$

For $j >> 1$, $(J^2)^{v=2}$ can be approximated as

$$(J^2)^{v=2}_{J_2} \simeq -2j(j+1)(2j+1) \left\{ \begin{array}{ccc} j & j & J_2 \\ j & j & 1 \end{array} \right\}. \tag{7.4.19}$$

Substituting this in Eqs. (7.4.18) will give,

$$\beta^{\Delta}[(J^2)^{v=2}] = 2j(j+1)(2j+1)\sqrt{2\Delta+1}\,(-1)^{\Delta} \left[\frac{1}{3}\delta_{\Delta,1} + (-1)^{\Delta+1} \left\{ \begin{array}{ccc} j & j & \Delta \\ j & j & 1 \end{array} \right\} \right]. \tag{7.4.20}$$

Now A in Eq. (7.4.16) takes a simple form,

$$\begin{aligned} A &= -8[j(j+1)(2j+1)]^2 \sum_{J_2} (2J_2+1)\ V^{v=2}_{J_2}\ X_{J_2}\,; \\ X_{J_2} &= \sum_{\Delta} (2\Delta+1) \left\{ \begin{array}{ccc} j & j & J_2 \\ j & j & \Delta \end{array} \right\} \left[\frac{1}{3}\delta_{\Delta,1} + (-1)^{\Delta+1} \left\{ \begin{array}{ccc} j & j & \Delta \\ j & j & 1 \end{array} \right\} \right]^2 \\ &= \frac{1}{3} \left\{ \begin{array}{ccc} j & j & J_2 \\ j & j & 1 \end{array} \right\} + 2 \left\{ \begin{array}{ccc} j & j & 1 \\ j & j & 1 \end{array} \right\} \left\{ \begin{array}{ccc} j & j & J_2 \\ j & j & 1 \end{array} \right\} + \left\{ \begin{array}{ccc} j & j & J_2 \\ j & 1 & j \\ 1 & j & j \end{array} \right\} \\ &= -\frac{(J^2)^{v=2}_{J_2}}{6Y_j} + \frac{(J^2)^{v=2}_{J_2}[j(j+1)-1]}{Y_j^2} + \frac{(J^2)^{v=2}_{J_2}[(J^2)^{v=2}_{J_2}+2]}{4Y_j^2}\,, \end{aligned} \tag{7.4.21}$$

where $Y_j = j(j+1)(2j+1)$. Above simplifications are obtained using the results given in [Ed-74, Br-94] for angular-momentum recoupling coefficients. Going further, Eq.

(7.4.16) will give the expression for $\left\langle (J^2)^{\nu=2}(H^{\nu=2})^2\right\rangle^m$ with A and B defined by

$$
\begin{aligned}
A &= \sum_{\Delta} \frac{(-1)^{\Delta}}{(2\Delta+1)^{\frac{1}{2}}} \left[\beta^{\Delta}(V^{\nu=2})\right]^2 \beta^{\Delta}((J^2)^{\nu=2}), \\
B &= \sum_{J_2} (2J_2+1)(V_{J_2}^{\nu=2})^2 (J^2)_{J_2}^{\nu=2}.
\end{aligned}
\tag{7.4.22}
$$

Using the expression for β^{Δ} for a $\nu=2$ operator from Eq. (7.4.18) and the simple formula for $\beta^{\Delta}((J^2)^{\nu=2})$ given by Eq. (7.4.20), the term A in Eq. (7.4.22) simplifies to,

$$
\begin{aligned}
A &= 8j(j+1)(2j+1) \sum_{J_2,J_2'} (2J_2+1)(2J_2'+1) V_{J_2}^{\nu=2} V_{J_2'}^{\nu=2} \\
&\times \left[\begin{Bmatrix} j & j & J_2 \\ j & j & 1 \end{Bmatrix} \begin{Bmatrix} j & j & J_2' \\ j & j & 1 \end{Bmatrix} - \begin{Bmatrix} J_2' & j & j \\ j & J_2 & 1 \end{Bmatrix}^2 \right]
\end{aligned}
\tag{7.4.23}
$$

$$
\begin{aligned}
&= \frac{2}{j(j+1)(2j+1)} \left[\sum_{J_2} (2J_2+1) V_{J_2}^{\nu=2} (J^2)_{J_2}^{\nu=2} \right]^2 \\
&- 2\sum_{J_2} (2J_2+1)(V_{J_2}^{\nu=2})^2 J_2(J_2+1).
\end{aligned}
\tag{7.4.24}
$$

Most complicated is the $k_{24}(m)$ cumulant that involves $\left\langle (J^2)^{\nu=2}(J^2)^{\nu=2}(H^{\nu=2})(H^{\nu=2})\right\rangle^m$. Equations (69) and (70) in [Wo-86] give a formula for this trace in a complex form. After carrying out the simplification of these equations and correcting errors at many places, it is seen that there will be 9 terms in the propagation equation. Table 7.5 gives the final result. We have verified that the results in Table 7.5 are correct by replacing $(J^2)^{\nu=2}$ with $H^{\nu=2}$ and then comparing with the formulas given in [No-72]. Results that are simple as in Table 7.5 for $k_{24}(m)$ for multi-j shell situation are not yet available and because of this, we have restricted our discussion to single-j shell in this section. For multi-j shell with realistic sp energies, the EGOE(1+2)-J is also called realistic TBRE (RTBRE) [Fl-00].

Table 7.5: Propagation equation for $\langle\langle (J^2)^{\nu=2}(J^2)^{\nu=2}H^{\nu=2}H^{\nu=2}\rangle\rangle^m$. Column 2 gives the input trace in a symbolic form and the corresponding expressions are given in the footnote. Column 3 gives the corresponding propagators. Multiplying the terms in column 2 with corresponding ones in column 3 and summing gives the propagation formula. Note that $N = 2j+1$.

term	Input Trace	Propagator
#1	$J^1H^1H^1J^1$	$\binom{N-8}{m-2}+\binom{N-8}{m-6}+4\binom{N-8}{m-4}$
#2	$J^1J^2H^2H^1$	$2\binom{N-8}{m-4}+\frac{4}{N}\left\{\binom{N-8}{m-3}+\binom{N-8}{m-5}\right\}-\frac{24}{N}\binom{N-8}{m-4}$
#3	$J^1H^2H^2J^1$	$\binom{N-8}{m-4}+\frac{4}{N}\left\{\binom{N-8}{m-3}+\binom{N-8}{m-5}\right\}-\frac{8}{N}\binom{N-8}{m-4}$
#4	$J^1\beta^2_J\beta^2_H H^1$	$-4\binom{N-8}{m-3}-4\binom{N-8}{m-5}+8\binom{N-8}{m-4}$
#5[a]	$J^1\beta^2_H\beta^2_H J^1$	$-2\binom{N-8}{m-3}-2\binom{N-8}{m-5}$
#6[a]	$H^1\beta^2_J\beta^2_J H^1$	$-2\binom{N-8}{m-3}-2\binom{N-8}{m-5}$
#7	$\beta^1_H\beta^1_H\beta^1_J\beta^1_J$	$3\binom{N-8}{m-4}$
#8	$\beta^1_J\beta^2_J\beta^2_H\beta^1_H$	$4\binom{N-8}{m-3}+4\binom{N-8}{m-5}-8\binom{N-8}{m-4}$
#9[a]	$\beta^1_H\beta^2_J\beta^2_J\beta^1_H$	$-8\binom{N-8}{m-4}$

$$\#1 = \left\langle\left\langle \left[(J^2)^{\nu=2}\right]^2 (H^{\nu=2})^2 \right\rangle\right\rangle^{m=2}, \quad \#2 = \left[\langle\langle (J^2)^{\nu=2}H^{\nu=2}\rangle\rangle^{m=2}\right]^2$$

$$\#3 = \left\langle\left\langle \left[(J^2)^{\nu=2}\right]^2 \right\rangle\right\rangle^{m=2} \langle\langle (H^{\nu=2})^2\rangle\rangle^{m=2}$$

$$\#4 = \sum_{\Gamma,\Delta}(2\Gamma+1)\left\{\begin{array}{ccc} j & j & \Gamma \\ j & j & \Delta \end{array}\right\}(J^2)^{\nu=2}_\Gamma\,\beta^\Delta((J^2)^{\nu=2})\,\beta^\Delta(H^{\nu=2})\,V^{\nu=2}_\Gamma$$

$$\#7 = \sum_\Delta \frac{1}{(2\Delta+1)}\left[\beta^\Delta(H^{\nu=2})\right]^2\left[\beta^\Delta((J^2)^{\nu=2})\right]^2$$

$$\#8 = \sum_\Delta \beta^\Delta((J^2)^{\nu=2})\,\beta^\Delta(H^{\nu=2})\sum_{\Gamma_2,\Gamma_3}(2\Gamma_2+1)(2\Gamma_3+1)\left\{\begin{array}{ccc} \Delta & \Gamma_2 & \Gamma_3 \\ j & j & j \end{array}\right\}^2 (J^2)^{\nu=2}_{\Gamma_2}\,V^{\nu=2}_{\Gamma_3}$$

[a]Terms $J^1\beta^2_H\beta^2_H J^1$ and $H^1\beta^2_J\beta^2_J H^1$ follow from appropriate permutations of $(J^2)^{\nu=2}$ and $H^{\nu=2}$ in the $J^1\beta^2_J\beta^2_H H^1$ expression. Similarly $\beta^1_H\beta^2_J\beta^2_J\beta^1_H$ follows by appropriate permutations in the $\beta^1_J\beta^2_J\beta^2_H\beta^1_H$ expression.

7.4.4 Structure of centroids and variances

Centroids $E_c(m,J)$

In the dilute limit with $m \to \infty$, $N \to \infty$ and $m/N \to 0$, the centroids $E_c(M,J)$ given by Eq. (7.4.9) take a simple form. Firstly, the constant term in the expansion for $E_c(M,J)$ is [after simplifying $k_{12}(m)$ and $k_{14}(m)$],

$$E_c(m) - 3\sigma(m)\frac{k_{12}(m)}{2} \simeq \frac{m^2}{N^2}\sum_{J_2}(2J_2+1)V_{J_2}\,. \qquad (7.4.25)$$

Similarly, the $J(J+1)$ term is

$$\sigma(m)\frac{k_{12}(m)}{2\sigma^2_{J_z}(m)} \simeq \frac{3}{2[j(j+1)]^2N^2}\sum_{J_2}(2J_2+1)\;V_{J_2}\;(J^2)^{v=2}_{J_2}\,. \qquad (7.4.26)$$

More remarkable is that the $[J(J+1)]^2$ term $\frac{\sigma(m)k_{14}(m)}{24\sigma^4_{J_z}(m)} - \frac{\sigma(m)k_{04}(m)k_{12}(m)}{6\sigma^4_{J_z}(m)}$ also takes a simple form. The results in Sec. 7.4.3 will give the expression for the first term as,

$$\begin{aligned}
\sigma(m)\,\frac{k_{14}(m)}{24\,\sigma^4_{J_z}(m)} &= \sum_{J_2}(2J_2+1)\;V^{v=2}_{J_2}\;S_{J_2}\,; \\
S_{J_2} &\simeq \frac{9}{40m^2(N-m)^2N^2[j(j+1)]^4}\left\{3\left[(J^2)^{v=2}_{J_2}\right]^2(N-2m)^2\right. \\
&\qquad \left. -4(J^2)^{v=2}_{J_2}\;j(j+1)\;\left[2N^2-2Nm+2m^2\right]\right\}.
\end{aligned} \qquad (7.4.27)$$

Similarly, we can write the expression for $\frac{\sigma(m)k_{04}(m)k_{12}(m)}{6\sigma^4_{J_z}(m)}$ and in the dilute limit this reduces exactly to the second piece in the expression for S_{J_2} in Eq. (7.4.27). Therefore, in the dilute limit, the term multiplying $[J(J+1)]^2$ in the $E_c(m,J)$ expansion is,

$$\begin{aligned}
&\frac{\sigma(m)}{\sigma^4_{J_z}(m)}\left\{\frac{k_{14}(m)}{24} - \frac{k_{04}(m)k_{12}(m)}{6}\right\} = \sum_{J_2}(2J_2+1)\;V^{v=2}_{J_2}\;R_{J_2}\,; \\
&R_{J_2} \sim \frac{9(N-2m)^2}{40m^2(N-m)^2N^2[j(j+1)]^4}\left\{3\left[J_2(J_2+1)-2j(j+1)\right]^2\right\}.
\end{aligned} \qquad (7.4.28)$$

It is already pointed out in [Ko-02a] that the constant term and the $J(J+1)$ term as given by Eqs. (7.4.25) and (7.4.26) are same as those derived by Mulhall et al [Mu-

00, Mu-02] using statistical mechanics approach that is completely different from the present moment method approach. For EGOE(2)-J ensemble, Eqs. (7.4.25)-(7.4.28) will give the distribution of the centroids over the ensemble as discussed in [Mu-02]. More remarkable is that the $[J(J+1)]^2$ term given by Eq. (7.4.28) is also very close to the formula given by Mulhall [Mu-02]. These results confirm that the approximations used in [Mu-00, Mu-02] are equivalent to the proposition that $\rho(E,M)$ is a Edgeworth corrected bivariate Gaussian as assumed in the present approach. The equivalence of Mulhall et al approach with the moment method approach in the dilute limit is further substantiated by the expansion for fixed-M occupancies; the results are given in Appendix I.

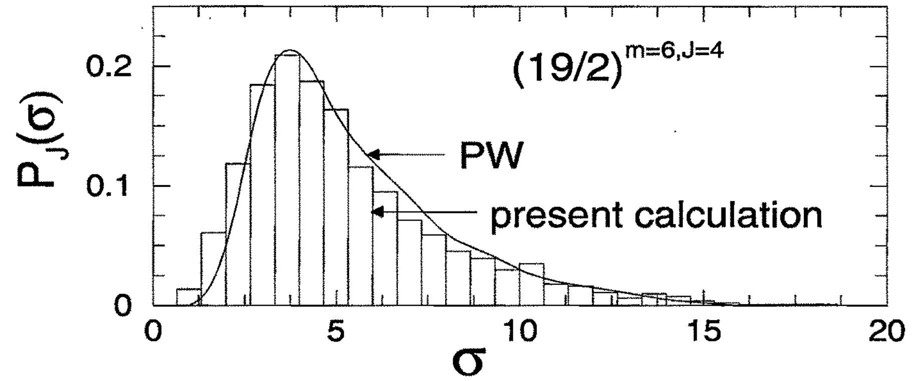

Figure 7.1: Probability distribution for widths σ for EGOE(2)-J ensemble; see text for details.

Variances $\sigma^2(m,J)$

In the dilute limit, simplifying $k_{22}(m)$ and $\sigma^2(m)$ will give

$$\begin{aligned}\sigma^2(m,J) &= \frac{m^2}{N^2}\sum_{J_2}(2J_2+1)(V_{J_2}^{\nu=2})^2 \\ &+ \frac{3\,J(J+1)}{2\,N^2[j(j+1)]^2}\sum_{J_2}(2J_2+1)(V_{J_2}^{\nu=2})^2\langle J^2\rangle^{\nu=2}\,.\end{aligned} \tag{7.4.29}$$

However to add $[(J(J+1)]^2$ correction, we need to simplify $k_{24}(m)$ and this is quite cumbersome. A quantity of interest, as pointed out by Papenbrock and Weidenmüller [Pa-04] (PW) is the probability distribution for the spectral widths $\sigma = \{\langle H^2\rangle^{m,J}\}^{1/2} = [\sigma^2(m,J) + E_c^2(m,J)]^{1/2}$ over the EGOE(2)-J ensemble. To compare with PW results, we have generated a EGOE(2)-J ensemble for $(\frac{19}{2})^{m=6}$ system with 2500 members, i.e we have used 2500 sets of V_{J_2}'s. Using the formalism described in Secs. 7.4.2 and 7.4.3 we have calculated the bivariate cumulants $k_{rs}(m)$. For our example $\sigma_{J_z}(m) = 12.124$ and $k_{04}(m) = -0.229$. The ensemble averaged cumulants $\overline{k_{12}(m)}$, $\overline{k_{14}(m)} \sim 0$ as expected. However $\overline{k_{22}(m)} = -0.053$ and $\overline{k_{24}(m)} = -0.114$. With these, it is clear that the expansions to order $[J(J+1)]^2$ are needed. Equation (7.4.10) is found to be good for $J < 30$. We have calculated $\langle H^2 \rangle^{m,J}$ for each member of the ensemble and then $P_J(\sigma)$ vs σ histograms are constructed for various J values. Results for $J = 4$ are shown in Fig. 7.1. The calculated histogram is in good agreement with the exact curve given by PW [Pa-04]; in [Pa-04] a completely different formalism is used. Though not shown in Fig. 7.1, we have noticed that for $J = 0$, the widths given by the exact results (they are reported in [Pa-04]) are somewhat larger than the numbers given by the present formalism. This could be because $J = 0$ is at one extreme end of the Edgeworth expansion and therefore, the truncation to $1/m^2$ terms may not be adequate.

7.5 Summary

To summarize, by extending the binary correlation approximation method for two different operators and for traces over two-orbit configurations, we have derived formulas for γ_1 and γ_2 parameters for EGOE(1+2)-π ensemble. Note that EGOE(1+2)-π is defined by the embedding algebra $U(N) \supset U(N_+) \oplus U(N_-)$ with the Hamiltonian breaking the symmetry in a particular way as discussed in Chapter 5. In addition, we have derived formula for the fourth order trace defining correlation coefficient of the bivariate transition strength of the transition operator relevant for 0ν-$\beta\beta$ decay. We have also derived the formulas for the fourth order cumulants in order to establish bivariate Gaussian form of the transition strength densities. Here also the embedding algebra is $U(N) \supset U(N_p) \oplus U(N_n)$ with the Hamiltonian preserving the symmetry and the transition operator breaking the symmetry in a particular way. Going further, we have considered an application to EGOE(2)-J for fermions in a single-j shell. Here the embedding algebra is $U(2j+1) \supset SO_J(3)$. Expansions to order $[J(J+1)]^2$ for energy centroids $E_c(m,J)$ and spectral variances $\sigma^2(m,J)$ are obtained. Formulas are derived for fixed-m bivariate cumulants and they are used to show the expansion to order $[J(J+1)]^2$ explain the structure of fixed-J centroids and variances. These results are important in the subject of regular structures generated by random interactions.