Chapter 9

Conclusions and Future Outlook

In this thesis, we have identified and systematically analyzed many different physically relevant EGEs with symmetries by considering a variety of quantities and measures that are important for finite interacting quantum systems such as nuclei, quantum dots, small metallic grains and ultracold atoms. The studies carried out and the main results obtained in the thesis are as follows.

Finite interacting Fermi systems with a mean-field and a chaos generating two-body interaction are modeled, more realistically, by one plus two-body embedded Gaussian orthogonal ensemble of random matrices with spin degree of freedom [called EGOE(1+2)-**s**]. Numerical calculations are used to demonstrate that, as λ, the strength of the interaction (measured in the units of the average spacing of the single particle levels defining the mean-field), increases, generically there is Poisson to GOE transition in level fluctuations, Breit-Wigner to Gaussian transition in strength functions (also called local density of states) and also a duality region where information entropy will be the same in both the mean-field and interaction defined basis. Spin dependence of the transition points λ_c, λ_F and λ_d, respectively, is described using the propagator for the spectral variances and the analytical formula for the propagator is derived. We have further established that the duality region corresponds to a region of thermalization. For this purpose we have compared the single particle entropy defined by the occupancies of the single particle orbitals with thermodynamic entropy and information entropy for various λ values and they are very close to each other at $\lambda = \lambda_d$. Chaos markers play an important role in quantum information science, statistical nuclear spectroscopy and thermalization in finite quantum systems.

EGOE(1+2)-**s** also provides a model for understanding general structures generated by pairing correlations. In the space defined by EGOE(1+2)-**s** ensemble for fermions, pairing defined by the algebra $U(2\Omega) \supset Sp(2\Omega) \supset SO(\Omega) \otimes SU_S(2)$ is identified and some of its properties are derived. Using numerical calculations it is shown that in the strong coupling limit, partial densities defined over pairing subspaces are close to Gaussian form and propagation formulas for their centroids and variances are derived. As a part of understanding pairing correlations in finite Fermi systems, we have shown that pair transfer strength sums (used in nuclear structure) as a function of excitation energy (for fixed S), a statistic for onset of chaos, follows, for low spins, the form derived for spinless fermion systems, i.e., it is close to a ratio of Gaussians. Going further, we have considered a quantity in terms of ground state energies, giving conductance peak spacings in mesoscopic systems at low temperatures, and studied its distribution over EGOE(1+2)-**s** by including both pairing and exchange interactions. This model is shown to generate bimodal to unimodal transition in the distribution of conductance peak spacings consistent with the results obtained using realistic calculations for small metallic grains.

For m fermions in Ω number of single particle orbitals, each four-fold degenerate, we have introduced and analyzed in detail embedded Gaussian unitary ensemble of random matrices generated by random two-body interactions that are $SU(4)$ scalar [EGUE(2)-$SU(4)$]. Here, the $SU(4)$ algebra corresponds to the Wigner's supermultiplet $SU(4)$ symmetry in nuclei. Embedding algebra for the EGUE(2)-$SU(4)$ ensemble is $U(4\Omega) \supset U(\Omega) \otimes SU(4)$. Exploiting the Wigner-Racah algebra of the embedding algebra, analytical expression for the ensemble average of the product of any two m-particle Hamiltonian matrix elements is derived. Using this, formulas for a special class of $U(\Omega)$ irreps are derived for the ensemble averaged spectral variances and also for the covariances in energy centroids and spectral variances. On the other hand, simplifying the tabulations available for $SU(\Omega)$ Racah coefficients, numerical calculations are carried out for general $U(\Omega)$ irreps. Spectral variances clearly show, by applying the so-called Jacquod and Stone prescription, that the EGUE(2)-$SU(4)$ ensemble generates ground state structure just as the quadratic Casimir invariant (C_2) of $SU(4)$. This is further corroborated by the calculation of the expectation values of

$C_2[SU(4)]$ and the four periodicity in the ground state energies. Secondly, it is found that the covariances in energy centroids and spectral variances increase in magnitude considerably as we go from EGUE(2) for spinless fermions to EGUE(2) for fermions with spin to EGUE(2)-$SU(4)$ implying that the differences in ensemble and spectral averages grow with increasing symmetry. Also for EGUE(2)-$SU(4)$ there are, unlike for GUE, non-zero cross-correlations in energy centroids and spectral variances defined over spaces with different particle numbers and/or $U(\Omega)$ [equivalently $SU(4)$] irreps. In the dilute limit defined by $\Omega \to \infty$, $r >> 1$ and $r/\Omega \to 0$, for the $\{4^r, p\}$ irreps, we have derived analytical results for these correlations. All correlations are non-zero for finite Ω and they tend to zero as $\Omega \to \infty$.

One plus two-body embedded Gaussian orthogonal ensemble of random matrices with parity [EGOE(1+2)-π] generated by a random two-body interaction (modeled by GOE in two particle spaces) in the presence of a mean-field, for spinless identical fermion systems, in terms of two mixing parameters and a gap between the positive ($\pi = +$) and negative ($\pi = -$) parity single particle states is introduced. Numerical calculations are used to demonstrate, using realistic values of the mixing parameters, that this ensemble generates Gaussian form (with corrections) for fixed parity state densities for sufficiently large values of the mixing parameters. The random matrix model also generates many features in parity ratios of state densities that are similar to those predicted by a method based on the Fermi-gas model for nuclei. We have also obtained a simple formula for the spectral variances defined over fixed-(m_1, m_2) spaces where m_1 is the number of fermions in the +ve parity single particle states and m_2 is the number of fermions in the −ve parity single particle states. The smoothed densities generated by the sum of fixed-(m_1, m_2) Gaussians with lowest two shape corrections describe the numerical results in many situations. The model also generates preponderance of +ve parity ground states for small values of the mixing parameters and this is a feature seen in nuclear shell-model results.

Turning to interacting boson systems, for m number of bosons, carrying spin ($\mathbf{s} = \frac{1}{2}$) degree of freedom, in Ω number of single particle orbitals, each doubly degenerate, we have introduced and analyzed embedded Gaussian orthogonal ensemble of random matrices generated by random two-body interactions that are spin (S) scalar [BEGOE(2)-$\mathbf{s}$]. The ensemble BEGOE(2)-$\mathbf{s}$ is intermediate to the BEGOE(2) for spinless bosons and for bosons with spin $\mathbf{s} = 1$ which is relevant for spinor BEC. Embedding algebra for the BEGOE(2)-**s** ensemble and also for BEGOE(1+2)-**s** that includes the mean-field one-body part is $U(2\Omega) \supset U(\Omega) \otimes SU(2)$ with $SU(2)$ generating spin. A method for constructing the ensembles in fixed-(m, S) spaces has been developed. Numerical calculations show that for BEGOE(2)-**s**, the fixed-(m, S) density of states is close to Gaussian and level fluctuations follow GOE in the dense limit. For BEGOE(1+2)-**s**, generically there is Poisson to GOE transition in level fluctuations as the interaction strength (measured in the units of the average spacing of the single particle levels defining the mean-field) is increased. The interaction strength needed for the onset of the transition is found to decrease with increasing S. Propagation formulas for the fixed-(m, S) space energy centroids and spectral variances are derived for a general one plus two-body Hamiltonian preserving spin. Derived also is the formula for the variance propagator for the fixed-(m, S) ensemble averaged spectral variances. Using these, covariances in energy centroids and spectral variances are analyzed. Variance propagator clearly shows that the BEGOE(2)-**s** ensemble generates ground states with spin $S = S_{max}$. This is further corroborated by analyzing the structure of the ground states in the presence of the exchange interaction $\hat{S}^2$ in BEGOE(1+2)-**s**. Natural spin ordering (S_{max}, $S_{max} - 1$, $S_{max} - 2$, ..., 0 or $\frac{1}{2}$) is also observed with random interactions. Going beyond these, we have also introduced pairing symmetry in the space defined by BEGOE(2)-**s**. Expectation values of the pairing Hamiltonian show that random interactions exhibit pairing correlations in the ground state region.

Parameters defining many of the important spectral distributions (valid in the chaotic region), generated by EGEs, involve traces of product of four two-body operators. For example, these higher order traces are required for calculating nuclear structure matrix elements for $\beta\beta$ decay and also for establishing Gaussian density of states generated by various embedded ensembles. Extending the binary correlation approximation method for two different operators and for traces over two-orbit configurations, we have derived formulas, valid in the dilute limit, for both skewness and excess parameters for EGOE(1+2)-π. In addition, we have derived a formula for the traces defining the correlation coefficient of the bivariate transition strength distribution generated by the two-body transition operator appropriate for calculating 0ν-$\beta\beta$ decay nuclear transition matrix elements and also for other higher order traces required for justifying the bivariate Gaussian form for the strength distribution. With applications in the subject of regular structures generated by random interactions, we have also derived expressions for the coefficients in the expansions to order $[J(J+1)]^2$ for the energy centroids $E_c(m,J)$ and spectral variances $\sigma^2(m,J)$ generated by EGOE(2)-J ensemble members for the single-j situation. These also involve traces of four two-body operators.

In order to establish random matrix structure of nuclear shell-model Hamiltonian matrices, we have presented a comprehensive analysis of the structure of Hamiltonian matrices based on visualization of the matrices in three dimensions as well as in terms of measures for GOE, banded and embedded random matrix ensembles. We have considered two nuclear shell-model examples, ^{22}Na with $J^\pi T = 2^+0$ and ^{24}Mg with $J^\pi T = 0^+0$ and, for comparison we have also considered SmI atomic example with $J^\pi = 4^+$. It is clearly established that the matrices are neither GOE nor banded. For the EGOE [strictly speaking, EGOE(2)-JT or EGOE(2)-J] structure we have examined the correlations between diagonal elements and eigenvalues, fluctuations in the basis states variances and structure of the two-body part of the Hamiltonian in the eigenvalue basis. Unlike the atomic example, nuclear examples show that the nuclear shell-model Hamiltonians can be well represented by EGOE.

In summary, in this thesis, large number of new results are obtained for embedded ensembles EGOE(1+2)-**s**, EGUE(2)-$SU(4)$, EGOE(1+2)-π and BEGOE(1+2)-**s**, with EGUE(2)-$SU(4)$ introduced for the first time in this thesis. Moreover, some results are presented for EGOE(2)-J and for the first time BEGOE(1+2)-**s** has been explored in detail in this thesis. In addition, formulas are derived, by extending the binary correlation approximation method, for higher order traces for embedded ensembles with $U(N) \supset U(N_1) \oplus U(N_2)$ embedding and some of these are needed for new applications of statistical nuclear spectroscopy. Results of the present thesis establish that embedded Gaussian ensembles can be used gainfully to study a variety of problems in many-body quantum physics.

Some of the future studies in embedded ensembles should include the following.

- It is important to examine the energy dependence of the transition markers generated by EGOE(1+2)-**s** and this will give new information about onset of chaos in interacting many-particle systems as we increase the excitation energy. In addition, going beyond the strength functions and occupancies, the distribution of transition strengths, generated by a general one-body transition operator, that is a vector in the spin space should be studied. This is important for producing a better random matrix basis for the smoothed forms for transition strength densities.

- Going beyond the measures employed in Chapter 2, new entanglement measures, introduced in the context of quantum information science, should be analyzed to characterize complexity in quantum many-body systems; for disordered spin-1/2 lattice systems, entanglement and delocalization are found to be strongly correlated [Br-08, Pi-08]. Besides the entanglement measures, further analysis of the thermodynamic region generated by two-body ensembles (defined by λ_d in Fig. 2.11) using long-time averages of various complexity measures as discussed in [Ca-09, Ri-08] is needed. This, besides being important in QIS, should lead to a deeper understanding of wavefunction thermalization in generic isolated many-body quantum systems [Ko-11, Ca-09, Ri-08, Ge-00, Fl-00].

- It is possible to apply the Hamiltonian in Eq. (3.5.5) with sp energies drawn from GOE (or GUE), adding a particle number dependent term and also by varying the interaction strength λ. Analysis with this H will generalize the results in Fig. 3.7 and also those reported in [Sc-08]. Furthermore, it will be useful to consider a generalized pairing operator by extending Eq. (3.2.1) to $P = \sum_i \beta_i P_i$ where β_i are free parameters.

- For evaluating $\gamma_2(m, f_m)$ for EGUE(2)-$SU(4)$, even for $f_m^{(p)}$ irreps, the needed $SU(\Omega)$ Racah coefficients are not available in analytical form nor there are tractable methods for their numerical evaluation. The mathematical problem here is challenging and its solution will establish the Gaussian form of the eigenvalue densities generated by EGUE(2)-$SU(4)$.

- It will also be interesting to analyze EGOE/EGUE with $SU(4) - ST$ symmetry.

With this, it will be possible to understand the role of random interactions in generating the differences in the gs structure of even-even and odd-odd N=Z nuclei.

- It is important to investigate EGOE(1+2)-π for proton-neutron systems and then we will have four unitary orbits (two for protons and two for neutrons). This extended EGOE(1+2)-π model with protons and neutrons occupying different sp states will be generated by a 10×10 block matrix for $V(2)$ in two-particle spaces with 14 independent variances. Therefore, parametrization of this ensemble is more complex.

- Further extension of BEGOE(1+2) including $\mathbf{s}=1$ (also spin 2 etc.) degree of freedom for bosons, as discussed in Appendix G, is relevant for spinor BEC studies [Pe-10, Yi-07] and this ensemble should be analyzed so that realistic applications of BEGOE can be attempted.

- Wavefunction structure should be analyzed for BEGOE(1+2)-**s** and with this, it is possible to address questions related to thermalization in finite interacting boson systems.

- Binary correlation theory for EGEs with symmetries [going beyond direct sum sub-algebra of $U(N)$] needs to be developed and then it is possible to derive results for the excess parameter γ_2 for EGOE(2)-**s** and EGUE(2)-$SU(4)$ ensembles.

- Extensions of binary correlation approximation to spinless boson systems and for boson systems with spin ($\mathbf{s}=1/2$ and 1) will be interesting and may prove to be useful in ultracold atom studies.

- Binary correlation results presented for EGOE(1+2)-π in Chapter 7 should be extended further for deriving spectral properties of partitioned EGOE [Ko-01, Ko-99].

- Applications of embedded ensembles to wider class of systems like quantum dots, BEC etc. should be carried out by deriving results for physically relevant quantities that can be confronted directly with experimental data.

- In systems like nuclei and quantum dots, it is important to find experimental signatures for cross-correlations (they are discussed in Chapters 4 and 6 and Appendix C) as they will give direct evidence for embedded ensembles. This requires identifying measures involving cross-correlations in lower order moments of the two-point function that can be used in data analysis.

- In future, it is important to analyze embedded ensembles with much larger matrix dimensions that are needed for particle number $m \geq 10$. This requires new numerical efforts.

- In literature and also in this thesis, embedded ensembles with only GOE and GUE embedding are explored. In future, embedded ensembles with GSE embedding (EGSE) should be attempted.

- New efforts in developing further the Wigner-Racah algebra for general $SU(N)$ groups are needed for more complete analytical tractability of embedded random matrix ensembles. For example, analytical form for the two-point function is not yet available even for the spinless EGOE/BEGOE.

- Starting with $U(N)$ algebra for m fermions/bosons in N sp states, we have identified EEs with embedding defined by some of the $U(N)$ sub-algebras. As $U(N)$ admits very large class of sub-algebras, it is possible to identify many more EEs that could be physically relevant and this exploration will enrich the subject of embedded random matrix ensembles.